

# Invariant Geometry

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Throughout this notes, the ground field will be  $\mathbb{C}$ .

## 1 Algebraic Group Actions and Quotients

### 1.1 Actions

Let  $G$  be an affine algebraic group and  $Y$  be an algebraic variety.

**Definition 1.1.** An action of  $G$  on  $Y$  is a morphism  $\sigma : G \times Y \rightarrow Y$ , such that

1.  $\forall g \in G, \sigma_g = \sigma|_{g \times Y} \in \text{Aut}(Y)$ ;
2.  $G \rightarrow \text{Aut}(Y)$  given by  $g \mapsto \sigma_g$  is a group morphism.

We usually denote by  $g \cdot y (= \sigma(g, y))$  the action.

For any  $y \in Y$ ,  $O^G(y) := \{g \cdot y | g \in G\}$  is called the orbit of  $y$ .  $S^G(y) := \{g \in G | g \cdot y = y\}$  is called the stabilizer of  $y$ .

*Example 1.2.* Let  $G = \mathbb{C}^*$  and  $Y = \mathbb{C}^2$ . The following is an action:

$$\begin{aligned} \sigma : \mathbb{C}^* \times \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ (\lambda, (x, y)) &\mapsto (\lambda x, \lambda^{-1} y). \end{aligned}$$

We have the following types of orbits:

- $O^G((x, y)) = \{(x', y') \mid x'y' = xy\}$ , for  $xy \neq 0$ ;
- $O^G((x, 0)) = \{(\lambda x, 0)\}$ , for  $x \neq 0$ ;
- $O^G((0, y)) = \{(0, \lambda^{-1}y)\}$ , for  $y \neq 0$ ;
- $O^G((0, 0)) = \{(0, 0)\}$ .

Note that  $O^G((x, 0))$  and  $O^G((0, y))$  are not closed, moreover

$$\overline{O^G((x, 0))} \cap \overline{O^G((0, y))} = O^G((0, 0)).$$

**Definition 1.3.** A morphism  $\varphi : X \rightarrow Y$  is a  $G$ -morphism if the following diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{Id \times \varphi} & G \times Y \\ \downarrow \sigma & & \downarrow \sigma \\ X & \xrightarrow{\varphi} & Y \end{array}$$

commutes. We say  $\varphi$  is  $G$ -invariant, if  $\varphi(O^G(x)) = y$  for any  $x \in X$ .

In other words, a  $G$ -invariant morphism is a  $G$ -morphism which is constant on all orbits.

## 1.2 Categorical quotients

Given an action, what is a quotient of the action, or what properties should a quotient satisfy. The reason that we study quotient is that we like the quotient to parametrize all orbits of the action. A property of a quotient is that the quotient morphism must be a  $G$ -invariant morphism. So to get a quotient, we may search it in the set of  $G$ -invariant morphisms. However, in general, the  $G$ -invariant morphisms may not be able to parametrize all orbits of the action. That means, a "good" quotient may not exist. However, if it can provide finest possible parametrization of the orbits, it still be useful. Under this philosophy, Here it comes the definition of categorical quotient:

**Definition 1.4** (Categorical Quotient). Assume that  $G$  acts on  $Y$ . A pair  $(Z, \varphi)$  consisting of a scheme  $Z$  and a morphism  $\varphi : Y \rightarrow Z$  is called a categorical quotient of  $Y$  by  $G$ , if it satisfies the following universal property, i.e. given any  $G$ -invariant morphism

$\varphi' : Y \rightarrow Z'$ , there exist a unique morphism  $\gamma : Z \rightarrow Z'$  such that the following diagram

$$\begin{array}{ccc} Y' & & \\ \downarrow \varphi & \searrow \varphi & \\ Z & \xrightarrow{\gamma} & Z' \end{array}$$

commutes.

*Example 1.5.* Consider the action given in Example 1.2. Then there is a categorical quotient given by  $\mathbf{C}^2 \xrightarrow{\pi} \mathbf{C}$  such that  $\pi(x, y) = xy$ . It is easy to see that  $\pi^{-1}(c) = \{(x, y) | xy = c\} = O^G(c)$  if  $c \neq 0$ , while  $\pi^{-1}(0) = O^G((x, 0)) \cap O^G((0, y)) \cap O^G((0, 0))$ . So  $\mathbf{C}$  parametrizes only closed orbits.

### 1.3 Good quotients

In some sense, to study the varieties is equivalent to study the regular functions on it. A function is constant on orbits if and only if it is  $G$ -invariant. So given a variety  $Y$  with an action  $G$ , it is naturally to consider the varieties  $X$  with property that  $A(X) = A(Y)^G$ , where  $A(X)$  is the coordinate ring.

A finer quotient is involving the invariant subring.

**Definition 1.6.** Let  $G$  be an affine algebraic group and  $Y$  be an algebraic variety. We say that a pair  $(Z, \varphi)$  consisting of an algebraic variety  $Z$  and a morphism  $\varphi : Y \rightarrow Z$  is a good quotient is if it satisfies the following properties:

1.  $\varphi$  is  $G$ -invariant.
2.  $\varphi$  is surjective and affine such that for each affine open subset  $U \subset Z$ ,  $A(U) = A(\varphi^{-1}(U))^G$ .
3. The images of  $G$ -invariant closed subsets are closed.
4. Any two disjoint  $G$ -invariant closed subsets have disjoint images.

*Example 1.7.* Let  $\pi : \mathbf{C}^2 \rightarrow \mathbf{C}$  be a map such that  $\pi(x, y) = xy$ . One can check easily that  $(\mathbf{C}, \varphi)$  is a good quotient.

Also, note that  $\mathbb{C}[x, y] \supset \mathbb{C}[x, y]^G = \mathbb{C}[xy] \stackrel{t=xy}{\cong} \mathbb{C}[t]$ . Then  $\pi$  is exactly the induced morphism  $\mathbb{C}^2 = \text{Spec}\mathbb{C}[x, y] \rightarrow \text{Spec}\mathbb{C}[x, y]^G = \mathbb{C}$ . We will see later, a morphism formed in this way is indeed a good quotient.

**Definition 1.8.** A good quotient is called a geometric quotient if each orbit is closed.

*Example 1.9.* Let  $\mathbb{C}^*$  act on  $\mathbb{C}^{n+1} \setminus \{0\}$  by multiplication. Then  $\varphi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$  is a geometric quotient.

## 2 Existence of geometric quotients

Let  $G$  be an algebraic group and  $V$  be a vector space. A representation of  $G$  on  $V$  is a homomorphism  $\rho : G \rightarrow GL(V)$ .  $V$  can be viewed as a  $G$ -module in the sense that,  $g \cdot v = \rho(g)v$ . A subspace  $W \subset V$  is called a  $G$ -submodule, if it is  $G$ -invariant, i.e.  $G \cdot W \subset W$ .

**Definition 2.1.**  $\rho$  is irreducible if  $V$  is a simple  $G$ -module.  $\rho$  is totally reducible if it decomposes in a direct sum of irreducible representations, equivalently,  $V$  is a semi-simple  $G$ -module, i.e.  $V$  is the direct sum of simple  $G$ -modules.  $G$  is linearly reductive if all the  $G$ -module  $V$  are semi-simple.

**Lemma 2.2.**  $G$  is linearly reductive if and only if for any finite dimensional  $G$ -module  $V$  and  $G$ -submodule  $W$ , there exists a  $G$ -submodule  $W'$  such that  $V = W \oplus W'$ .

**Lemma 2.3.** Let  $G$  be an affine algebraic group. Suppose that  $G$  contains a compact subgroup  $K$  (in the analytic topology) such that  $K$  is Zariski-dense in  $G$ . Then  $G$  is linearly reductive.

*Example 2.4.*  $GL(n)$ ,  $PGL(n)$ ,  $SL(n)$  are all linearly reductive. Because they contain Zariski-dense compact subgroup  $U(n)$ ,  $SU(n)$  and  $PU(n)$  respectively.

### 2.1 Affine case

**Theorem 2.5** (Hilbert-Mumford). Let  $G$  be a linearly reductive algebraic group acting on the affine algebraic variety  $Y = \text{Spec}A$ . Then the  $\mathbb{C}$ -algebra  $A^G$  is of finite type. Let  $Z = \text{Spec}A^G$  and  $\varphi : Y \rightarrow Z$  be the morphism induced by the inclusion  $A^G \hookrightarrow A$ . Then  $(Z, \varphi)$  is a good quotient.

## 2.2 Projective case

Let  $Y \subset \mathbf{CP}^n$  and  $A(Y)$  be its polynomial ring. Let  $G$  be an affine algebraic group acting on  $Y$ . To construct a good quotient of  $Y$ , it is natural to consider  $\text{Proj}A^G$ . However, in general, the induced map  $Y = \text{Proj}A(Y) \rightarrow \text{Proj}A(Y)^G$  may not well-defined.

*Example 2.6.* Consider the  $\mathbf{C}^*$  action on  $\mathbf{CP}^2$  given by  $\lambda \cdot [x, y, z] = [\lambda x, \lambda^{-1} y, z]$ . Then  $\mathbf{C}[x, y, z]^G = \mathbf{C}[xy, z] \cong \mathbf{C}[t, z]$ .  $\text{Proj}\mathbf{C}[t, z] = \mathbf{CP}^1$ . The induced map  $[x, y, z] \mapsto [xy, z]$  is not well-defined at  $[1, 0, 0]$  and  $[0, 1, 0]$ . So we have to remove some points. Note that all the  $G$ -invariant non-trivial polynomials are vanish at those two points. Those are the unstable points in the following sense.

**Definition 2.7.** A point  $y \in Y$  is said to be semi-stable under the action of  $G$ , if there exist a homogeneous  $G$ -invariant polynomial  $f$  such that  $\deg f > 0$  and  $f(y) \neq 0$ . Otherwise,  $y$  is called unstable.

Let  $Y^{ss}$  be the open subset of semi-stable points of  $Y$  under the action of  $G$  and  $Z = \text{Proj}A(Y)^G$ . Then we have a morphism of algebraic varieties

$$\varphi : Y^{ss} \rightarrow Z.$$

**Theorem 2.8.** *The morphism  $\varphi : Y^{ss} \rightarrow Z$  induced by  $A(Y)^G \hookrightarrow A(Y)$  is a good quotient of the open set  $Y^{ss}$  of the semi-stable points of  $Y$  under the action of  $G$ .*

**Definition 2.9.** A point  $y \in Y$  is said to be stable under the action of  $G$  if it is semi-stable and the orbit map  $G \rightarrow Y^{ss}$  given by  $g \mapsto g \cdot y$  is proper.

## 3 The Hilbert-Mumford criterion

### 3.1 Numerical criterion

**Definition 3.1.** A one parameter subgroup of an affine algebraic group  $G$  is a non-trivial morphism  $\lambda : G_m = \mathbf{C}^* \rightarrow G$ .

**Definition 3.2.** Let  $V$  be a  $G$ -module and  $v \in V$ . We say that  $v$  is

- semi-stable if  $0 \notin O^G(v)$ ;

- stable if the orbit  $O^G(v)$  is closed and the stabilizer  $S^G(v)$  is finite.

**Theorem 3.3.** *Let  $G$  be a linearly reductive group and  $V$  be a  $G$ -module. A point  $y \in V$  is semi-stable (stable) under the action of  $G$  if and only if for all one parameter subgroups  $\lambda$  it is semi-stable (respectively stable) under the action of  $\mathbf{C}^*$  given by  $\lambda$ .*

This theorem gives easily a numerical criterion for semi-stability.

Given a one parameter subgroup  $\lambda : \mathbf{C}^* \rightarrow G$  acting on  $\mathbf{C}^{n+1}$ . Then  $\lambda$  can be diagonalized. Let  $e_0, \dots, e_n$  be the basis diagonalizes  $\lambda$ . Then for each  $t \in \mathbf{C}^*$ ,  $t \cdot e_i := \lambda(t) \cdot e_i = t^{r_i} e_i$  for some integer  $r_i$ . Let  $x \in \mathbf{CP}^n$  which is represented by  $\hat{x} := \sum_{i=0}^n x_i e_i \in \mathbf{C}^{n+1} \setminus \{0\}$ . We define  $\mu(\lambda, x) := \max\{-r_i | x_i \neq 0\}$ .

**Theorem 3.4.** *Let  $G$  be a linearly reductive group acting on  $\mathbf{C}^{n+1}$ . Let  $Y \subset \mathbf{CP}^n$  be a closed  $G$ -invariant subvariety. A point  $y \in Y$  is semistable (stable) if  $\mu(\lambda, y) \geq 0$  (respectively,  $\mu(\lambda, y) > 0$ ) for all one parameter subgroups  $\lambda$*

*Remark 3.5.* From the definition of  $\mu(\lambda, y)$ , we see that  $\mu(\lambda, g \cdot y) = \mu(g^{-1} \lambda g, y)$ . It allows us to replace  $\lambda$  by its conjugate. In particular, for  $g \in SL(n+1)$ , it is well know that every one parameter subgroup is conjugate to a diagonal matrix of the form

$$\begin{pmatrix} t^{r_0} & 0 & \dots & 0 \\ 0 & t^{r_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{r_n} \end{pmatrix}$$

with  $\sum r_i = 0, r_0 \geq r_1 \geq \dots \geq r_n$  and not all  $r_i = 0$ .

## 3.2 Computations of semi-stable points

### 3.2.1 Binary forms

**Proposition 3.6.** *A point  $(x_1, \dots, x_d) \in \mathbf{CP}^{1(d)}$  is semi-stable under the induced action of  $SL(2)$  on  $\mathbf{CP}^1$  if no more than half of  $x_1, \dots, x_n$  coincide.*

*Proof.* Note that we can view  $(x_1, \dots, x_n)$  as the zero set of a binary form of degree  $n$ :

$$F_n(x, y) = \prod_{i=1}^d (b_i x - a_i y),$$

where  $x_i = (a_i, b_i)$ . In this view,  $\mathbf{CP}^{1(d)} = \mathbf{P}(\mathbf{C}[x, y]_d)$ . The quotient we are consider is then  $\mathbf{P}(\mathbf{C}[x, y]_d) // SL(2)$ . Any one parameter subgroup  $\lambda$  of  $SL(2)$  is conjugate to one of the form

$$\lambda_r(t) = \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}.$$

Let  $\lambda_r(t)$  act on a degree  $d$  binary form  $f := \sum_{i=0}^d a_i x^i y^{d-i}$  is given by

$$\lambda_r(t) \cdot f = \sum_{i=0}^d a_i t^{r(2i-d)} x^i y^{d-i}.$$

So  $\mu(\lambda_r, f) = r(n - 2i_0)$ , where  $i_0$  is the smallest  $i$  such that  $a_i \neq 0$ . Thus,  $\mu\lambda_r, f \geq 0$  if and only if  $i_0 \leq \lfloor \frac{n}{2} \rfloor$ . Equivalently,  $a_i \neq 0$  for  $i > \lfloor \frac{n}{2} \rfloor$ , equivalent to no point of  $\mathbf{CP}^1$  occurs as a point of multiplicity  $\lfloor \frac{n}{2} \rfloor$ . Therefore,  $(x_1, \dots, x_d) \in \mathbf{CP}^{1(d)}$  is semi-stable if no more that half of them coincide.  $\square$

### 3.2.2 Grassmannian

Let  $H$  and  $V$  be two  $\mathbf{C}$ -vector spaces of dimension  $m$  and  $n$  respectively. Let  $G(r, H \otimes V)$  be the Grassmannian of vector sub-spaces of dimension  $r$  in  $H \otimes V$ . The linear reductive group  $SL(H)$  acts on  $H \otimes V$  by  $g \cdot (h \otimes v) = (g \cdot h) \otimes v$ . It follows that  $SL(H)$  acts on  $G(r, H \otimes V)$  by  $g \cdot k = (g \otimes Id)(k)$ . We get a linear action of  $SL(H)$  on  $\wedge^r(H \otimes V)$  by  $g \cdot (x_1 \wedge \dots \wedge x_r) = g \cdot x_1 \wedge \dots \wedge g \cdot x_r$ . So an action on the projective space  $\mathbf{P}(\wedge^r(H \otimes V))$  The Plucker embedding  $G(r, H \otimes V) \hookrightarrow \mathbf{P}(\wedge^r(H \otimes V))$  is then  $SL(H)$ -equivariant, i.e. it commutes with the group action. As a closed invariant subvariety, we want to determine the semi-stable and stable points of  $G(r, H \otimes V)$  under the  $SL(H)$  action.

**Proposition 3.7.** *Let  $K \in G(r, H \otimes V)$ . Then the following are equivalent.*

1.  $K$  is semi-stable (resp. stable) under  $SL(H)$ .

2. for each proper non-zero sub-space  $H' \subset H$ , let  $K' = H' \otimes V \cap K$ , we have

$$\frac{\dim K'}{\dim H'} \leq \frac{\dim K}{\dim H} \text{ ( resp. } < \text{ )}.$$

*Proof.* Let  $\lambda$  be a one parameter subgroup of  $SL(H)$ , we need to compute  $\mu(\lambda, K)$ . From Remark 3.5, we can assume that

$$\lambda(t) = \begin{pmatrix} t^{r_0} & 0 & \cdots & 0 \\ 0 & t^{r_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{r_n} \end{pmatrix}$$

with  $\sum r_i = 0$ ,  $r_0 \geq r_1 \geq \cdots \geq r_n$  and not all  $r_i = 0$ . We can assume that there are only  $s$  different  $r_i$ , say  $r_{i_1} > \cdots > r_{i_s}$ . Let  $\bigoplus_{i=1}^s H_i$  be the irreducible decomposition of  $H$  associated to the action  $\lambda(t)$ . Then  $K = \bigoplus [(H_i \otimes V) \cap K]$  and

$$\wedge^r (H \otimes V) = \bigoplus_{j_1 + \cdots + j_s = r} \wedge^{j_1} (H_1 \otimes V) \otimes \cdots \otimes \wedge^{j_s} (H_s \otimes V).$$

Denote by  $K_i = (H_i \otimes V) \cap K$ . Then  $\wedge^r K \cap \wedge^{j_1} (H_1 \otimes V) \otimes \cdots \otimes \wedge^{j_s} (H_s \otimes V) \neq 0$  if  $j_i = \dim K_i$ . Therefore

$$\begin{aligned} \lambda(t) \cdot K &= \max \left\{ - \sum_{i=1}^s r_i j_i \mid \wedge^r K \cap \wedge^{j_1} (H_1 \otimes V) \otimes \cdots \otimes \wedge^{j_s} (H_s \otimes V) \neq 0 \right\} \\ &= \sum_{i=1}^s r_i \dim K_i. \end{aligned}$$

Suppose that

$$\frac{\dim K'}{\dim H'} \leq \frac{\dim K}{\dim H}$$

for any  $H' \subset H$  and  $K' = H' \otimes V \cap K$ . We see that for all  $1 \leq t \leq s$ ,

$$\sum_{i=1}^t \dim K_i \leq \frac{\dim K}{\dim H} \sum_{i=1}^t \dim H_i.$$

Note that  $\sum_{i=1}^r r_i \dim H_i = 0$ . Then we have

$$\begin{aligned}
-\sum_{i=1}^r r_i \dim K_i &= -r_1 \dim K_1 + \sum_{t=2}^{s-1} r_t \left( \sum_{i=1}^t \dim K_i - \sum_{i=1}^{t-1} \dim K_i \right) \\
&= -\sum_{t=1}^{s-1} [(r_t - r_{t+1}) \sum_{i=1}^t \dim K_i + r_s r] \\
&\geq -\frac{\dim K}{\dim H} \left\{ \sum_{t=1}^{s-1} [(r_t - r_{t+1}) \sum_{i=1}^t \dim H_i + r_s m] \right\} \\
&= -\frac{\dim K}{\dim H} \sum_{i=1}^r r_i \dim K_i = 0.
\end{aligned}$$

Therefore  $K$  is semi-stable.

Conversely, let  $H_1 \subset H$  be a subspace and  $H_2$  its complement, i.e.  $H_1 \oplus H_2 = H$ . If  $K$  is semi-stable for all one parameter subgroups of  $SL(H)$ . It is semistable for the one parameter subgroup

$$\begin{pmatrix} t^{n-\dim H_1} I_{H_1} & 0 \\ 0 & t^{n-\dim H_2} I_{H_2} \end{pmatrix}$$

associated to the direct sum  $H_1 \oplus H_2$ , where  $I_{H_1}$  and  $I_{H_2}$  are the identity matrix over  $H_1$  and  $H_2$  respectively. Let  $K_1 = H_1 \otimes V \cap K$ . Then we have

$$\begin{aligned}
\mu(\lambda, K) &= -(n - \dim H_1) \dim K_1 - (n - \dim H_2) \dim K_2 \\
&= -(n - \dim H_1) \dim K_1 - (\dim H_1) \dim K_2 \\
&= m \dim H_1 - n \dim K_1 \geq 0.
\end{aligned}$$

Therefore,  $m \dim H_1 \geq n \dim K_1$ , equivalently,  $\frac{\dim K_1}{\dim H_1} \leq \frac{\dim K}{\dim H}$ .

The equivalence in stability case is clear by replacing the inequalities by strict inequalities.  $\square$