Filtrations of Vector Bundles on Smooth Projective Algebraic Curves

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The bundles we will talk about are all algebraic vector bundles on a smooth projective algebraic curve.

Definition 0.1. The slope of a vector bundle E, denoted by $\mu(E)$, is defined to be $\frac{\deg E}{\mathrm{rk} E}$. E is semi-stable (stable) if $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$) for any sub-bundle $F \subset E$. Equivalently, $\mu(E/F) \geq \mu(E)$ (resp. $\mu(E/F) > \mu(E)$).

Let X be a smooth projective algebraic curve. Given a rational number μ, we define $\mathscr{C}(\mu) = \{E \text{ semi-stable bundle of slope } \mu \text{ on } X\}$

Proposition 0.2. *$\mathscr{C}(\mu)$ is an abelian sub-category of category of vector bundles.*

Proof. We only need to show that the following two properties hold.

(1) Additivity:

It is easy to show that $\mu(E \oplus F) = \mu$ if $\mu(E) = \mu(F) = \mu$. The semi-stability follows from the following fact:

Lemma 0.3. *Let $E, F \in \mathscr{C}(\mu)$ and $0 \to E \to G \to F \to 0$ exact. Then G is semi-stable and $\mu(G) = \mu$.*

Proof. Clearly, $\mu(G) = \mu$. Take a sub-bundle G' of G, we will get a short exact sequence $0 \to E' = E \cap G' \to G' \to F' \to 0$ with $E' \subset E$ and $F' \subset F$. Hence, $\mu(G') \leq \mu(F') = \mu(F) = \mu = \mu(G)$. Hence G is semi-stable. □

(2) Kernel and cokernel:

Let $E, F \in \mathscr{C}(\mu)$ and $f : E \to F$ is a bundle map. Then we have the following short exact sequences: $0 \to \ker f \to E \to \operatorname{im} f \to 0$ and $0 \to \operatorname{im} f \to F \to \operatorname{coker} f \to 0$

By definition, $\mu = \mu(E) \leq \mu(\operatorname{im} f) \leq \mu(F) = \mu$. Hence $\mu(\operatorname{im} f) = \mu$, so that $\mu(\ker f) = \mu$. This implies that $\ker(f)$ is semi-stable, since any subbundle of $\ker f$ is also a subbundle of E.

Similarly, $\mu(\operatorname{coker} f) = \mu$. $\operatorname{coker} f$ is a vector bundle follows from the following result:

Lemma 0.4. *Let $\mu(E) = \mu(F)$ and $E \subset F$. Then F/E is torsion free, hence a vector bundle*

Proof. It is clear that E/F can be written as $E/F = Q \oplus T$, where Q is a vector bundle and T is the torsion subsheaf of E/F. Then the following diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & \longrightarrow & 0 & \longrightarrow & T & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & E/F & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker} f & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

commutes, in which the rows and columns are all exact. Hence by snake lemma, we know $T = \operatorname{coker} f$. If $T \neq 0$. then $\deg T > 0$ and $\operatorname{rk} T = 0$. Therefore $\mu = \mu(F) \geq \mu(G) > \mu(E) = \mu$. Contradiction! □

Note that any sub-bundle S of $\operatorname{coker} f$ can be realized as a quotient $B/\operatorname{im} f$ of a subbundle B of F by $\operatorname{im} f$. If $\mu(S) > \mu$ then $\mu(B)$ must be strictly greater than μ. A contradiction! Therefore $\operatorname{coker} f$ is semi-stable. □

It is easy to see, in $\mathscr{C}(\mu)$, a descending filtration will be eventually stationary, because the rank decrease.

Lemma 0.5. *Let $E \in \mathscr{C}(\mu)$. If E is not stable, then E has a stable sub-bundle F of slope μ.*

Proof. pick up a any sub-bundle $E_1 \in \mathscr{C}(\mu)$ of E. If E_1 is stable, then we are done. If not we can pick up a sub-bundle $E_1 \in \mathscr{C}(\mu)$ of E_1. If E_2 is stable, then we are done. If E has no stable sub-bundle at all, keep searching in this way, we will eventually get an descending chain $E \supset E_1 \supset E_2 \supset \cdots \supset \cdots$ of infinite length. But the $\mathrm{rk} E$ is fixed. So we must have a stable sub-bundle. □

Theorem 0.6 (Jordan-Holder Filtration)**.** *Let F be a semi-stable bundle of slope μ. Then there exists a filtration, called Jordan-Holder filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_k = F$ such that $gr_i := F_i/F_{i-1}$ is stable. If $0 \subset F_1' \subset F_2' \subset \cdots \subset F_{k'}' = F$ is another Jordan-Holder filtration, then $k = k'$ and $\{gr_i\} = \{gr_i'\}$.*

Proof. We will only discuss the existence. By the above lemma, there exist a $F_1 \subset F$ such that F_1 is stable of slope μ, hence F/F_1 is semi-stable of slope μ. Then there exists a sub-bundle $F \supset F_2 \supset F_1$, such that F_2/F_1 is stable in F/F_1. Repeat this procedure, since the rank of F is finite, we will eventually get the filtration. □

Jordan-Holder filtration allows us to deal with semi-stable bundles in terms of stable bundle. The next filtration will help us classify un-stable bundles in terms of semi-stable bundle.

Lemma 0.7. *Let E be a bundle on X. Then $\deg F$ is bounded function over $\{F | F \subset E\}$.*

Proof. If $X = \mathbf{P}^1$, then $E = \oplus_{i=1}^{\mathrm{rk} E} \mathscr{O}(a_i)$ and $F = \oplus_{j=1}^{\mathrm{rk} E} \mathscr{O}(b_i)$. Since F is a sub-bundle of E then $\max\{b_j\} \leq \max\{a_i\}$. So $\deg F \leq \mathrm{rk} E \cdot \max\{a_i\}$. Assume that $g(X) \geq 1$. It is well know that there exists a finite morphism $f : X \to \mathbf{P}^1$. Note that in this case f is flat thanks to the following theorem:

Theorem 0.8 (Hartshorne Proposition III.9.7)**.** *Let X and Y be two smooth projective variety and* $\dim Y = 1$*. If $f : X \to Y$ is finite covering then f is flat.*

Proposition 0.9 (see Eisenbud)**.** *A coherent flat sheaf is locally free.*

Therefore f_*F is a sub-bundle of the vector bundle f_*E. Notice that $\chi(F) = \chi(f_*F)$, since f is finite. By Riemann-Roch, $\deg F$ is bounded because $\deg f_*F$ is bounded. □

Theorem 0.10. *Let E be a vector bundle on X. Then there is a filtration, called Harder-Narasimhan filtration, $0 \subset E_1 \subset E_2 \subset \cdots \subset E_k = E$, such that the grading $gr_i := E_i/E_{i-1}$ satisfies:*

1. *gr_i is semi-stable,*
2. *$\mu(gr_i) > \mu(gr_{i+1})$.*

Moreover the filtration is unique.

Proof. We will only show the existence. If E is already semi-stable, we are done. If not, let E_1 be the sub-bundle of E with maximal slope and maximal rank among the sub-bundles of E. Clearly, $E_1 \neq 0$. We claim that E_1 is semi-stable. Let G be any sub-bundle of E. By the choice of E_1, $\mu(G) \leq \mu(E_1)$. Therefore E_1 is semi-stable. If E/E_1 is semi-stable, we are done. Otherwise, there exists a non-zero semi-stable sub-bundle E_2/E_1 of E/E_1. We will show that $\mu(E_2/E_1) < \mu(E_1)$. Otherwise, that $\mu(E_2/E_1) \geq \mu(E_1)$ implies that $\mu(E_2) \geq \mu(E_1)$ and $rkE_2 > rkE_1$, however, E_1 is the maximal element among the sub-bundles, therefore $E_2 = E_1$. Contradiction! By repeating this procedure, we will get the finite filtration. □