Connectedness of Moduli Spaces of Certain Types of Hyperplane Arrangements

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Arrangements of Hyperplanes: Definitions

- A r-arrangement is $\mathcal{A} = \{H_1, H_2, \cdots, H_n | H_i\}$, a finite collection of hyperplanes in $\mathbb{CP}^r$.
- The complement of $\mathcal{A}$ is define as $M(\mathcal{A}) := \mathbb{CP}^r - \bigcup_{H \in \mathcal{A}} H$.
- $L(\mathcal{A}) = \{ \bigcap_{\substack{i \in I \\ I \subset \{1, \cdots, k\}}} H_i | H_i \in \mathcal{A}\}$ is called the intersection lattice of $\mathcal{A}$.

Introduction

In the study of hyperplane arrangements, a central topic is to determine the relations between the topology of $M(\mathcal{A})$ and the combinatorial geometry of $L(\mathcal{A})$.

- In 1980, Orlik and Solomon computed the cohomology algebra of $M(\mathcal{A})$ in terms of $L(\mathcal{A})$. They further conjuctured that various homotopy invariants of the complement depend only on the intersection lattice of $\mathcal{A}$.
- Falk shows that there exist pairs of central arrangements in $\mathbb{C}^3$ with different intersection lattices but the complements are homotopy.
- Yau and Jiang show that the lattices of line arrangements in $\mathbb{CP}^2$ are topological invariant.

Introduction (continue)

- On the other hand, Rybnikov shows that there exists a pair of line arrangements in $\mathbb{CP}^2$ with same lattice but the fundamental groups of their complements are not isomorphic. Such pairs are called Zariski pairs.

Question:

Explore the relations between the combinatorial of intersection lattices and topology of complements of arrangements

- Hacking, Keel and Tevelev show that the moduli spaces of hyperplane arrangements exists and have compactifictions. However, it seems we don't know much of those moduli spaces.

Question:

The moduli of what kind of hyperplane arrangements is connected.

Some Results

Theorem (Jiang-Yau, Wang-Yau and Yau-)

There are certain types of arrangements, called nice arrangements or nice point arrangements, in projective spaces whose moduli spaces with respect to a fixed lattice is connected.

Examples of Nice Arrangements in $\mathbb{CP}^2$

- Line arrangements of n, $n \leq 5$, are all nice arrangements hence their moduli spaces are connected. Line arrangements of 6 lines are also nice arrangements except the arrangements $\mathcal{A}_3$ whose lattice contains 4 triple points.
- The line arrangement $\mathcal{A}_3$ is a fiber-type line arrangement. Roughly speaking, an arrangement is of fiber-type if the complement is linearly fibered over $\mathbb{CP}^1 \setminus \{x_1, \cdots, x_m\}$.

Properties of Fiber-type Projective Line Arrangements

- The defining equation of a projective fiber-type arrangement in $\mathbb{P}^2$ can be written as $z(x-k_1z)(x-k_2z)\cdots(x-k_mz)(y-a_1x-b_1z)(y-a_2x-b_2z)\cdots(y-a_nx-b_nz)=0$.
- Given two line arrangement with the same lattice as $L(\mathcal{A}_3)$, the complements are diffeomorphic. However, I don't know the moduli of line arrangements with fixed lattice $L(\mathcal{A}_3)$ is connected or not.

Question:

Is the moduli space of fiber-type projective line arrangements with fixed lattice connected?

Idea: Constructing One-parameter Family

- Let $\mathcal{A}_1 = \{H_1, \cdots, H_n\}$ and $\mathcal{A}_2 = \{G_1, \cdots, G_n\}$ be two arrangements such that $L(\mathcal{A}_1) = L(\mathcal{A}_2)$. To construct a one-parameter family, a very natural starting points is to consider

$$\mathcal{A}(X) = \{x_1 H_1 + y_1 G_1, \cdots, x_n H_n + y_n G_n\}, \ (x_i, y_i) \in \mathbb{CP}^1.$$

  This is a family of arrangements over $(\mathbb{P}^1)^n$.
- Extract the main information of the lattices and convert the information into linear conditions. More precisely, we show that $L(\mathcal{A}(X)) \cong L(\mathcal{A}_1)$ if $X = ((x_1, y_1), \cdots, (x_n, y_n))$ is a solution of a bunch of polynomial equations $P((x_1, y_1), \cdots, (x_n, y_n)) = 0$ and inequalities $Q((x_1, y_1), \cdots, (x_n, y_n)) \neq 0$.
- We have to show that after removing some hypersurface sections defined by $Q((x_1, y_1), \cdots, (x_n, y_n)) = 0$ the variety defined by the polynomial equations $P((x_1, y_1), \cdots, (x_n, y_n)) = 0$ is connected.

Thank You !