For each problem give a complete written solution which states the method being used and clearly shows all the mathematical steps. Numerical answers can be given in terms of constants such as $e, \pi, \ln 2, \sqrt{5}$. A calculator is not necessary. A graphing calculator with elementary functions may be used. Calculators with symbolic algebra and calculus capability cannot be used.

1. Differentiate with respect to x. Write your answers showing the use of the appropriate techniques. Do **not** simplify.

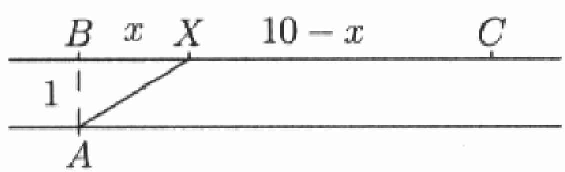
$$\text{(a)}x^{2007} - x^{2/3} \qquad \text{(b) } (x^2 - 2x + 2)e^x, \qquad \text{(c) } \ln(x^2 + 4).$$

2. For the curve $y^2 + xy - x^3 = 5$,
   (a) use implicit differentiation to find the derivative $\dfrac{dy}{dx}$,
   (b) find the equation of the line tangent to this curve at the point $(1, 2)$.

3. Use calculus to find the exact x-coordinates of any local maxima, local minima, and inflection points of the function $f(x) = 3x^5 - 20x^3 + 14$.

4. Find
$$\lim_{x \to 0} \frac{e^x - x - 1}{3x^2}.$$
Explain how you obtain your answer.

5. An electrical company at point A needs to run a wire from a generator to a factory that is on the other side of a one mile wide river and 10 miles downstream at point C. It costs \$600 per mile to run the wire on towers across the river and \$400 per mile to run the wire over land along the river. The wire will cross the river from A to a point X and then travel over land from X to C. Let x be the distance from B to X.
   (a) Find the total cost as a function $f(x)$ of the variable x.
   (b) Use calculus to find the value of x that minimizes the cost.



6. Differentiate with respect to x. Write your answers showing the use of the appropriate techniques. Do **not** simplify.

$$\text{(a)}\ \frac{x^2+1}{x^2+x+1}, \qquad \text{(b)}\ \sin^3(5x+2), \qquad \text{(c)}\ \arctan\left(\frac{x}{2}\right).$$

7. (a) Calculate the left and right Riemann sums with three subdivisions, L_3 and R_3 for the integral

$$\int_0^6 f(x)\,dx.$$

Some values of the function f are given in the table:

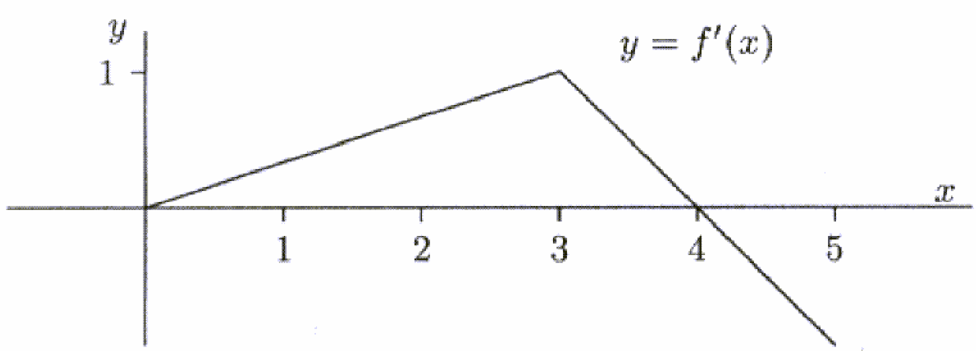
x	0	2	4	6
$f(x)$	1.6	1.9	2.4	3.1

(b) If the function f is increasing, could the integral be greater than 15? Explain why or why not?

8. (a) Write the integral which gives the area of the region between $x = 0$ and $x = 1$ above the x-axis, and below the curve $y = x - x^3$.
(b) Evaluate your integral exactly to find the area.

9. Evaluate the integral $\displaystyle\int_0^1 \frac{1}{\sqrt{3x+1}}\,dx$ by finding an antiderivative.

10. The graph below represents the derivative, $f'(x)$.



(a) On what interval is f increasing?
(b) On what interval is f increasing?
(c) For what value of x is $f(x)$ a maximum?
(d) What is $\displaystyle\int_0^5 f'(x)\,dx$?
(e) What is $f(5) - f(0)$?

1. (a) $\left(x^{2007} - x^{\frac{2}{3}}\right)' = 2007x^{2006} - \frac{2}{3}x^{-\frac{1}{3}}$

(b) $(x^2-2x+2)e^x = (x^2-2x+2)'e^x + (x^2-2x+2)(e^x)'$

$= (2x-2)e^x + (x^2-2x+2)e^x \quad (= x^2e^x)$

(c) $\left(\ln(x^2+4)\right)' = \frac{1}{x^2+4} \cdot (x^2+4)' = \frac{2x}{x^2+4}$

2. $\frac{d}{dx}(y^2+xy-x^3) = \frac{d}{dx}(5) = 0$

$2y\frac{dy}{dx} + y + x\frac{dy}{dx} - 3x^2 = 0$

$(2y+x)\frac{dy}{dx} - 3x^2 + y = 0$

$(2y+x)\frac{dy}{dx} = 3x^2 - y$

$\frac{dy}{dx} = \frac{3x^2-y}{2y+x} \Rightarrow \frac{dy}{dx}\Big|_{(1,2)} = \frac{3\cdot1^2-2}{2\cdot2+1} = \frac{1}{5}$

$\Rightarrow \quad y-2 = \frac{1}{5}(x-1)$

3. $f'(x) = 15x^4 - 60x^2 = 0 \Leftrightarrow 15x^2(x^2-4) = 0 \Rightarrow x = 0, 2 \text{ or } -2$

$f''(x) = 60x^3 - 120x = 0 \Leftrightarrow 60x(x^2-2) = 0 \Rightarrow x = 0, \sqrt{2} \text{ or } -\sqrt{2}$

So inflection points are

$(0, f(0)) = (0, 14)$, $(\sqrt{2}, f(\sqrt{2})) = (\sqrt{2}, 14-28\sqrt{2})$ and

$(\sqrt{2}, f(-\sqrt{2})) = (-\sqrt{2}, 28\sqrt{2}+14)$

Using FDT to determine local extrema.

	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,2)$	2	$(2,+\infty)$
sign of $f'(x)$	$f(-3) = 675$ $+$	0	$f(-1) = -45$ $-$		$f(1) = -45$ $-$		$f(3) = 675$ $+$
$f(x)$	$\uparrow$	Max	$\downarrow$		$\downarrow$	min	$\uparrow$

So $(-2, f(-2)) = (-2, 78)$ is the local max

$(2, f(2)) = (2, -50)$ is the local min

4. L'Hôpital's rule twice

$\lim_{x\to 0} (e^x - x - 1) = e^0 - 0 - 1 = 1 - 1 = 0$ $\quad \lim_{x\to 0} 3x^2 = 3\cdot 0 = 0$

$\lim_{x\to 0} (e^x - x - 1)' = \lim_{x\to 0} (e^x - 1) = e^0 - 1 = 0$ $\quad \lim_{x\to 0} (3x^2)' = \lim_{x\to 0} 6x = 0$

$$\lim_{x\to 0} \frac{(e^x-1)'}{(6x)'} = \lim_{x\to 0} \frac{e^x}{6} = \frac{e^0}{6} = \frac{1}{6}$$

by L'Hôpital's rule

$$\lim_{x\to 0} \frac{e^x - 1}{6x} = \lim_{x\to 0} \frac{(e^x-1)'}{(6x)'} = \frac{1}{6}$$

by L'Hôpital's rule again

$$\lim_{x\to 0} \frac{e^x - x - 1}{3x^2} = \lim_{x\to 0} \frac{(e^x - x - 1)'}{(3x^2)'} = \lim_{x\to 0} \frac{e^x - 1}{6x} = \frac{1}{6}$$

5. (a) The cost $= 600 \cdot AX + 400 \cdot XC$

We know that $XC = 10 - x$ $\quad AX = \sqrt{AB^2 + BX^2} = \sqrt{1+x^2}$

So $f(x) = 600\sqrt{1+x^2} + 400(10-x)$ with $0 \le x \le 10$

(b) To find the minimal cost, we need to find the global minimum of $f(x)$ over $0 \le x \le 10$.

Step 1: Find critical points between 0 and 10

$$f'(x) = \left(600\sqrt{1+x^2} + 400(10-x)\right)'$$
$$= 600\left(\sqrt{1+x^2}\right)' + 400(10-x)'$$
$$= 600\,\frac{2x}{2\sqrt{1+x^2}} - 400 = \frac{600x}{\sqrt{1+x^2}} - 400$$

Solve $f'(x) = 0$ to get critical points.

$$\frac{600x}{\sqrt{1+x^2}} - 400 = 0 \iff 600x - 400\sqrt{1+x^2} = 0$$

$$\iff 3x - 2\sqrt{1+x^2} = 0$$
$$\sqrt{1+x^2} = \frac{3x}{2}$$
$$1 + x^2 = \frac{9x^2}{4}$$
$$1 = \frac{5x^2}{4}$$
$$\frac{4}{5} = x^2$$
$$x = \frac{2\sqrt{5}}{5}$$

Step 2:

$f(0) = 600 + 400\cdot 10 = 4600$

$f(10) = 600\sqrt{101} + 400 > 6400$

$f\left(\frac{2\sqrt{5}}{2}\right) = 600\sqrt{1+\frac{4}{5}} + 400\left(10 - \frac{2\sqrt{5}}{5}\right)$
$= 600\cdot\frac{3\sqrt{5}}{5} + 4000 - 400\cdot\frac{2\sqrt{5}}{5}$
$= 4000 + 200\sqrt{5}$

Note: $200\sqrt{5} < 600$ because $\sqrt{5} < 3$

$\boxed{\text{Hence } f_{min} = f\left(\frac{2\sqrt{5}}{2}\right) = 4000 + 200\sqrt{5}}$

6. (a) $$\left(\frac{x^2+1}{x^2+x+1}\right)' = \frac{(x^2+1)'(x^2+x+1) - (x^2+x+1)'(x^2+1)}{(x^2+x+1)^2}$$

$$= \frac{2x(x^2+x+1) - (2x+1)(x^2+1)}{(x^2+x+1)^2}$$

(b) $$\left(\sin^3(5x+2)\right)' = 3\sin^2(5x+2)\cdot\left(\sin(5x+2)\right)'$$

$$= 3\sin^2(5x+2)\cos(5x+2)\cdot(5x+2)'$$

$$= 15\sin^2(5x+2)\cos(5x+2)$$

(c) $$\left(\arctan\frac{x}{2}\right)' = \frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\left(\frac{x}{2}\right)' = \frac{1}{2}\cdot\frac{1}{1+\frac{x^2}{4}}$$

7. (a) $L_3 = 2\,(f(0) + f(2) + f(4)) = 2\cdot(1.6 + 1.9 + 2.4) = 11.8$

$R_3 = 2\,(f(2) + f(4) + f(6)) = 2\,(1.9 + 2.4 + 3.1) = 14.8$

(b) If f is increasing, then

$$11.8 = L_3 \leqslant \int_0^6 f(x)\,dx \leq R_3 = 14.8$$

So $\int_0^6 f(x)\,dx$ can not be greater that 15.

8. $$A = \int_0^1 (x - x^3)\,dx = \int_0^1 x\,dx - \int_0^1 x^3\,dx$$

$$= \frac{1}{2}x^2\Big|_0^1 - \frac{1}{4}x^4\Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

9. $$\int_0^1 \frac{1}{\sqrt{3x+1}}\,dx \overset{u=3x+1}{=\!=} \frac{1}{3}\int_1^4 \frac{1}{\sqrt{u}}\,du = \frac{1}{3}\cdot 2\sqrt{u}\Big|_1^4 = \frac{2}{3}(\sqrt{4} - \sqrt{1}) = \frac{2}{3}$$

an antiderivative can be found by substitution method, i.e. let $u = 3x+1$

10 (a) f increasing $\Leftrightarrow f' > 0 \Leftrightarrow 0 \leq x \leq 4$

(b) f decreasing $\Leftrightarrow f' < 0 \Leftrightarrow 4 \leq x \leq 5$

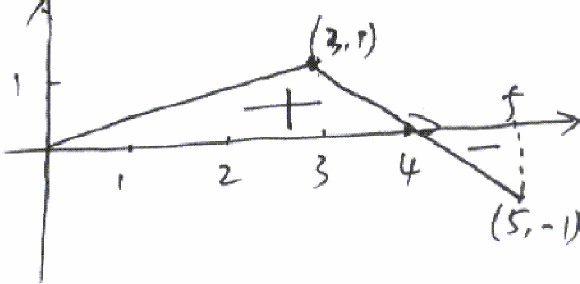
(c) $f(x)$ is max $\Leftrightarrow f'(x) = 0$ and the sign of $f'(x)$ changes from $+$ to $-$

$\Leftrightarrow x = 4$

(d) $\int_0^5 f'(x)\,dx$ = area

$= \frac{1}{2}\cdot 4\cdot 1 - \frac{1}{2}\,1\cdot 1$

$= 2 - \frac{1}{2} = \frac{3}{2}$



(e) By Fundamental Theorem of Calculus Part I

$$f(5) - f(0) = \int_0^5 f'(x)\,dx = \frac{3}{2}$$

because $f(x)$ is an antiderivative of $f'(x)$.