Mathematics 180 - Hour Exam Two - Solutions

1.

Solution.
(a) $(x \arctan x)' = (x)' \arctan x + x(\arctan x)' = \arctan x + x \cdot \frac{1}{x^2+1} = \arctan x + \frac{x}{1+x^2}$.
(b)$(x^{1/2} - x^{-1/2})' = (x^{1/2})' - (x^{-1/2})' = \frac{1}{2}x^{-1/2} - (-\frac{1}{2})x^{-3/2} = \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2}$.
(c)$(e^{-x^2})' = e^{-x^2} \cdot (-x^2)' = e^{-x^2} \cdot (-2x) = -2xe^{-x^2}$.

2.

Solution.
(a) Take the implicit differentiation of the equation $2x^2 - 3xy + 2y^2 = 4$ with respect to x, i.e. regard y as a function of x.
$\frac{\mathrm{d}}{\mathrm{d}x}(2x^2 - 3xy + 2y^2) = 0 \implies \frac{\mathrm{d}}{\mathrm{d}x}(2x^2) - \frac{\mathrm{d}}{\mathrm{d}x}(3xy) + \frac{\mathrm{d}}{\mathrm{d}x}(2y^2) = 4x - (3y + 3x\frac{\mathrm{d}y}{\mathrm{d}x}) + 4y\frac{\mathrm{d}y}{\mathrm{d}x} = (4x - 3y) + (4y - 3x)\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y - 4x}{4y - 3x} \implies \frac{\mathrm{d}y}{\mathrm{d}x}|_{(2,1)} = \frac{3 \cdot 1 - 4 \cdot 2}{4 \cdot 1 - 3 \cdot 2} = -\frac{5}{2}$, i.e. the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $(2, 1)$ is $-\frac{5}{2}$. In other words, the slope of tangent line at point at $(2, 1)$ is $-\frac{5}{2}$.

(b)The equation of the tangent line of a function $f(x)$ at a point $(a, f(a))$ is given by $y - f(a) = f'(a)(x - a)$. In this question, $a = 2$, $f(a) = f(2) = 1$ and $f'(a) = f'(2) = \frac{\mathrm{d}y}{\mathrm{d}x}|_{(2,1)} = -\frac{5}{2}$, the slope. Hence the defining equation of the tangent line is $y - 1 = -\frac{5}{2}(x - 2)$ or $y = -\frac{5}{2}x + 6$.

3

Solution.
(a) $f'(x) = (x^3 - 6x^2 + 20)' = 3x^2 - 12x$. Set $f'(x) = 0$ and solve the equation $3x^2 - 12x = 0$ for x. The solutions are the critical points. $3x^2 - 12x = 0 \implies 3x(x - 4) = 0 \implies x = 0$ or $x = 4$.

(b)To determine the decreasing interval, flows the following steps. Step 1: use the critical points 0 and 4 to divides the real line into three intervals $(-\infty, 0)$,$(0, 4)$ and $(4, +\infty)$. Step 2: determine the sign of $f'(x)$ over each interval (in terms of the test point method). $f'(-1) = 3(-1)^2 - 12(-1) = 15 > 0 \implies f'(x)$ is positive over $(-\infty, 0)$. $f'(1) = 3(1)^2 - 12 \cdot 1 = -12 < 0 \implies f'(x)$ is negative over $(0, 4)$. $f'(5) = 3(5)^2 - 12 \cdot 5 = 15 > 0 \implies f'(x)$ is positive over $(4, +\infty)$. Hence $f(x)$ is decreasing over $(0, 4)$ because $f'(x)$ has negative sign over it.

(c)To find the global minimum of $f(x)$ over the interval $(-1, 5)$, we need to compare the values of the function at the critical points and endpoints of the interval. $f(-1) = (-1)^3 - 6(-1)^2 + 20 = 13$. $f(0) = 20$. $f(4) = 4^3 - 6 \cdot 4^2 + 20 = -12$. $f(5) = 5^3 - 6 \cdot 5^2 + 20 = -5$. Hence the minimum value of $f(x)$ over the interval $(-1, 5)$ is $f(4) = -12$.

4

Solution.

Since the limit is of $\frac{0}{0}$ form and $\lim_{x\to 1} \frac{(x^3-1)'}{(x^2-1)'} = \lim_{x\to 1} \frac{3x^2}{2x^2} = \frac{3}{2}$, then by L'Hopital's Rule, $\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{(x^3-1)'}{(x^2-1)'} = \frac{3}{2}$.

5

Solution.

Recall the distance formula: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)}$, where (x_1, y_1) and (x_2, y_2) are the two points that we want to compute the distance.

To find the point on the line $y = 2x$ closest to $(1, 0)$ means we need to find the minimum distance between $(1, 0)$ and a point on $y = 2x$. The distance between a point $(x, 2x)$ on the line and $(1, 0)$ is a function of x, $d(x) = \sqrt{(x-1)^2 + (2x-0)^2} = \sqrt{5x^2 - 2x + 1}$.

To find the global minimum, we need to find the critical points and compare the values of the function at critical points and endpoints.

$d'(x) = (\sqrt{5x^2 - 2x + 1})' = \frac{1}{2\sqrt{5x^2 - 2x + 1}}(5x^2 - 2x + 1)' = \frac{10x - 2}{2\sqrt{5x^2 - 2x + 1}}$. Set $d'(x) = 0$ and solve for x. $\implies 10x - 2 = 0 \implies x = \frac{1}{5}$. Since $f(+\infty) = +\infty$, $f(-\infty) = +\infty$ and $f(1/5) = 2\sqrt{5}/5$, the minimal distance is between $(1/5, 2/5)$ and $(1, 0)$.

6

Solution. Note: the graph is for the derivative, ***but not*** for the original function.

(a) f increasing $\iff$ f' positive $\iff x \in (-\infty, a) \cup (c, e)$.

(b) f concave up $\iff$ f'' positive $\iff$ f' increasing $\iff$ $x \in (b, d)$.

(c) x is a local maximum $\iff$ x is a critical point and the sign of change of $f'(x)$ is from $+$ to $-$. $\iff$ x is a root of f', equivalently the $x-$intercept and $f'(x)$ goes from the above to the below of the $x-$axis as x goes to right direction $\iff$ $x = a$ or $x = e$.

(d) x is a inflection point $\iff$ x is a root of f'' $\iff$ the slope of tangent line of the function f' at x is zero. In the graph, the only two points are b and d.