1. Differentiate with respect to $x$. Write your answers showing the use of the appropriate techniques. Do *not* simplify.

$$\text{(a) } x^{1066} + x^{1/2} - x^{-2}, \qquad \text{(b) } e^{\sqrt{x}}, \qquad \text{(c) } \frac{\sin(x)}{5+x^2}.$$

2. Differentiate, writing your answers as in problem 1.

$$\text{(a) } e^{3x}\cos(5x), \qquad \text{(b) } \ln(x^2+x+1), \qquad \text{(c) } \tan\left(\frac{1}{x}\right).$$

3. Differentiate, writing your answers as in problem 1.

$$\text{(a) } x^{2005} + x^{2/3}, \qquad \text{(b) } \cos(\pi x), \qquad \text{(c) } \frac{1+2x}{3+x^2}.$$

4. Differentiate, writing your answers as in problem 1.

$$\text{(a) } x^2 e^{-3x}, \qquad \text{(b) } \arctan(x), \qquad \text{(c) } \ln(\cos(x)).$$

5. Use implicit differentiation to find the slope of the line tangent to the curve

$$x^2 + xy + y^2 = 7$$

at the point $(2, 1)$.

6. Use calculus to find the exact $x$- and $y$-coordinates of any local maxima, local minima, and inflection points of the function $f(x) = x^3 - 12x + 5$.

7. Use calculus to find the $x$- and $y$-coordinates of any local maxima, local minima, and inflection points of the function $f(x) = xe^{-x}$ on the interval $0 \le x < \infty$. The $y$-coordinates may be written in terms of $e$ or as a 4-place decimal.

8. Estimate the integral $\int_0^{40} f(t)\,dt$ using the left Riemann sum with four subdivisions
Some values of the function $f$ are given in the table:

| $t$ | 0 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| $f(t)$ | 5.3 | 5.1 | 4.6 | 3.7 | 2.3 |

If the function $f$ is known to be decreasing, could the integral be larger than your estimate? Explain why or why not?

9. Write the integral which gives the area of the region between $x = 0$ and $x = 2$, above the $x$-axis, and below the curve $y = 9 - x^2$.
Evaluate your integral exactly to find the area.

10. Write an integral which gives the area of the region between $x = 1$ and $x = 3$, above the $x$-axis, and below the curve $y = x - \frac{1}{x^2}$.
Evaluate your integral to find the area.

11. The average value of the function $f(x)$ on the interval $a \le x \le b$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Find the average value of the function $f(x) = \frac{1}{x^2}$ on the interval $2 \le x \le 6$.

12. Find

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}.$$

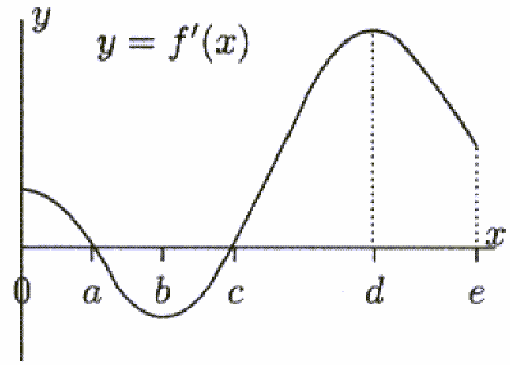
Explain how you obtain your answer.

13. Find

$$\lim_{x\to 0} \frac{1-\cos(3x)}{x^2}.$$

Explain how you obtain your answer.

14. The graph below represents the derivative, $f'(x)$.




(i) On what interval is the original $f$ decreasing?

(ii) At which labeled value of $x$ is the value of $f(x)$ a global minimum?

(iii) At which labeled value of $x$ is the value of $f(x)$ a global maximum?

(iv) At which labeled values of $x$ does $y = f(x)$ have an inflection point?

15. The function $f(x)$ has the following properties:

- $f(5) = 2$,
- $f'(5) = 0.6$,
- $f''(5) = -0.4$.

(a) Find the tangent line to $y = f(x)$ at the point $(5, 2)$.

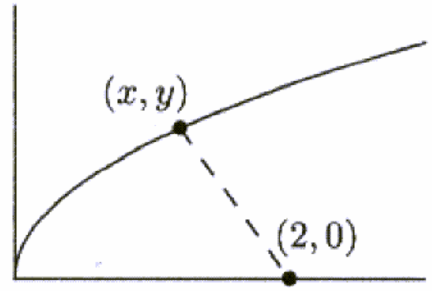
(b) Use (a) to estimate $f(5.2)$.

(c) If $f$ is known to be concave down, could your estimate in (b) be greater than actual $f(5.2)$? Give a reason supporting your answer.

16. The point $(x, y)$ lies on the curve $y = \sqrt{x}$.

(a) Find the distance from $(x, y)$ to $(2, 0)$ as a function $f(x)$ of $x$ alone.

(a) Find the value of $x$ that makes this distance the smallest.



17. You have 24 feet of rabbit-proof fence to build a rectangular garden using one wall of a house as one side of the garden and the fence on the other three sides. What dimensions of the rectangle give the largest possible area for the garden.

18. Find $\int xe^{x^2-1}\,dx$.

19. Find $\int \sin^2 x \cos x\,dx$.

20. Evaluate $\int_2^5 \frac{2x-3}{\sqrt{x^2-3x+6}}\,dx$.

1. (a) $(x^{1066}+x^{\frac{1}{2}}-x^{-2})' = 1066x^{1065}+\frac{1}{2}x^{-\frac{1}{2}}+2x^{-3}$

(b) $(e^{\sqrt{x}})' = e^{\sqrt{x}}\cdot(\sqrt{x})' = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

(c) $\left(\frac{\sin x}{5+x^2}\right)' = \frac{(\sin x)'(5+x^2)-(5+x^2)'\sin x}{(5+x^2)^2} = \frac{\cos x(5+x^2)-2x\sin x}{(5+x^2)^2}$

2. (a) $(e^{3x}\cos(5x))' = (e^{3x})'\cos(5x)+e^{3x}(\cos(5x))'$

$= 3e^{3x}\cos(5x) - 5e^{3x}\sin(5x)$

(b) $(\ln(x^2+x+1))' = \frac{1}{x^2+x+1}\cdot(x^2+x+1)' = \frac{2x+1}{x^2+x+1}$

(c) $(\tan(\frac{1}{x}))' = \sec^2(\frac{1}{x})\cdot(\frac{1}{x})' = -\frac{\sec^2 x}{x^2}$

3. (a) $(x^{2005}+x^{\frac{2}{3}})' = 2005x^{2004}+\frac{2}{3}x^{-\frac{1}{3}}$

(b) $(\cos(\pi x))' = -\pi\sin(\pi x)$

(c) $\left(\frac{1+2x}{3+x^2}\right)' = \frac{2(3+x^2)-2x(1+2x)}{(3+x^2)^2}$

4 (a) $(x^2e^{-3x})' = 2xe^{-3x}-3x^2e^{-3x}$

(b) $(\arctan x)' = \frac{1}{1+x^2}$

(c) $(\ln(\cos x))' = \frac{1}{\cos x}\cdot(\cos x)' = -\frac{\sin x}{\cos x} = -\tan x$

5 $\frac{d}{dx}(x^2+xy+y^2) = \frac{d}{dx}7 = 0$

Equation of tangent line

$y-y_0 = f'(x_0)(x-x_0)$

$2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx} = 0$

$(2x+y)+(x+2y)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x+y}{x+2y}$

$\Rightarrow y-1 = -\frac{5}{4}(x-2)$

$\left.\frac{dy}{dx}\right|_{(2,1)} = -\frac{2\cdot2+1}{2+2\cdot1} = -\frac{5}{4}$

6 $f'(x) = 3x^2 - 12 = 0 \Rightarrow x = 2$ or $-2$

$f''(x) = 6x = 0 \Rightarrow x = 0 \Rightarrow (0, f(0)) = (0, 5)$ is the inflection point

| | $(-\infty, -2)$ | $-2$ | $(-2, 2)$ | $2$ | $(2, +\infty)$ |
|---|---|---|---|---|---|
| sign of $f'(x)$ | $f'(-3) = 15$ + | 0 | $f'(0) = -12$ − | 0 | $f'(3) = 15$ + |
| $f(x)$ | ↑ | Max $f(-2) = 21$ | ↓ | min $f(2) = -11$ | ↑ |

$\Rightarrow (2, f(2)) = (2, -11)$ is the local min

$(-2, f(-2)) = (-2, 21)$ is the local max

7. $f'(x) = e^{-x} - xe^{-x} = 0 \Rightarrow x = 1$

$f''(x) = -e^{-x} - (1-x)e^{-x} = 0 \Rightarrow x = 2 \Rightarrow (2, f(2)) = (2, \frac{2}{e^2})$ is the inflection point

| | $(0, 1)$ | $1$ | $(1, +\infty)$ |
|---|---|---|---|
| sign of $f'(x)$ | $f(\frac{1}{2}) = \frac{1}{2}e^{-\frac{1}{2}}$ + | 0 | $f(2) = -e^{-2}$ − |
| $f(x)$ | ↑ | | ↓ |

$\Rightarrow (1, f(1)) = (1, \frac{1}{e})$ is the local max

8. $L_4 = 10 \cdot (f(0) + f(10) + f(20) + f(30))$

$= 10\ (5.3 + 5.1 + 4.6 + 3.7)$

$= 187$

If $f$ is decreasing, then $R_4 < L_4$. But we know that $\int_0^{40} f(t)dt$ is bounded by $L_4$ and $R_4$. Hence $R_4 < \int_0^{40} f(t)dt < L_4$. It follows that the integral can not be larger than 187.

9. $A = \int_0^2 (9 - x^2)dx = \int_0^2 9dx - \int_0^2 x^2 dx$

$= 9x\Big|_0^2 - \frac{1}{3}x^3\Big|_0^2$

$= 18 - \frac{8}{3}$

$= \frac{46}{3} = 15\frac{1}{3}$

10 $A = \int_1^3 (x - \frac{1}{x^2}) dx = \int_1^3 x\,dx - \int_1^3 x^{-2} dx$

$= \frac{1}{2}x^2\Big|_1^3 + \frac{1}{x}\Big|_1^3 = \frac{1}{2}\cdot 3^2 - \frac{1}{2} + \frac{1}{3} - \frac{1}{1} = \frac{10}{3}$

11 The average value of the function $f(x) = \frac{1}{x^2}$ on the interval $2 \le x \le 6$ is given by

$$\frac{1}{6-2}\int_2^6 \frac{1}{x^2}\,dx$$

$$= \frac{1}{4}\cdot\left(-\frac{1}{x}\right)\Big|_2^6 = \frac{1}{4}\cdot\left[-\left(\frac{1}{6} - \frac{1}{2}\right)\right] = \frac{1}{4}\cdot\frac{1}{3} = \frac{1}{12}$$

12. Method 1: $\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x\to 0} \frac{(\sqrt{1+x} - 1)(\sqrt{1+x} + 1)}{x(\sqrt{1+x} + 1)} = \lim_{x\to 0} \frac{x}{x(\sqrt{1+x}+1)}$

$= \lim_{x\to 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}$

Method 2: L'Hôpital's rule

Since $\lim_{x\to 0}(\sqrt{1+x} - 1) = \sqrt{1+0} - 1 = 0$ $\quad \lim_{x\to 0} x = 0$

and $\lim_{x\to 0} \frac{(\sqrt{1+x} - 1)'}{(x)'} = \lim_{x\to 0} \frac{1}{2\sqrt{1+x}} = \frac{1}{2\sqrt{1+0}} = \frac{1}{2}$

Then by L'Hôpital's rule

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x\to 0} \frac{(\sqrt{1+x} - 1)'}{(x)'} = \frac{1}{2}$$

13 L'Hôpital's rule

Step1: $\begin{cases} \lim_{x\to 0} (1 - \cos 3x) = 1 - \cos(3\cdot 0) = 1 - 1 = 0 \\ \lim_{x\to 0} x^2 = 0 \\ \lim_{x\to 0} \frac{(1-\cos 3x)'}{(x^2)'} = \lim_{x\to 0} \frac{3\sin 3x}{2x} \end{cases}$

Step2: $\begin{cases} \lim_{x\to 0} 3\sin 3x = 0 \quad \lim_{x\to 0} 2x = 0 \\ \lim_{x\to 0} \frac{(3\sin 3x)'}{(2x)'} = \lim_{x\to 0} \frac{9\cos 3x}{2} = \frac{9\cos 0}{2} = \frac{9}{2} \end{cases}$

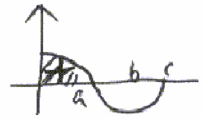
So by L'Hôpital's rule $\lim_{x\to 0} \frac{1-\cos 3x}{x^2} = \lim_{x\to 0} \frac{(1-\cos 3x)'}{(x^2)'} = \lim_{x\to 0} \frac{(3\sin 3x)'}{(2x)'} = \frac{9}{2}$

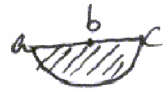
14. (i) $f$ decreasing $\iff f' < 0 \iff x \in (b, c)$

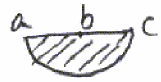
(ii) $f(x)$ is a <u>global min</u> at $x = e$ because, by FTC Part II

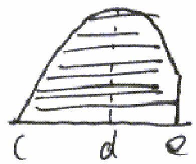
$f(x) = \int_0^x f'(t)\,dt$

$f(x)$ has two critical points $x = a$ and $x = c$

$f(0) = 0$ $f(a)$ = the area indicated in the graph 

$f(c)$ = the area of  − the area of 

$f(e)$ = the area of  − the area of 

+ the area of 

Clearly $f(e)$ is the greatest value.

(iii) Similarly, $f(c)$ is the smallest value of $f(x)$ over $[0, e]$

(iv) It is clear that $f'(x)$ has local extrema at $x=b$ and $x=d$.

Then $f''(b) = 0$ and $f''(d) = 0$ So at $x=b$ and $x=d$.

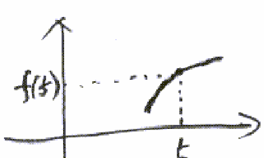
$f(x)$ has inflection points.

15. (a) $y - 2 = f'(5)(x-5) = 0.6(x-5)$ i.e. $y = 0.6x - 1$

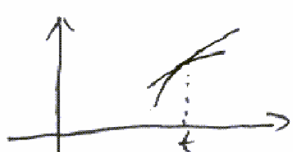
(b) $f(5.2) \approx L(5.2) = 0.6 \cdot 5.2 - 1 = 2.12$

Note: The linear approximation and tangent line have the same function rule i.e. $L(x) = f'(x_0)(x - x_0) + y_0$

(c) Locally, $f(x)$ is increasing concave down around $x = 5$. i.e. the graph looks like



So the tangent line at $(5, f(5))$ is locally above the graph of $f(x)$,



Therefore $f(5.2) < L(5.2) = 2.12$ So the answer is YES.

16. (a) Distance formula: Given two point $(x_0, y_0)$ and $(x_1, y_1)$ the distance between them is $d = \sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$

$$f(x) = \sqrt{(x-2)^2+(y-0)^2} = \sqrt{(x-2)^2+(\sqrt{x}-0)^2} = \sqrt{x^2-3x+4} \qquad x \geq 0$$

(b) $f'(x) = \left(\sqrt{x^2-3x+4}\right)' = \dfrac{2x-3}{2\sqrt{x^2-3x+4}}$

$f'(x) = 0 \Rightarrow x = \frac{3}{2}$

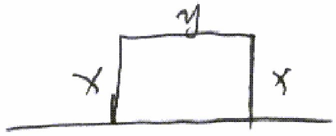
$f(0) = \sqrt{0-0+4} = 2 \qquad f(-\infty) = \lim_{x\to+\infty} \sqrt{x^2-3x+4} = +\infty$

$f(\frac{3}{2}) = \sqrt{(\frac{3}{2})^2 - 3\cdot\frac{3}{2}+4} = \sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{2}$

since $\sqrt{7} < 4$ so $\frac{\sqrt{7}}{2} < 2$

Hence the minimal distance is $f(\frac{3}{2}) = \frac{\sqrt{7}}{2}$

17. Assume as indicated in the picture the values




Then we have $2x + y = 24$

we need $A = xy$

$\Rightarrow A = A(x) = x(24-2x) = -2x^2+24x$

$0 \leq x \leq 12$

$A'(x) = -4x+24 = 0 \Rightarrow x = 6$

$A(0) = 0 \qquad A(6) = -2\cdot 6^2+24\cdot 6 = 2\cdot 6^2 = 72$

$A(12) = -2\cdot 12^2 + 24\cdot 12 = 0$

So $A_{max} = A(6) = 72$

18. $\int x e^{x^2-1}\,dx = \frac{1}{2}\int e^{x^2-1}\,d(x^2-1) = \frac{1}{2}e^{x^2-1} + C$

19. $\int \sin^2 x \cos x\,dx = \int \sin^2 x\,d(\sin x) = \frac{1}{3}\sin^3 x + C$

20. $\int_2^5 \frac{2x-3}{\sqrt{x^2-3x+6}}\,dx$

$= \int_2^5 \frac{d(x^2-3x+6)}{\sqrt{x^2-3x+6}}$

$= 2\sqrt{x^2-3x+6}\Big|_2^5$

$= 2\left(\sqrt{5^2-3\cdot5+6} - \sqrt{2^2-3\cdot2+6}\right)$

$= 2\cdot 4 - 2$

$= 6$

<u>Note:</u> $2\sqrt{x^2-3x+6}$ is an antiderivative of $\frac{2x-3}{\sqrt{x^2-3x+6}}$

So you don't have to change the limit. In fact, the idea is from substitution method but we didn't really do substitution