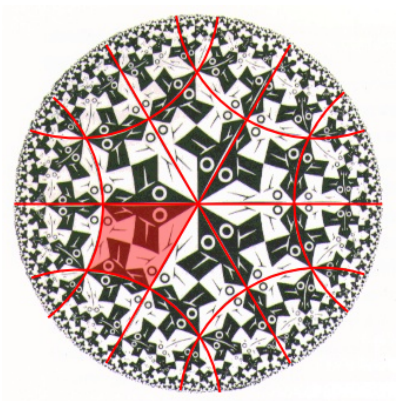
Superrigidity and Measure Equivalence, Part I

Alex Furman

University of Illinois at Chicago

Institut Henri Poincaré, Paris, June 20 2011

Poincaré disc and surfaces



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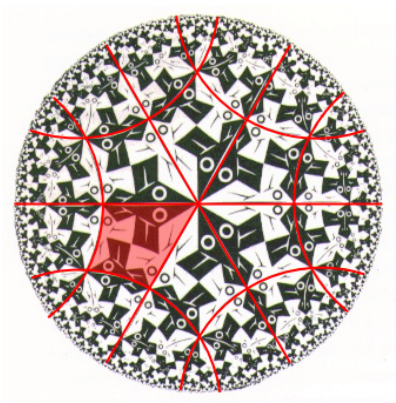

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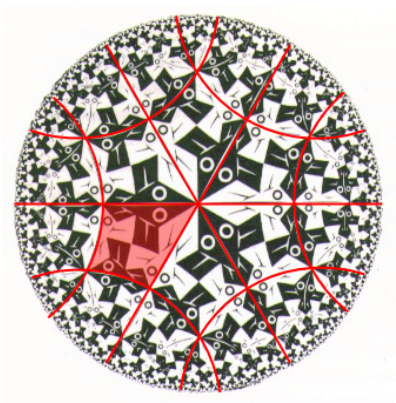

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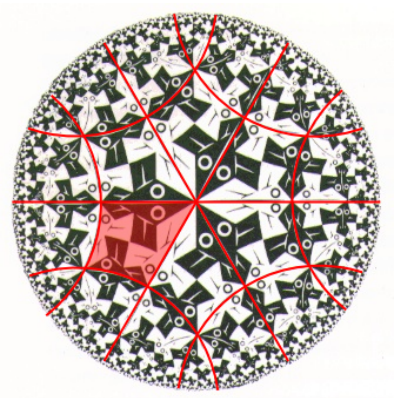

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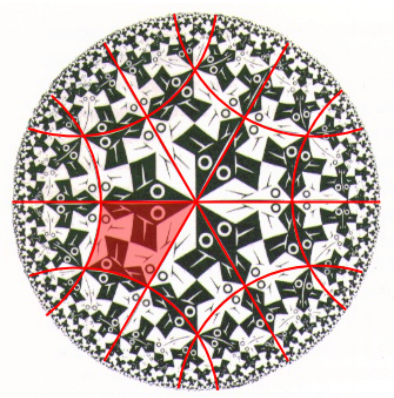

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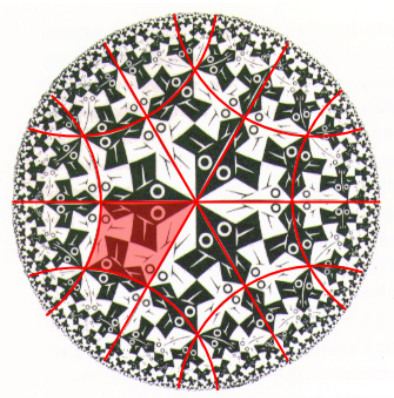

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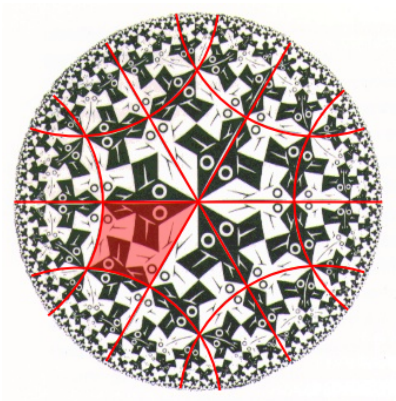

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Defn: Teichmüller space = moduli of hyperbolic metrics on Σ

$$\mathrm{Teich}(\Sigma) = \{\text{lattice embeddings} \quad \rho : \Gamma \to G\}/G$$

Flexibility of lattices in $\mathrm{SL}_2(\mathbb{R})$

Theorem (Riemann ?, Poincaré, Teichmüller ?)

For a closed surface of genus $g \geq 2$ one has

$$\mathrm{Teich}(\Sigma) \cong \mathbb{R}^{6 \cdot g - 6}$$

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$$\Gamma \to G = \mathrm{PSL}_2(\mathbb{R})$$

where

$$\Gamma = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

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Remarks

- $\forall\ \rho_1, \rho_2 : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ lattice embeddings

$$\exists! f \in \mathrm{Homeo}(S^1 = \partial \mathbf{H}^2) \qquad \rho_2(\gamma) = f^{-1} \circ \rho_1(\gamma) \circ f$$

- Similar results apply to non-uniform lattices.

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  Given $j : \Gamma \cong \Gamma'$ there $\exists! g \in G$ with $\mathbf{j}(\gamma) = \mathbf{g}^{-1}\gamma\mathbf{g}$.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\cong} & \Gamma' \\ \downarrow{\scriptstyle \subset} & & \downarrow{\scriptstyle \subset} \\ G & \overset{\cong}{\dashrightarrow} & G' \end{array}$$

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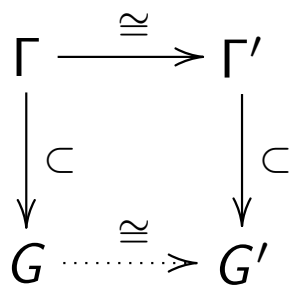
$G = \mathrm{Isom}(\mathbf{H})$, $G' = \mathrm{Isom}(\mathbf{H}')$ where $\mathbf{H}, \mathbf{H}' \in \{\mathbf{H}^n, \mathbf{H}^n_{\mathbb{C}}, \mathbf{H}^n_{\mathbb{H}}, \mathbf{H}^2_{\mathbb{O}}\} \setminus \mathbf{H}^2$.
Let $\Gamma < G$, $\Gamma' < G'$ be uniform lattices and $j : \Gamma \cong \Gamma'$ an isomorphism.
Then $j : \Gamma \cong \Gamma'$ extends to an isomorphism $G \cong G'$.



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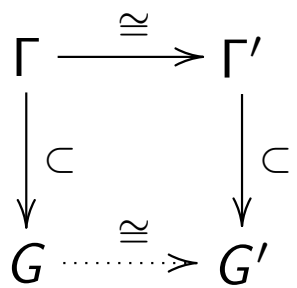
Theorem 3 (Mostow '73)

Same for any (semi)-simple G, $G' \not\cong \mathrm{SL}_2(\mathbb{R})$
and uniform (irreducible) lattices $\Gamma < G$, $\Gamma' < G'$.



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Given:

- $\Gamma, \Gamma' \curvearrowright \mathbf{H}^n$ properly discontinuous cocompact isometric actions.
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- f is j-equivariant

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- Furman ('01): simple $\mathrm{rk}(G) \geq 2$, or $G = \mathrm{Isom}(\mathbf{H}^n_K)$ and H/Γ' compact.
- Bader-Furman-Sauer ('12): all cases (including $\mathrm{SL}_2(\mathbb{R})$) and more...

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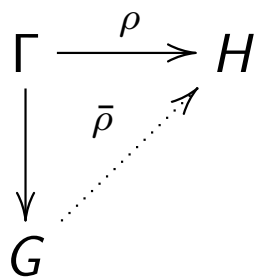

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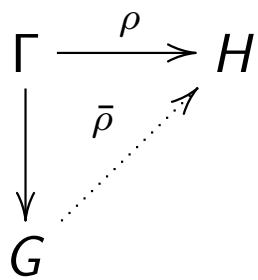

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Let $G = \prod G_i$ semi-simple, $\sum \mathrm{rk}(G_i) \geq 2$; H - simple, center-free.
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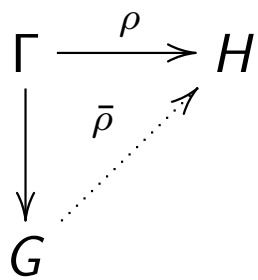

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Margulis' Arithmeticity Theorem ('75)

All (irreducible) lattices in higher rank (semi)-simple Lie groups are arithmetic.

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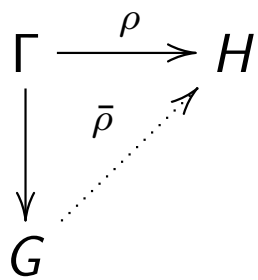

Margulis' Superrigidity Theorem (∼74)

Let $G = \prod G_i$ semi-simple, $\sum \mathrm{rk}(G_i) \geq 2$; H - simple, center-free.
$\Gamma < G$ an irreducible lattice, and $\rho : \Gamma \to H$, with Z-dense unbdd image.
Then $\rho : \Gamma \cong \Gamma'$ extends to an epimorphism $G \to H$.

Margulis' Arithmeticity Theorem ('75)

All (irreducible) lattices in higher rank (semi)-simple Lie groups are arithmetic.

Arith lattice ?

Something like
$\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbb{R})$

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$\Gamma < G$ lattice, G' - simple, $\rho : \Gamma \to G'$ hom with Z-dense unbounded image.
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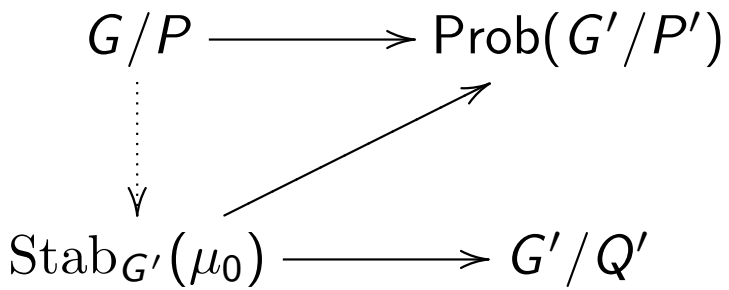

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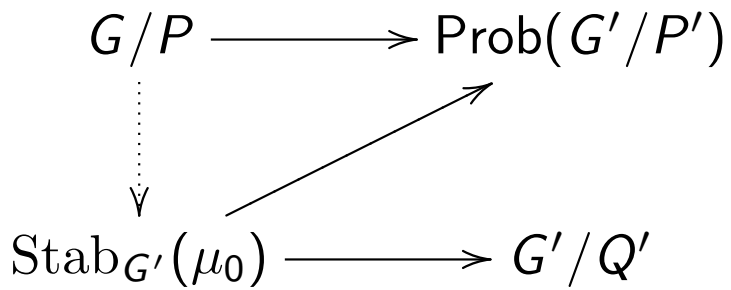

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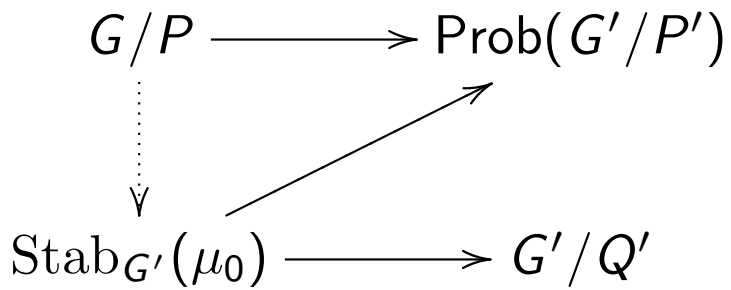

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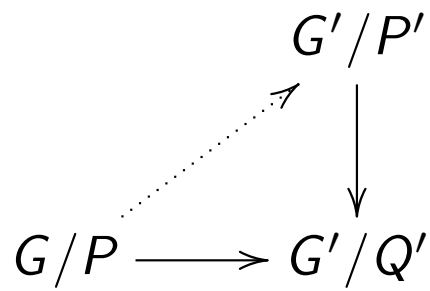
Theorem 1 can be strengthened to

- μ_0 is Dirac, $Q' = P'$ minimal parabolic.
- Γ-equiv. msbl $f : G/P \to G'/P'$ is unique.



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Special case of Margulis' superrigidity

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Further superrigidity phenomena (long list of names...)

- **Other** H: CAT(-1), Gromov-hyp, Homeo(S^1), $\mathrm{MCG}(\Sigma)$, $\mathcal{C}_{reg}$, $\mathcal{S}$,...
- **Other** G: products $G = G_1 \times \cdots \times G_n$, $n \geq 2$, of general lcsc grps, $\tilde{A}_2$ groups

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* Everything is measurable, taken up to null sets !

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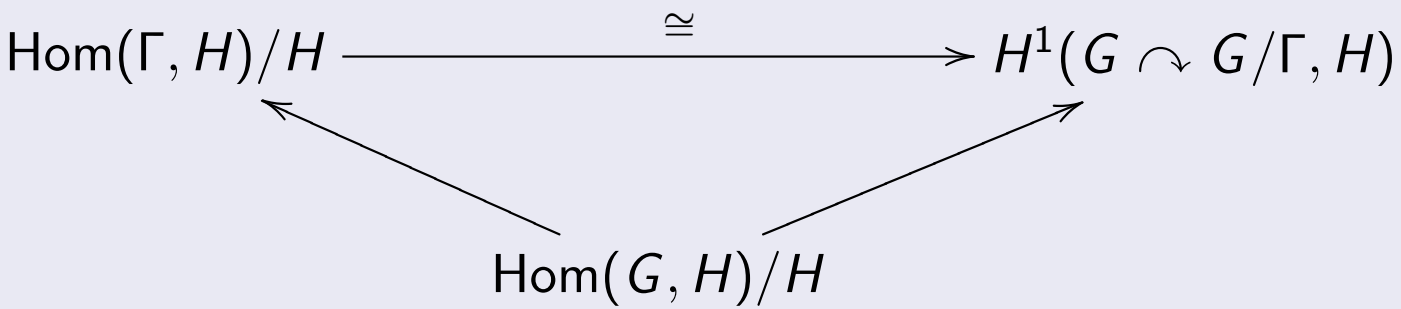
Proposition/observation. For a lattice $\Gamma < G$ and any H



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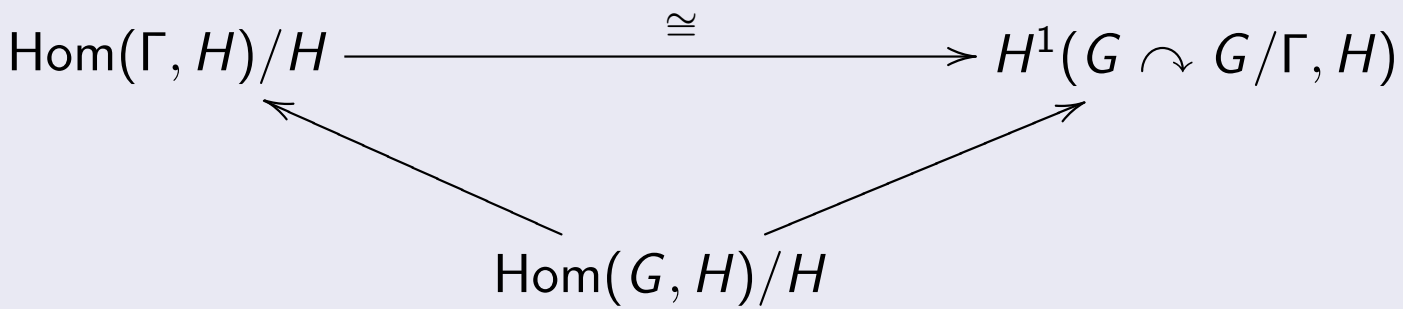

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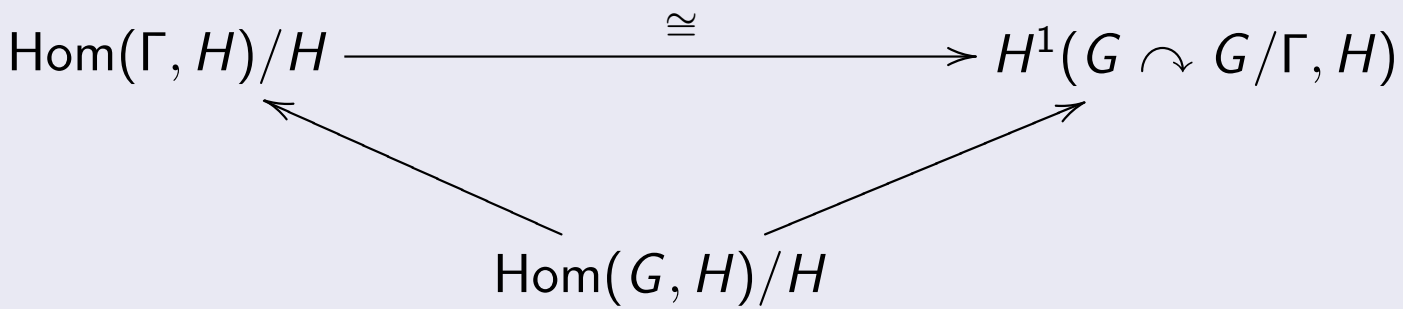

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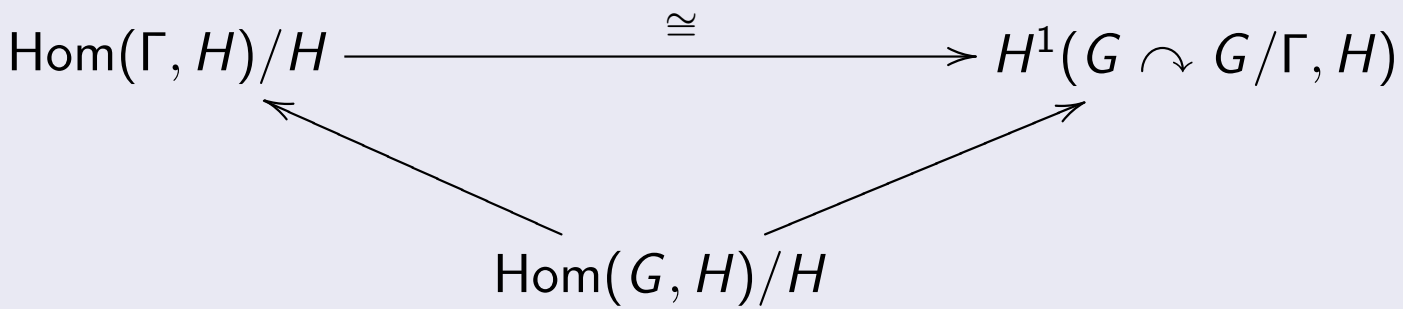

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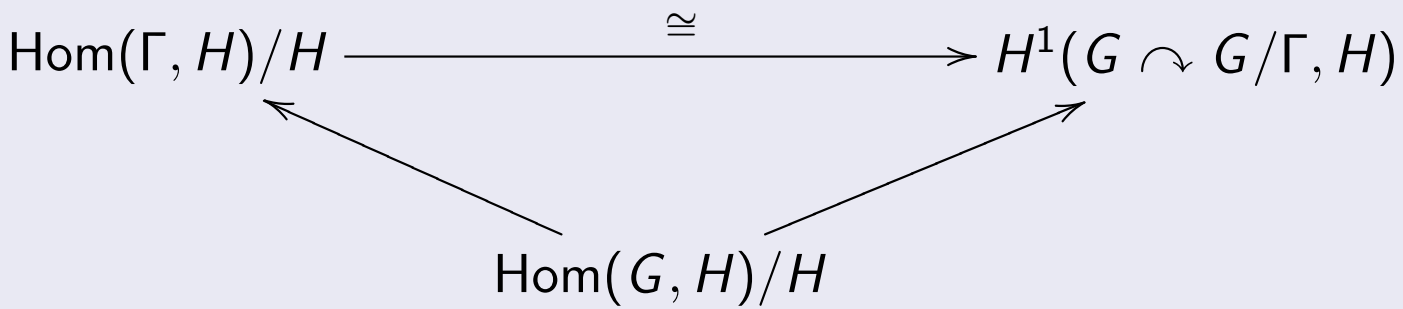

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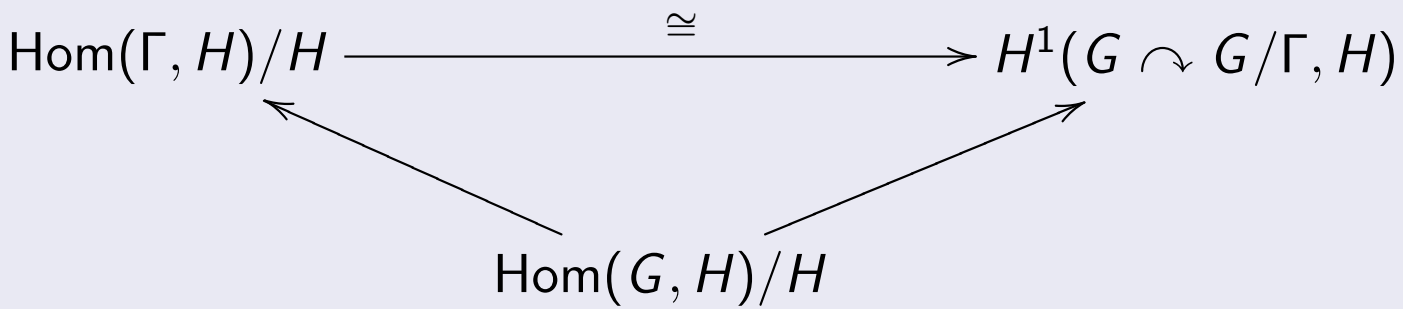

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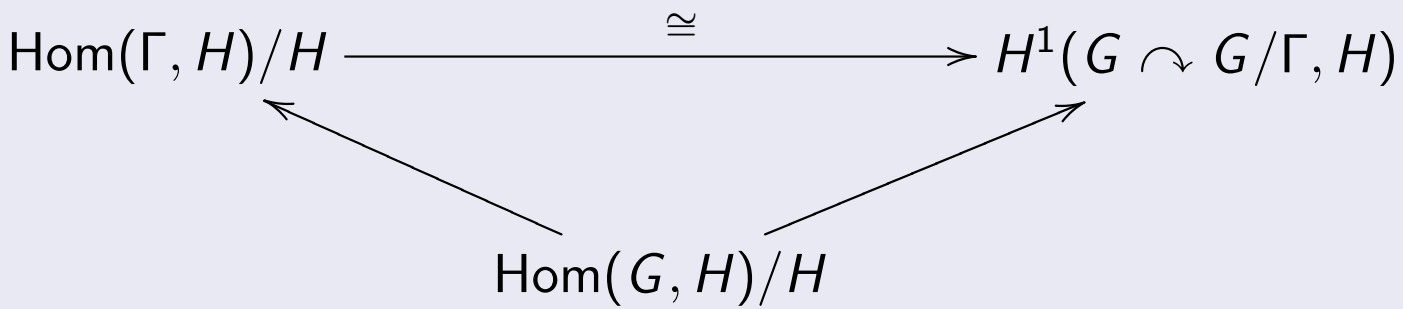

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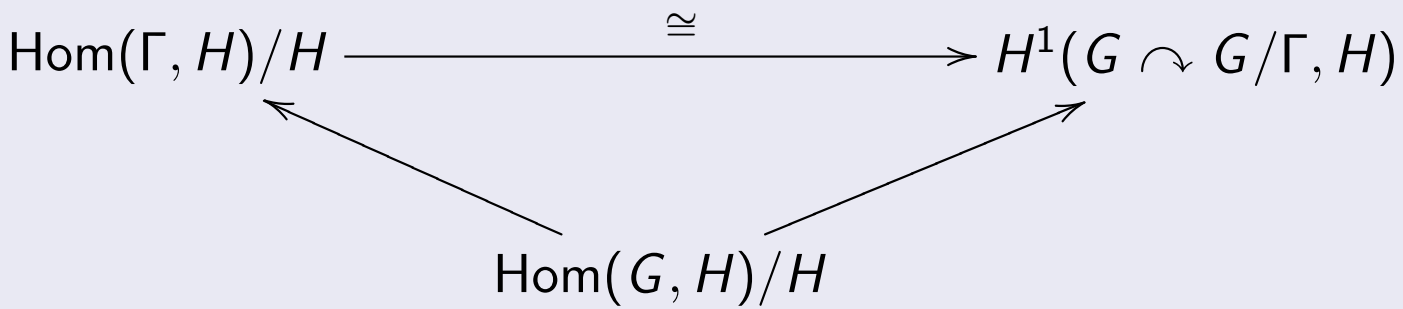

1. $\rho : \Gamma \to H$ up to H-conj $\quad\leftrightarrow\quad$ cocycle $c : G \times G/\Gamma \to H$ up to conj
2. ρ extends to $\bar{\rho} : G \to H$ $\quad\leftrightarrow\quad$ $c(g, x) \sim \rho(g)$

Proof of (1). Choose a Borel cross-section $\sigma : G/\Gamma \to G$ of $g \mapsto g\Gamma$.

- $\sigma(x)\Gamma\sigma(x)^{-1} = \mathrm{Stab}_G(x)$ for $x \in G/\Gamma$
- $c(g, x) = \sigma(g.x)^{-1} g\, \sigma(x) \in \Gamma$. Note $c : G \times G/\Gamma \to \Gamma$ is a cocycle.
- $\rho : \Gamma \to H$ a hom $\qquad\rightsquigarrow\qquad$ $\rho \circ c : G \times G/\Gamma \to \Gamma \to H$ is a cocycle

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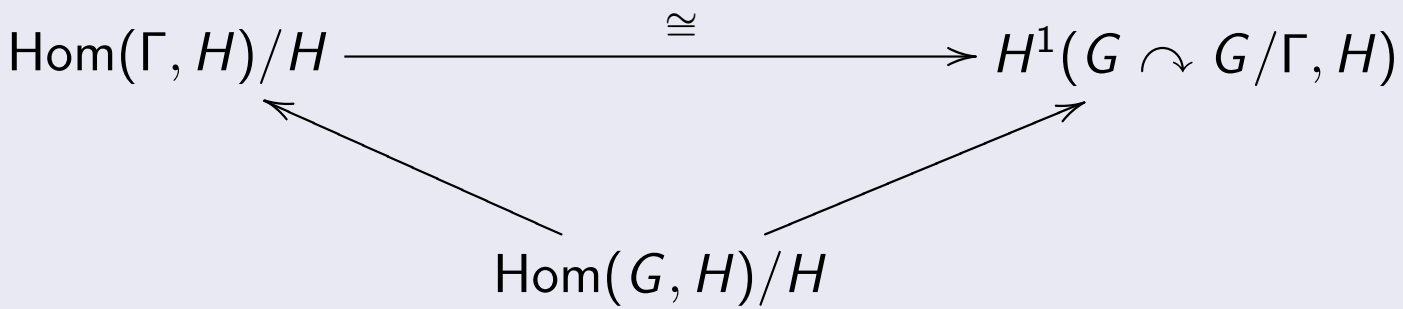

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- $\alpha : G \times G/\Gamma \to H$ $\quad\rightsquigarrow\quad$ $\rho_x(\gamma) = \alpha(\sigma(x)\gamma\,\sigma(x)^{-1}, x)$ is a hom.

Zimmer's cocycle superrigidity

Cocycle Superrigidity Theorem (Zimmer '81)

Let G (semi)-simple, H be simple Lie groups, $\mathrm{rk}(G) \geq 2$
$G \curvearrowright (X, \mu)$ (irred) ergodic p.m.p. $c : G \times X \to H$ cocycle
where c is Zariski-dense, not compact.
Then $\exists$ epimor $\pi : G \to H$ and measurable map $f : X \to H$

$$c(g, x) = f(g.x)^{-1} \pi(g) f(x).$$

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Strategy of the proof

- Boundary map: $f : X \times G/P \to H/Q$ s.t. $f_{g.x}(g\xi) = c(g, x) f_x(\xi)$.
- Ergodicity vs. smoothness of algebraic actions
- Regularity as in Margulis' proof.

Cocycles in nature

(stable) Orbit Equivalence

$\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ freely, and $T : X \cong Y$ with $T(\Gamma.x) = \Lambda.T(x)$
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Volume preserving actions on manifolds

$\Gamma \to \mathrm{Diff}_+(M^n, \mathrm{vol})$ defines the **derivative cocycle** $c : \Gamma \times (M, \mathrm{vol}) \to \mathrm{SL}_n(\mathbb{R})$.

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Other geometric cocycles

G connected and simply connected $\curvearrowright M \quad \rightsquigarrow \quad$ a cocycle $c : G \times M \to \pi_1(M)$.

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Popa's cocycle superrigidity

Invest in the action $\Gamma \curvearrowright (X, \mu)$ rather than in Γs and Gs
(program in flux - follow the arXiv closely...)