

# Equidistribution for groups of toral automorphisms

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# Basic dynamical questions

## General goal

$T : X \rightarrow X$  homeomorphism of a compact space  $X$

Understand the *distribution* of  $x, T^N x, \dots, T^{N-1} x$  as  $N \rightarrow \infty$ .

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- ▶ **Closed Invariant sets**

## Toral automorphisms

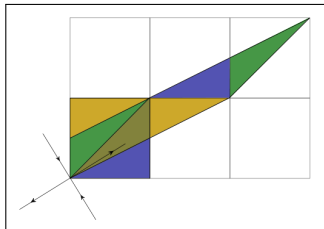
$A \in \mathrm{SL}_d(\mathbb{Z})$  acts on  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$  by  $A : x + \mathbb{Z}^d \mapsto Ax + \mathbb{Z}^d$

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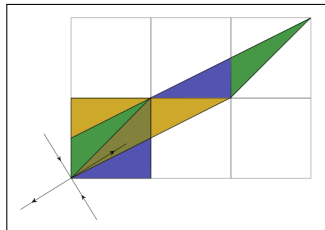


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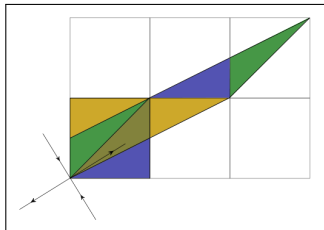
Periodic points =  $\left\{ \left( \frac{p_1}{q}, \dots, \frac{p_d}{q} \right) + \mathbb{Z}^d : \gcd(p_1, \dots, p_d, q) = 1 \right\}$

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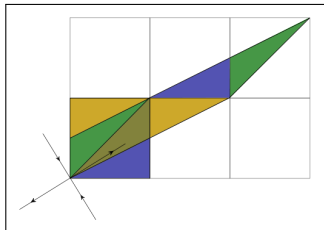
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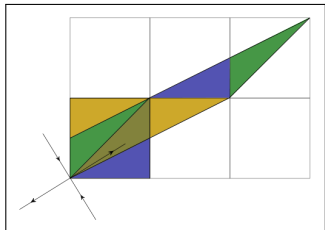
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- 3 **Equidistribution:** **no chance !**

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- 4 **Equidistribution** (in fact, **quantitative!**)  
BLFM (07, 10).

[BFLM] *Stationary measures and equidistribution for orbits of non-abelian semi-groups on the torus*, JAMS to appear.

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$$\|x - \frac{\mathbf{p}}{q}\| < e^{-\lambda n} \quad \text{with} \quad q < \left( \frac{2\|a\|}{t} \right)^c$$

- ▶ where  $c > 0, \lambda > 0, C$  depend only on  $\nu$ ,
- ▶  $\widehat{\mu}(a) = \int_{\mathbb{T}^d} e^{2\pi i \langle a, x \rangle} d\mu(x)$  for  $a \in \mathbb{Z}^d$ .

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### Theorem (M. Burger)

Let  $\mu \in \mathcal{P}(\mathbb{T}^d)$  be **invariant** under a **finite index** subgroup  $\Gamma < \mathrm{SL}_d(\mathbb{Z})$ .  
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Make the proof for the following effective

If  $\mu = \nu * \mu$  has  $|\widehat{\mu}(a)| = t > 0$  for some  $a \in \mathbb{Z}^d \setminus \{0\}$ . Then  $\mu$  has atoms.

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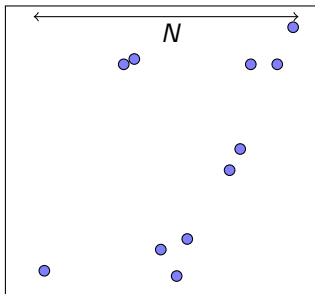
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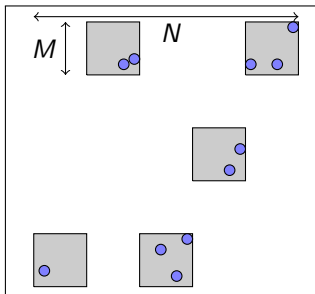
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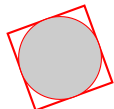
A

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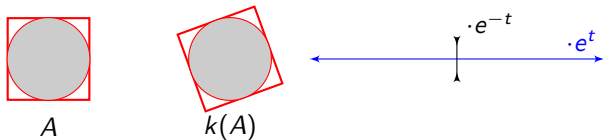
$A$



$k(A)$

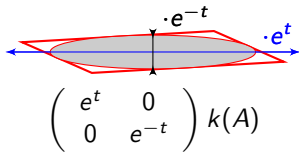
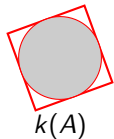
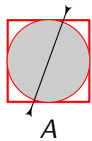
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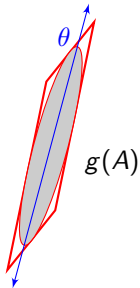
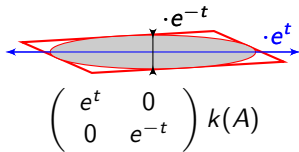
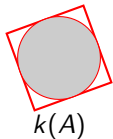
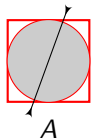
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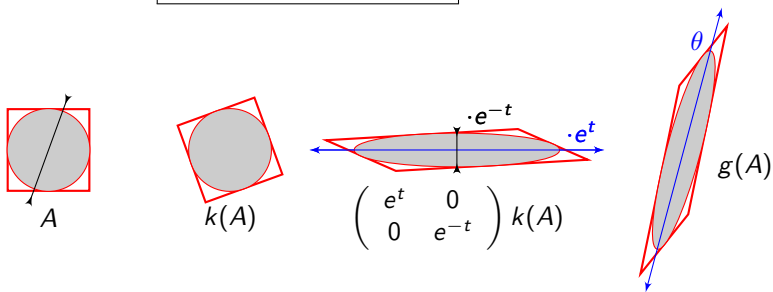
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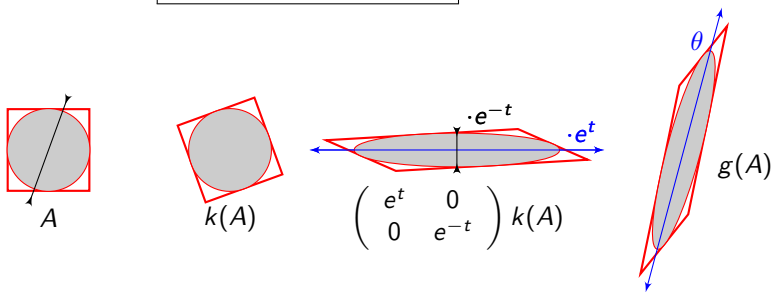
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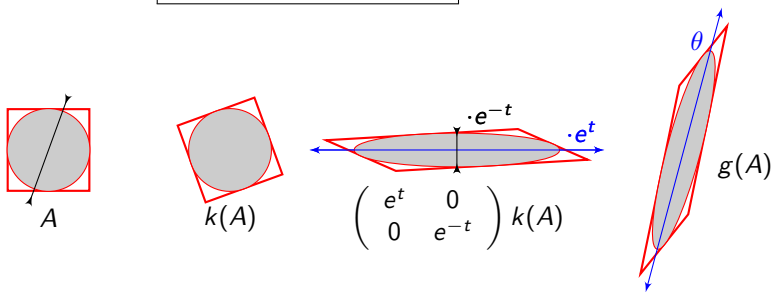


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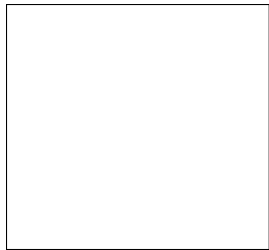
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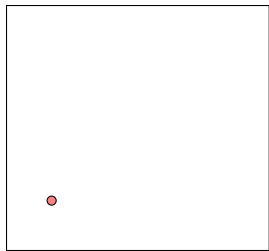
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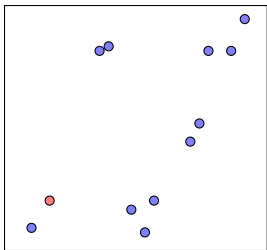
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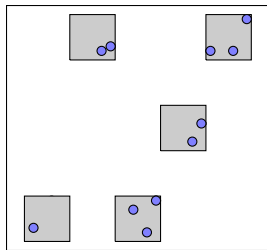
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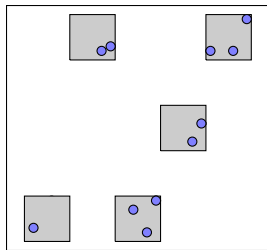
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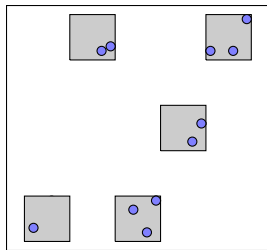
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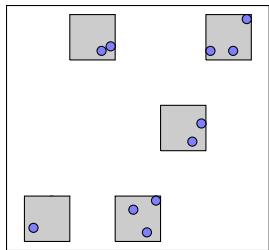
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Need to improve the **dimension**  $\alpha = \alpha_0$  to  $\alpha = d$  in steps  $\alpha_j \rightarrow \alpha_{j+1}$

$$\mathcal{N}_{M_i}(A_{t_i} \cap B_{0,N_i}) > \tilde{c}_i(t) \left(\frac{N_i}{M_i}\right)^{\alpha_i}$$

# Additive structure of Fourier coefficients

Lemma

$$\frac{1}{|A|^2} \sum_{a,b \in A} \widehat{\mu}(a-b) \geq \left| \frac{1}{|A|} \sum_{a \in A} \widehat{\mu}(a) \right|^2$$

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Proof.

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Proof.

$$\begin{aligned} \frac{1}{|A|^2} \sum_{a,b \in A} \widehat{\mu}(a-b) &= \frac{1}{|A|^2} \sum_{a,b \in A} \int_{\mathbb{T}^d} e^{2\pi i \langle a-b, x \rangle} d\mu(x) \\ &= \int_{\mathbb{T}^d} \left( \frac{1}{|A|} \sum_{a \in A} e^{2\pi i \langle a, x \rangle} \right) \cdot \overline{\left( \frac{1}{|A|} \sum_{b \in A} e^{2\pi i \langle b, x \rangle} \right)} d\mu(x) \\ &= \int_{\mathbb{T}^d} \left| \frac{1}{|A|} \sum_{a \in A} e^{2\pi i \langle a, x \rangle} \right|^2 d\mu(x) \geq \left| \int_{\mathbb{T}^d} \frac{1}{|A|} \sum_{a \in A} e^{2\pi i \langle a, x \rangle} d\mu(x) \right|^2 \\ &= \left| \frac{1}{|A|} \sum_{a \in A} \widehat{\mu}(a) \right|^2 \quad \square \end{aligned}$$

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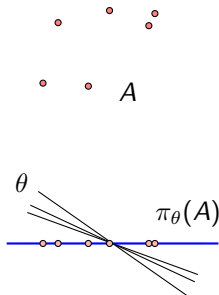
## Theorem (Bourgain)

$\forall \beta, \gamma > 0$ ,  $\exists \delta > 0$  so that  $\forall \alpha \in [\beta, d - \beta]$

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- ▶  $A, \eta$  not too degenerate

Then for  $\eta$ -most  $\theta \in \mathbb{P}^{d-1}$  s.t.

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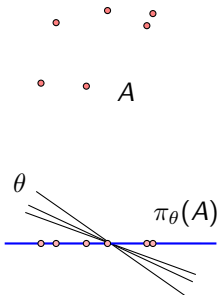
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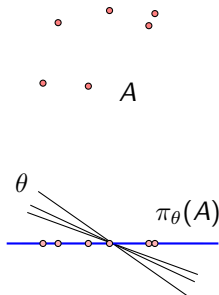
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## Amplifying the dimension



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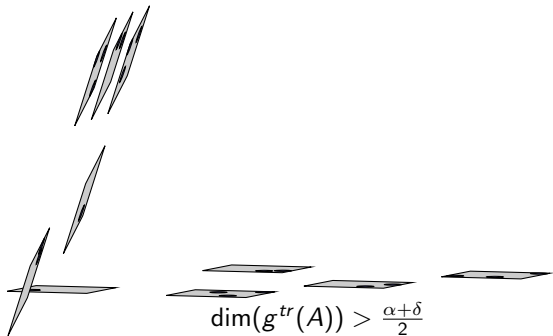
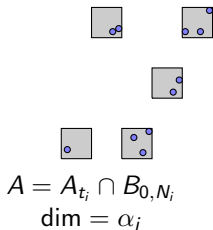
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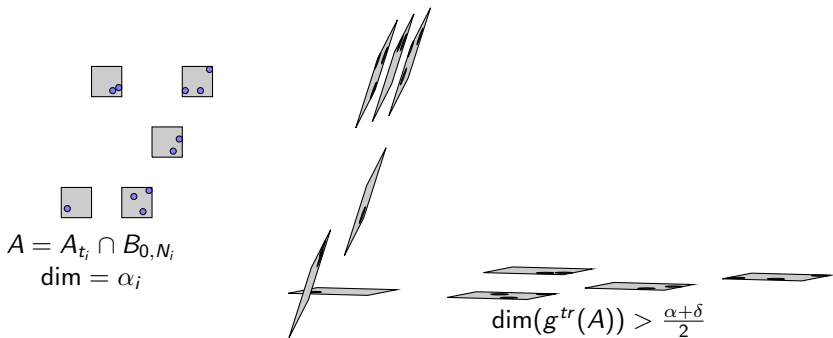
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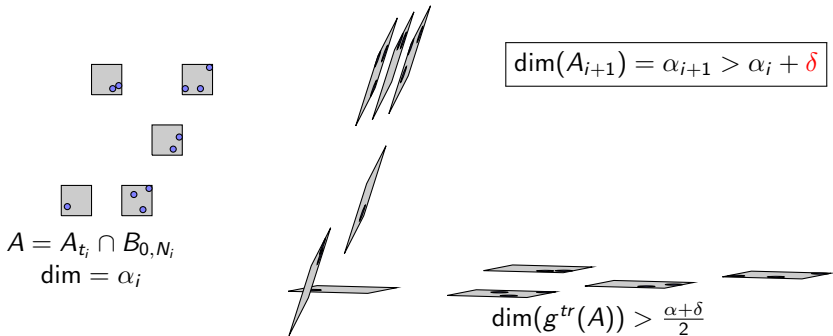
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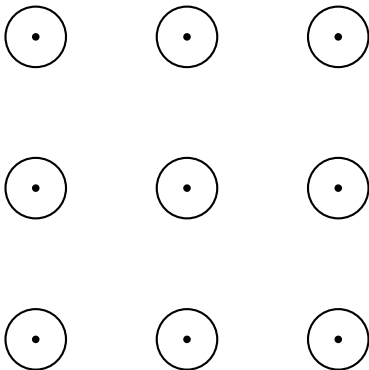
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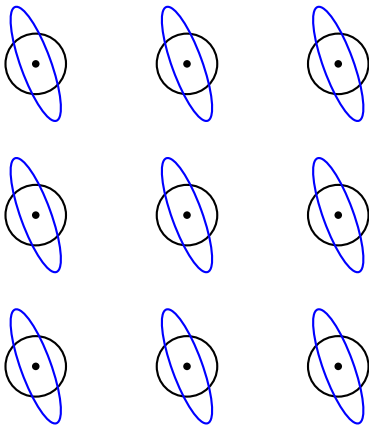


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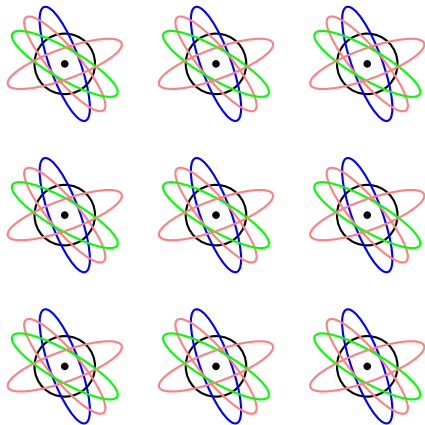
## Self packing of dense balls



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