Orbit Equivalence since
Zimmer’s Cocycle Superrigidity Theorem

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Orbit Structures of $\mathbb{II}_1$ Group Actions

$\mathbb{II}_1$ actions $(X, \mu, \Gamma)$:

- $\Gamma$ – discrete countable group
- $(X, \mathcal{B}, \mu)$ – std prob space $\cong ([0, 1], \text{Borel}, \text{Lebesgue})$
- $\Gamma \curvearrowright (X, \mu)$ – ergodic m.p. ($\gamma_*\mu = \mu, \quad \forall \gamma \in \Gamma$)

Orbit Equivalence $(X, \mu, \Gamma)$ $\text{OE} \cong (Y, \nu, \Lambda)$ if

$\exists T : (X, \mu) \sim (Y, \nu)$ s.t. $T(\Gamma \cdot x) = \Lambda \cdot T(x)$

Denoting by $R_\Gamma, X = \{ (x, y) \in X \times X | \Gamma \cdot x = \Gamma \cdot y \}$ the orbit relation

$T : (X, \mu, \Gamma)$ $\text{OE} \cong (Y, \nu, \Lambda) \Leftrightarrow T \times T (R_\Gamma, X) \sim = R_{\Lambda, Y}$

Stable, or weak, OE: $(X, \mu, \Gamma)$ $\text{sOE} \cong (Y, \nu, \Lambda)$

$X \supset A \overset{T}{\rightarrow} B \subset Y$ $T \times T (R_X, \Gamma | A \times A) \sim = R_{Y, \Lambda | B \times B}$
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Stable, or weak, OE: \( (X, \mu, \Gamma) \overset{sOE}{\sim} (Y, \nu, \Lambda) \)

\[ X \ni A \xrightarrow{T} B \subset Y \quad T \times T(R_{X,\Gamma}|_{A \times A}) \cong R_{Y,\Lambda}|_{B \times B} \]
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**Theorem (Zimmer, 1981)**

Let $G_1, G_2$ be center free simple Lie groups, $\text{rk}(G_1) \geq 2$, $\Gamma_i < G_i$ lattices $\Gamma_i \bowtie (X_i, \mu_i)$ ess. free $\text{II}_1$ actions $(X_1, \mu_1, \Gamma_1) \overset{\text{sOE}}{\sim} (X_2, \mu_2, \Gamma_2)$. 
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For free $\text{II}_1$ actions of higher rank lattices $\Gamma < G$ any $\mathcal{R}_{X,\Gamma}$ remembers $G$!
Cocycles and Orbit Equivalence

Definition

$c : G \times X \to H$ is a measurable **cocycle** if for all $g_1, g_2 \in G$ a.e. on $X$

$$c(g_2 g_1, x) = c(g_2, g_1 \cdot x) \cdot c(g_1, x)$$
Cocycles and Orbit Equivalence

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c : G × X → H is a measurable cocycle if for all g_1, g_2 ∈ G a.e. on X

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any measurable f : X → H defines a conjugate cocycle

\[ c^f(g, x) = f(gx) c(g, x) f(x)^{-1} \]
## Cocycles and Orbit Equivalence

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Any measurable $f : X \to H$ defines a **conjugate** cocycle

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### Example

Any Orbit Equivalence $T : (X, \mu, G) \to (Y, \nu, H)$ of **free** $\text{II}_1$ actions, defines

$$c : G \times X \to H \quad \text{by} \quad T(g . x) = c(g, x) . T(x)$$
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If \( c^f(g, x) = \rho(g) \), then \( \rho : G → H \) is a group homomorphism,
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$$c: G \times X \to H \quad \text{by} \quad T(g \cdot x) = c(g, x) \cdot T(x)$$

If $c^f(g, x) = \rho(g)$, then $\rho: G \to H$ is a group homomorphism, and

$$T'(x) = f(x) \cdot T(x) \quad \text{satisfies} \quad T'(g \cdot x) = \rho(g) \cdot T'(x)$$
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Any Orbit Equivalence \( T : (X, \mu, G) \rightarrow (Y, \nu, H) \) of free II\(_1\) actions, defines

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\]

Moreover, \( T' : (X, \mu) \cong (Y, \nu) \) is a measure space iso and \( \rho \) is a group iso.
Zimmer’s Cocycle Superrigidity Theorem

Theorem (Zimmer, 1981)

Let $G$, $H$ be (semi)simple Lie groups with $\text{rk}(G) \geq 2$, $G \bowtie (X, \mu)$ an (irr) erg. p.m.p. action, $\alpha : G \times X \rightarrow H$ a non-compact Zariski dense cocycle.

Then $\alpha$ is conjugate to a homomorphism $\rho : G \rightarrow H$.

Same for cocycles $\Gamma \bowtie (X, \mu)$ where $\Gamma < G$ is a lattice.

A generalization of

Theorem (Margulis, 1973)

Let $G$, $H$ be (semi)simple Lie groups, $\text{rk}(G) \geq 2$, $\Gamma < G$ an (irr) lattice, $\rho : \Gamma \rightarrow H$ a homomorphism with unbounded Zariski dense $\rho(\Gamma)$.

Then $\rho$ extends to $G \rightarrow H$. 
Definition (Gromov)

Let $\Gamma_1, \Gamma_2$ be two groups

1. **Topological Equivalence** is a loc cpt space $\Sigma$ with cont action of $\Gamma_1 \times \Gamma_2$ with $\Gamma_i \curvearrowright \Sigma$ properly disc and cocompact.

2. **Measure Equivalence** is a measure space $(\Omega, m)$ with a m.p. action of $\Gamma_1 \times \Gamma_2$ s.t. $\Gamma_i \curvearrowright \Omega$ has a finite measure fundamental domain.
Measurable Group Theory

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Theorem

1. $\Gamma_1 \overset{TE}{\sim} \Gamma_2$ if and only if $\Gamma_1 \overset{qi}{\sim} \Gamma_2$ (Gromov).

2. $\Gamma_1 \overset{ME}{\sim} \Gamma_2$ if and only if $\exists$ free $(X_1, \Gamma_1) \overset{sOE}{\sim} (X_2, \Gamma_2)$. 

Example

1. Uniform lattices $\Gamma_1, \Gamma_2$ in a loc cpt group $G$: $\Gamma_1 \bowtie G \bowtie \Gamma_2$.

2. Arbitrary lattices $\Gamma_1, \Gamma_2$ in a loc cpt group $G$: $\Gamma_1 \bowtie G \bowtie \Gamma_2$. 
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Measure Equivalence and Higher Rank Lattices

Theorem (F. 1999)

Let $G$ be simple $rk(G) \geq 2$, $\Gamma < G$ and $\Lambda$ any group with $\Gamma \sim^{ME} \Lambda$.
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Let $G$ be simple $\text{rk}(G) \geq 2$, $\Gamma < G$ and $\Lambda$ any group with $\Gamma \overset{\text{ME}}{\sim} \Lambda$

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Let $\Gamma \curvearrowright (X, \mu)$ be a II$_1$ action of lattice $\Gamma < G$, simple $rk(G) \geq 2$.

Let $\Lambda \curvearrowright (Y, \nu)$ be any free II$_1$ action with $(X, \Gamma) \overset{sOE}{\sim} (Y, \Lambda)$. Then

- If $X \not\rightarrow G/\Gamma'$, then $\Gamma \simeq \Lambda$ and $\Gamma \curvearrowright X \simeq \Lambda \curvearrowright Y$. 

Other applications

Feldman-Moore question, computations of $\text{Out}(R^X, \Gamma) = \text{Aut}/\text{Inn}$, Enveloping grps for lattices

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- Otherwise, for any $\pi : X \to G/\Gamma_\pi$ there is $(X_\pi, \Gamma_\pi) \overset{sOE}{\sim} (X, \Gamma)$.
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Rigidity for Products of Hyperbolic-like groups

Theorem (Monod-Shalom 2005)

Let \( \Gamma = \prod_{i}^{n} \Gamma_i \acts (X, \mu) \) free \( n \geq 2 \), where \( \Gamma_i \) are “hyperbolic-like” and \( \Gamma_i \acts (X, \mu) \) erg. \( i = 1, 2 \).
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- $R_{X,\Gamma}$ remembers the number of factors: $n$.
- If $\Lambda \curvearrowright (Y, \nu)$ is any free and mildly mixing, and $R_{X,\Gamma} \sim R_{Y,\Lambda}$ then $\Gamma \cong \Lambda$ and $\Gamma \curvearrowright X \cong \Lambda \curvearrowright Y$.
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Results of Kida 2006,+

For $\Gamma$ Mapping Class Group

- Full rigidity: $\text{ME}(\Gamma) = \{ \Gamma \}$, $\text{OE} = \text{isom}$ (up to finite)
- $\Gamma$ is not a lattice in any loc comp $G$ (except trivial)
- Computations of $\text{Out}(\mathcal{R}_X,\Gamma)$

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Ingredients
- Boundary theory on Thurston’s compactification (amenability,+)
- Ivanov’s $\Gamma = Aut(Curve Cpx)$ for groupoids
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- Treeability, anti-treeability (Adams)
- $\text{cost}(\mathcal{R}_{X,F_n}) = n, \ldots$ (Levitt, Gaboriau)
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- Homological invariants (Sauer)

Applications

- Descriptive Set Theory: Adams-Kechris, Hjorth, Thomas, ...

Applications to QI of amenable groups

- Shalom, Sauer
New Cocycle Superrigidity (after Sorin Popa)

Theorem (Popa 2006)

Let $\Gamma$ have (T) and $\Gamma \curvearrowright X = (X_0, \mu_0)^{\Gamma}$ be a Bernoulli action. Let any discrete, or cpt (or $\in U_{\text{fin}}$) group. Then any cocycle $\alpha : \Gamma \times X \to \Lambda$ is conjugate in $\Lambda$ to a homomorphism $\rho : \Gamma \to \Lambda$.
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Corollary

For \( \Gamma \curvearrowright (X, \mu) \) as above, \( R_{X,\Gamma} \) remembers \( \Gamma \) and \( \Gamma \curvearrowright X \).
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Corollary

For $\Gamma \curvearrowright (X, \mu)$ as above, $R_{X, \Gamma}$ remembers $\Gamma$ and $\Gamma \curvearrowright X$.

Theorem (Ioana 2007)

Let $\Gamma$ have $(T)$, $K = \lim \Gamma / \Gamma_i$ be a profinite completion. $\alpha : \Gamma \times K \to \Lambda$ any cocycle into any $\Lambda \in U_{\text{fin}}$. Then $\exists i$, and $\rho : \Gamma_i \to \Lambda$ so that $\alpha|_{\Gamma_i \times K_i}$ is conjugate to $\rho$, $K_i = \Gamma_i < K$. 

A.Furman () Zimmer's 60th birthday conference September 8, 2007 11 / 14
Theorem (F. after Ioana, 2007)

Let $\Gamma$ have $(\mathcal{T})$, $K$ compact, $\tau: \Gamma \to K$ dense hom.
$\alpha: \Gamma \times K \to \Lambda$ any cocycle into any discrete group.

Then $\exists$ a hom $\rho: \Gamma' \to \Lambda$ from a fin ind $\Gamma' \subset \Gamma$, and a finite cover $\hat{K}' \to K' = \overline{\tau(\Gamma')}$ so that $\alpha: \Gamma' \times \hat{K}' \to \Lambda$ is conjugate to $\rho$. 
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Proof using deformation - rigidity ideas

Proposition (Local Rigidity, after Popa, Hjorth)

Let $\Gamma$ have $(T)$, $\Lambda$ discrete, and $\Pi_1$ action $\Gamma \curvearrowright (X, \mu)$. Then close cocycles are conjugate:
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\mu\{x \in X \mid \forall s \in S : \alpha(s, x) = \beta(s, x)\} > 1 - \epsilon
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Deform \( \alpha : \Gamma \times K \to \Lambda \) by \( \alpha_t(\gamma, x) = \alpha(\gamma, xt^{-1}) \) \( (t \in K) \).
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For \( t \in U \) small there is \( f_t : K \to \Lambda \) so that

\[ \alpha_t = \alpha^{f_t} \quad \text{and} \quad \mu\{x \mid f(x) = e\} > 3/4. \]
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If $t, s, ts \in U$ then both

$$f_t(xs^{-1})f_s(x) \quad \text{and} \quad f_{ts}(x)$$

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\( f_{ts}(x) = f_t(xs^{-1})f_s(x) \) on a set of meas > 0, hence a.e.

Try to propagate to \( K' = \langle U \rangle \). May need to lift to a finite cover \( \hat{K}' \to K' \).

On \( \hat{K}' \) we have \( f_t(x) = \phi(xt^{-1})^{-1}\phi(x) \)

\[ \phi(\gamma xt^{-1})\alpha(\gamma, xt^{-1})\phi(xt^{-1}) = \phi(\gamma x)\alpha(\gamma, x)\phi(x)^{-1} = \rho(\gamma) \]
Proof of the Local Rigidity Statement

\[ \Gamma \overset{\sim}{\rightarrow} X \times \Lambda \text{ by } g : (x, \lambda) \mapsto (gx, \alpha(g, x)\lambda\beta(g, x)^{-1}). \]
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Let \(f : X \rightarrow \Lambda\) denote the location of the peak: \(F(x, f(x)) = p\).

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