SZEMERÉDI'S THEOREM FOR STABLE SETS

GABRIEL CONANT AND FRANK WAGNER

In [1, Theorem 8.8], it is shown that if $A \subseteq \mathbb{N}$ is definable in a stable expansion of $(\mathbb{Z}, +, 0)$ then A has upper Banach density 0. Question 8.6 of [1] asks if one can strengthen this conclusion to: A does not contain arbitrarily large finite arithmetic progressions. This question was later posed at the 2017 AIM meeting, Nonstandard Methods in Combinatorial Number Theory, and a proof was suggested by the second author using his results on "substable groups" from [5]. In subsequent conversations, we adapted the method to the "local case" (i.e., when only assuming $x + y \in A$ is a stable formula), and also reformulated the result as a characterization, in terms of arithmetic progressions, of syndetic subsets $A \subseteq \mathbb{Z}$ such that $x + y \in A$ is stable. In particular, we proved the following theorem.

Theorem 1. *Suppose $A \subseteq \mathbb{Z}$ is stable. The following are equivalent.*

(i) *A is syndetic.*
(ii) *A is piecewise syndetic.*
(iii) $\mathrm{BD}(A) > 0$.
(iv) *A contains arbitrarily large finite arithmetic progressions.*

Before getting into the proof, we recall the definitions and make some remarks.

Definition 2. Fix $A \subseteq \mathbb{Z}$.

(1) A is **syndetic** if $\mathbb{Z} = F + A$ for some finite set $F \subseteq \mathbb{Z}$.
(2) A is **piecewise syndetic** if $A \cup X$ is syndetic for some non-sydentic set X.
(3) A is **stable** if, for some $k \geq 1$, there does not exist $a_1, \ldots, a_k, b_1, \ldots, a_k \in \mathbb{Z}$ such that $a_i + b_j \in A$ if and only if $i \leq j$.
(4) The **upper Banach density of** A is

$$\mathrm{BD}(A) = \lim_{n\to\infty} \sup_{m \in \mathbb{Z}} \frac{|A \cap \{m+1, \ldots, m+n\}|}{n}.$$

Remark 3. Note that $A \subseteq \mathbb{Z}$ is stable if and only if the formula $x + y \in A$ is stable with respect to the structure $(\mathbb{Z}, +, A)$. In model theory, syndetic sets are called *generic* and piecewise syndetic sets are called *weakly generic*. It is easy to show that $A \subseteq \mathbb{Z}$ is piecewise syndetic if and only if there is a finite set $F \subseteq \mathbb{Z}$ such that $A + F$ contains arbitrarily large intervals. In particular, if $A \subseteq \mathbb{Z}$ is piecewise syndetic then $\mathrm{BD}(A) > 0$.

The proof of Theorem 1 uses arguments similar to [5], which deals with "substable groups" (i.e. undefinable subgroups of stable groups). We adapt these arguments to the local case of a stable formula.

Proof of Theorem 1. First, note that $(i) \Rightarrow (ii)$ is trivial, $(ii) \Rightarrow (iii)$ follows from Remark 3, and $(iii) \Rightarrow (iv)$ is Szemerédi's Theorem [4] for uniform density (see also Remark 4 below). We show $(iv) \Rightarrow (i)$.

Date: November 1, 2018
Updated: October 24, 2022.

Suppose A contains arbitrarily large finite arithmetic progressions. We work in the theory $T = \mathrm{Th}(\mathbb{Z}, +, A)$, and let $\varphi(x, y) := x + y \in A$, which is stable. Let G^* be a monster model of T. By assumption, the type $\pi(x, y) := \{x + ky \in A : k \in \mathbb{Z}\}$ (in variables x and y) is finitely satisfiable in $(\mathbb{Z}, +, A)$. So we may fix $a, b \in G^*$ such that $\pi(a, b)$ holds. So $\Gamma := b\mathbb{Z}$ is an infinite cyclic subgroup of G^* contained in the translate $B = A(G^*) - a$ of $A(G^*)$.

Let X be the (nonempty) set of types in $S_\varphi(G^*)$, which are finitely satisfiable in Γ. Then X is closed set in $S_\varphi(G^*)$ and hence a compact Hausdorff space. By stability of $\varphi(x, y)$, we may let S be the nonempty finite set of types in X of maximal Cantor-Bendixson rank.

Fix $p \in S$ and note that $B \in p$ since p is finitely satisfiable in Γ. Let $H = \mathrm{Stab}_\varphi(p)$. Since φ is stable, p is definable, and so H is definable. Any translate of p by an element of Γ is still finitely satisfiable in Γ (and of maximal rank), and so $H \cap \Gamma$ has finite index in Γ since S is finite. In particular, H is nontrivial. Since $H = \mathrm{Stab}_\varphi(p)$, we have $B - h \in p$ for all $h \in H$, and so:

(†) for any finite $K \subseteq H$ there is some $g \in G^*$ such that $g + K \subseteq B$.

For a contradiction, suppose H is not covered by finitely many translates of B. We inductively build sequences $(g_i)_{i=0}^\infty$ from G^* and $(h_i)_{i=0}^\infty$ from H such that $g_i + h_j \in B$ if and only if $i > j$, which contradicts stability of $x + y \in B$. Fix $n \in \mathbb{N}$ and suppose we have constructed $(g_i)_{i<n}$ and $(h_i)_{i<n}$ as desired. By (†), there is $g_n \in G^*$ such that $g_n + h_i \in B$ for all $i < n$. By our assumption, H is not contained in $\bigcup_{i \le n} \text{-}g_i + B$, and so there is $h_n \in H$ such that $g_i + h_n \notin B$ for all $i \le n$.

Altogether, we have built a nontrivial definable subgroup H of G^* covered by finitely many translates of B, and hence covered by finitely many translates of $A(G^*)$. This fact can be expressed as a first-order formula (without extra parameters), and so finitely many translates of A cover a nontrivial subgroup of $\mathbb{Z}$. Since any nontrivial subgroup of $\mathbb{Z}$ is syndetic, it follows that A is syndetic. □

Remark 4. In the previous proof, one can use more local stability to replace the appeal to Szemerédi's Theorem with Van der Waerden's Theorem, which is much easier to prove. In particular, fix $A \subseteq \mathbb{Z}$ and assume A is stable and $\mathrm{BD}(A) > 0$. Then there is a sequence $(I_n)_{n=0}^\infty$ of intervals in $\mathbb{Z}$ such that $|I_n| \to \infty$ and $\lim_{n\to\infty} |A \cap I_n|/|I_n| = \mathrm{BD}(A)$. Let μ be the left-invariant finitely-additive probability measure on $\mathbb{Z}$ given by $\lim_{\mathcal{U}} \mu_n$, where $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ and μ_n is the normalized counting measure on I_n (i.e. $\mu_n(X) = |X \cap I_n|/|I_n|$). Then $\mu(A) = \mathrm{BD}(A) > 0$. By stability of A, it follows that A is syndetic (see [2, Theorem 2.3]). So A contains arbitrarily large finite arithmetic progressions by Van der Waerden's Theorem.

References

[1] G. Conant, *Stability and sparsity in sets of natural numbers*, Israel J. Math. **230** (2019), no. 1, 471–508.
[2] G. Conant, A. Pillay, and C. Terry, *A group version of stable regularity*, Math. Proc. Cambridge Philos. Soc. **168** (2020), no. 2, 405–413.
[3] Bruno Poizat, *A course in model theory*, Universitext, Springer-Verlag, New York, 2000, An introduction to contemporary mathematical logic, Translated from the French by Moses Klein and revised by the author.
[4] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245, Collection of articles in memory of Juriĭ Vladimirovič Linnik.
[5] Frank Wagner, *Subgroups of stable groups*, J. Symbolic Logic **55** (1990), no. 1, 151–156.