Math 180 sample problems for Hour Exam Two

1. Differentiate the following functions:
   (a) $x^2 \ln x$,
   (b) $\sin(a + bx)$,
   (c) $\arctan(3x)$.

2. Differentiate the following functions:
   (a) e^{2-x^2},
   (b) $x \cos(x)$,
   (c) $\arcsin(x/2)$.

3. Let $y = f(x)$ be the function defined implicitly by $y^3 - y + x = 0$ and $f(6) = -2$.
   (a) Find $\dfrac{dy}{dx}$ at the point $(6, -2)$.
   (b) Find the equation of the tangent line at $(6, -2)$.

4. Use the information in the table about f and g to find:
   (a) $h'(0)$, where $h(x) = f(g(x))$.
   (b) $k'(2)$, where $k(x) = f(x)g(x)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	−1	2	5
1	−1	2	4	3
2	7	3	1	4

5. Find the critical points of the function $f(x) = x^3 + 3x^2 - 9x - 11$ and find the global minimum of $f(x)$ on the interval $-4 \le x \le 3$.

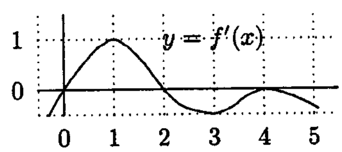
6. Find the x- and y-coordinates of all local maxima, local minima, and inflection points of $f(x) = x^3 - 3x + 2$.

7. Find $\displaystyle\lim_{x \to 0} \frac{1 - e^x}{x - x^2}$.

8. You wish to enclose a 400 square-foot rectangular garden with shrubs costing \$40 per foot on three sides and a wall costing \$20 per foot on the fourth side. Find the dimensions that minimize the total cost.

9. Find the largest interval on which $f(x) = (x^2 + 1)e^{-x}$ is concave down.

10. The graph below is of the *derivative* $f'(x)$ on the interval $(-0.5, 5.5)$. Determine the intervals on which the original function f is:
    (a) increasing,
    (b) decreasing,
    (c) concave up,
    (d) concave down.
    (e) Give one value of x at which f has a local maximum.



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(Solutions)

1. a) $(x^2 \ln x)' = (x^2)' \ln x + x^2 (\ln x)' = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x$

   b) $[\sin(a+bx)]' = \cos(a+bx) \cdot (a+bx)' = b \cdot \cos(a+bx)$

   c) $(\arctan 3x)' = \frac{1}{1+(3x)^2} \cdot (3x)' = \frac{3}{1+9x^2}$

2. a) $(e^{2-x^2})' = e^{2-x^2} \cdot (2-x^2)' = -2x e^{2-x^2}$

   b) $(x \cos x)' = \cos x - x \sin x$

   c) $\arcsin \frac{x}{2} = \frac{1}{\sqrt{1-(\frac{x}{2})^2}} \cdot (\frac{x}{2})' = \frac{1}{\sqrt{1-\frac{x^2}{4}}} \cdot \frac{1}{2} = \frac{1}{\sqrt{4-x^2}}$

3. $0 = \frac{d}{dx}(y^3 - y - x) = \frac{d y^3}{dx} - \frac{dy}{dx} - \frac{dx}{dx} = 3y^2 \frac{dy}{dx} - \frac{dy}{dx} - 1 \Rightarrow$

$(3y^2-1)\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{3y^2-1} \quad \Rightarrow \quad \frac{dy}{dx}\Big|_{(6,-2)} = \frac{1}{3(-2)^2-1} = \frac{1}{11}$

The equation of tangent line

$$y-(-2) = \frac{1}{11}(x-6) \iff y = \frac{1}{11}x - \frac{28}{11}$$

4. a) $h'(0) = f'(g(0)) \cdot g'(0) = f'(2) \cdot 5 = 3 \cdot 5 = 15$

   b) $k'(2) = f'(2)g(2) + f(2)g'(2) = 3 \cdot 1 + 7 \cdot 4 = 3 + 28 = 31$

5. $f'(x) = 3x^2 + 6x - 9 = 0 \Rightarrow 3(x^2+2x-3) = 0 \Rightarrow x^2+2x-3=0$

$\Rightarrow (x-1)(x+3) = 0 \Rightarrow x = 1 \text{ or } x = -3$

$f(-4) = (-4)^3 + 3(-4)^2 - 9(-4) - 11 = -64 + 48 + 36 - 11 = 9$

$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 11 = -3^3 + 3^3 + 27 - 11 = 16$

$f(1) = 1^3 + 3 \cdot 1^2 - 9 \cdot 1 - 11 = 1 + 3 - 9 - 11 = -16$

$f(3) = 3^3 + 3 \cdot 3^2 - 9 \cdot 3 - 11 = 27 + 27 - 27 - 11 = 16$

So $f_{min} = f(1) = -16$

6. $f'(x) = 3x^2 - 3 = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1$ or $x = -1$

$f''(x) = (3x^2 - 3)' = 6x = 0 \Rightarrow x = 0$

So $(0, f(0)) = (0, 2)$ is the only inflection point

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, +\infty)$
$f'(x)$	$f'(-2) = 9$ $+$	0	$f'(0) = -3$ $-$	0	$f'(2) = 9$ $+$
$f(x)$	$\nearrow$	max	$\searrow$	min	$\nearrow$

$(-1, f(-1)) = (-1, 4)$ is the local max

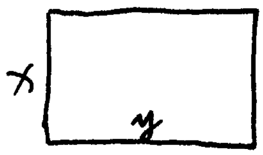
$(1, f(1)) = (1, 0)$ is the local min

7. Since $\lim\limits_{x \to 0} 1 - e^x = 0$ $\lim\limits_{x \to 0} x - x^2 = 0$, so

$\lim\limits_{x \to 0} \frac{1 - e^x}{x - x^2}$ is of $\frac{0}{0}$ type.

Hence $\lim\limits_{x \to 0} \frac{1 - e^x}{x - x^2} = \lim\limits_{x \to 0} \frac{(1 - e^x)'}{(x - x^2)'} = \lim\limits_{x \to 0} \frac{e^x}{1 - 2x} = \frac{1}{1} = 1$

8.



Assume x is the length of the wall

$$\left.\begin{array}{l} 400 = A = xy \\ \text{Cost} = 20x + 40(x + y + y) \end{array}\right\} \Rightarrow$$

$\text{Cost} = C(x) = 60x + 80 \cdot \frac{400}{x}$

$C'(x) = 60 - \frac{32000}{x^2} = 0 \Rightarrow 60x^2 - 32000 = 0$

$\Rightarrow x^2 = \frac{1600}{3} \Rightarrow x = \frac{40}{\sqrt{3}} = \frac{40}{3}\sqrt{3}$

$\Rightarrow y = \frac{400}{x} = \frac{400}{\frac{40}{\sqrt{3}}} = 10\sqrt{3}$

$\Rightarrow \text{Cost} = 60 \cdot \frac{40}{3}\sqrt{3} + \frac{32000}{\frac{40}{\sqrt{3}}} = 800\sqrt{3} + 800\sqrt{3} = 1600\sqrt{3}$

9. $f'(x) = (2x - x^2 - 1)e^{-x}$

$f''(x) = (x^2 - 2x + 1)e^{-x} + (2 - 2x)e^{-x} = (x^2 - 4x + 3)e^{-x}$

Let $f''(x) = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0$

$\Rightarrow x = 1$ or $x = 3$

x	$(-\infty, 1)$	1	$(1, 3)$	3	$(3, +\infty)$
$f''(x)$	$f''(0) = 3$ $+$	0	$f''(2) = -e^{-2}$ $-$	0	$f''(4) = 3e^{-4}$ $+$
$f(x)$	$\cup$	I	$\cap$	I	$\cup$

The largest interval on which $f(x)$ is concave down is $(1, 3)$.

10. a) increasing $\Leftrightarrow f'(x)$ positive $\Leftrightarrow x \in (0, 2)$
b) decreasing $\Leftrightarrow f'(x)$ negative $\Leftrightarrow x \in (-0.5, 0) \cup (2, 5.5)$
c) concave up $\Leftrightarrow f''(x)$ positive $\Leftrightarrow f'(x)$ increasing $\Leftrightarrow x \in (-0.5, 1) \cup (3, 4)$
d)concave down $\Leftrightarrow f''(x)$ negative $\Leftrightarrow f'(x)$ decreasing $\Leftrightarrow x \in (1, 3) \cup (4, 5.5)$
e) local maximum must be a critical point. a critical point is a root of $f'(x) = 0$. a root is a x-intercept. So the possible local maximum are 1,2,4. A critical point is a local maximum if and only if the change of sign of $f'(x)$ is $+ \to -$. So f(2) is the local maximum.