The space of metrics on Gromov hyperbolic groups

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Negatively Curved Manifolds: Geometry

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- Deformation rigidity (Guillemin-Kazhdan '80)
- Surfaces (Otal '90, Croke '90)
- (M, g) loc. symmetric (Hamenstädt '99, using BCG)

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  3. has conditionals on stable/unstable scaled by $e^{\pm ht}$ where $h = h_{\text{top}}$

$$d\phi^t_* \mu^{(s)}_{\text{BM}} = e^{-ht} \cdot d\mu^{(s)}_{\text{BM}}, \qquad d\phi^t_* \mu^{(u)}_{\text{BM}} = e^{+ht} \cdot d\mu^{(u)}_{\text{BM}}$$

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- Bowen-Margulis measure μ_{BM} vs. Patterson-Sullivan current m_{PS}

$$\mathrm{Meas}(SM)^{\phi^t} \quad\leftrightarrow\quad \mathrm{Meas}(S\tilde{M})^{\phi^t\times\Gamma} \quad\leftrightarrow\quad \mathrm{Meas}(\partial\widetilde{M} \times \partial\tilde{M})^{\Gamma}$$

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Theorem

Given $[d] \in D_\Gamma$ there is a Radon measure $m_{[d]}$ on $\partial^{(2)}\Gamma = \partial\Gamma \times \partial\Gamma \setminus \Delta$

- *$m_{[d]}$ is Γ-invariant and ergodic*
- *$dm_{[d]}(x, y) = e^{2h_{[d]} \cdot F(x,y)}\, d\nu(x)\, d\nu(y)$ where*

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Based on Coornaert's Patterson-Sullivan theory for Gromov hyperbolic groups, and If $c : \Gamma \times X \to \mathbb{R}$ cocycle with $|c(-, x)| \leq M(x)$, then $c(\gamma, z) = b(\gamma \cdot z) - b(z)$.

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(1) $\implies$ (2) by construction, (3) $\implies$ (4) from ergodicity. (4) $\implies$ (1) ...
(2) $\implies$ (3) is proved using an analogue of Bowen's construction - weak limits of

$$\frac{1}{\#\{\langle\gamma\rangle \in C_\Gamma \mid \ell_{[d]}(\langle\gamma\rangle) < R\}} \cdot \sum_{\{\gamma\in\Gamma \mid \ell_{[d]}(\gamma) < R\}} \delta_{(\gamma_-, \gamma_+)}.$$

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- *The spaces*
  1. Teich(Σ)
  2. QF$(\Sigma) \setminus$ Teich(Σ)
  3. Riem$_{<0}(\Sigma) \setminus$ Teich(Σ)
  4. *Word metrics on* $\Gamma = \pi_1(\Sigma)$

  have disjoint images in D_Γ.

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2. $\Gamma < \mathrm{Isom}(\mathbf{H}^n_K)$ with $d = \mathrm{dist}_{\mathbf{H}^n_K}$ $\rightsquigarrow$ $H_{[d]} = \mathrm{Isom}(\mathbf{H}^n_K)$ with $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

The most symmetric groups and metrics

Theorem

Let $\Gamma = \pi_1(M)$ *where* M *admits n.c. metric. Then*

- *either* $H_{[d]}$ *is discrete and* $[H_{[d]} : \Gamma] < \infty$*,*
- *or* Γ *is a uniform lattice in* $\mathrm{Isom}(\mathbf{H}^n_K)$ *where* $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$
  $M = \Gamma \backslash \mathbf{H}^n_K$ *and* $d \sim c \cdot \mathrm{dist}_{\mathbf{H}^n_K}$ *and* $H_{[d]} \simeq \mathrm{Isom}(\mathbf{H}^n_K)$.

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Theorem

Let $\Gamma = F_n$ and $[d] \in D_\Gamma$. Then

- *either $H_{[d]}$ is discrete and $[H_{[d]} : F_n] < \infty$,*
- *or $d \sim d_S$ – word metric; in which case $H_{[d]} \simeq \mathrm{Aut}(T)$.*

Related results on general lattices

Theorem (Bader-Furman-Sauer)

Let H be a lcsc group containing $\Gamma = F_n$ as a lattice.
Then, up to finite index and compact kernel

- *either $H \simeq \Gamma$ (trivial lattice),*
- *or $H \simeq \mathsf{PSL}_2(\mathbb{R})$ (non-uniform lattice),*
- *or H is a non-discrete closed subgroup of $\mathrm{Aut}(\mathrm{Tree})$ (uniform lattice).*

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Theorem (Bader-Furman-Sauer)

Let Γ be a Gromov-hyperbolic PD-group, H a lcsc group, $\Gamma < H$ lattice.
Then, up to finite index and compact kernel

- *either $H \simeq \Gamma$,*
- *or Γ is a cocompact rank one lattice and $H \simeq \mathrm{Isom}(\mathbf{H}^n_K)$.*

The last result uses recent results of Mahan Mj on Hilbert-Smith conjecture.

Minimal entropy characterization

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Definition

Let Γ be Gromov-hyperbolic group, $[d] \in D_\Gamma$. Let

$$\kappa_{[d]} = \inf\{\kappa > 0 \mid \exists \text{ rough isometric embedding } (\Gamma, \kappa \cdot d) \to \mathbf{H}_{\mathbb{R}}^{\infty}\}$$

After Bonk - Schramm.

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After Bonk - Schramm.

Theorem (after Bourdon)

Let $M = \Gamma \backslash \mathbf{H}^n_K$ where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Then

$$\frac{h_{[d]}}{\kappa_{[d]}} \geq kn + k - 2, \qquad k = \dim_\mathbb{R} K$$

with equality attained iff $d \sim c \cdot \mathrm{dist}_{\mathbf{H}^n_K}$.

Thanks

Thank you.

Thanks

Thank you.

Applause to the Organizers!

Keith Burns, John Franks, Bryna Kra,
Clark Robinson, Amie Wilkinson, Jeff Xia

Thank you!