FORKING AND DIVIDING IN CONTINUOUS LOGIC

GABRIEL CONANT & CAROLINE TERRY

Abstract. We investigate an open question concerning properties of algebraic independence in continuous theories (see Section 4). The rest of the work is essentially a translation to continuous logic of popular notions of independence (in particular, forking and dividing). Much of the time we are simply "copying" classical proofs from well-known sources, while along the way making the necessary adjustments for continuous languages.

1. Forking and Dividing

We assume that the reader is familiar with the construction of continuous languages and theories of metric structures. A full introduction to these concepts can be found in [2].

We begin with the usual definition of dividing, which does not need to be altered to work in the continuous setting.

Definition 1.1. *A partial type $\pi(x,b)$ **divides over** C if there is a C-indiscernible sequence $(b_i)_{i<\omega}$, with $b_0 \equiv_C b$, such that*

$$\bigcup_{i<\omega} \pi(x,b_i) := \bigcup_{i<\omega} \{\varphi(x,b_i) = 0 : \text{“}\varphi(x,b) = 0\text{”} \in p(x,b)\}$$

*is unsatisfiable. A formula $\varphi(x,b)$ **divides over** C if $\{\varphi(x,b) = 0\}$ divides over C.*

Basic familiar properties of dividing from classical logic can now be proved with little difficulty.

Proposition 1.2. *If $\varphi(x,b)$ divides over C and $b' \equiv_C b$ then $\varphi(x,b')$ divides over C.*

Proof. Let $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$ such that $\sigma(b) = b'$. Let $(b_i)_{i<\omega}$ be C-indiscernible, with $b_0 = b$, such that $\{\varphi(x,b_i) = 0 : i < \omega\}$ is inconsistent. Then $(\sigma(b_i))_{i<\omega}$ is C-indiscernible and $\{\varphi(x,\sigma(b_i)) = 0 : i < \omega\}$ is unsatisfiable, witnessing that $\varphi(x,b')$ divides over C. □

Proposition 1.3. *Suppose $\pi(x,b)$ divides over C. Then there are formulas $\varphi_1(x,b), \ldots \varphi_k(x,b)$ such that "$\varphi_j(x,b) = 0$" $\in \pi(x,b)$ for all j, and $\max_{1\leq j\leq k} \varphi_j(x,b)$ divides over C.*

Date: March 2, 2013. *Revised:* October 18, 2021 & November 2, 2023.

Proof. Suppose $\pi(x,b)$ divides over C, witnessed by the C-indiscernible sequence $(b_i)_{i<\omega}$. By compactness there are $\varphi_1(x,b), \ldots \varphi_k(x,b)$ such that "$\varphi_i(x,b)=0$" $\in \pi(x,b)$ for all i, and

$$\{\varphi_j(x,b_i)=0 : i<\omega,\ 1\leq j\leq k\}$$

is unsatisfiable. This implies

$$\{\max_{1\leq j\leq k}\varphi_j(x,b_i)=0 : i<\omega\}$$

is inconsistent, and so $(b_i)_{i<\omega}$ witnesses that $\max_{1\leq j\leq k}\varphi_j(x,b)$ divides over C. $\square$

Corollary 1.4. *A complete type $p\in S(B)$ divides over C if and only if there is some formula $\varphi(x,b)$ such that $\varphi(x,b)=0\in p$ and $\varphi(x,b)$ divides over C.*

Proof. The reverse direction is obvious. Conversely, if $\varphi_1(x,b)=0,\ldots,\varphi_k(x,b)=0\in p$ and p is a complete type then $\max_{1\leq j\leq k}\varphi_j(x,b)=0\in p$, so the result follows from Proposition 1.3. $\square$

Proposition 1.5. *Suppose p is C-invariant. Then p does not divide over C.*

Proof. Fix a formula $\varphi(x,b)$ such that "$\varphi(x,b)=0$" $\in p$. Let $(b_i)_{i<\omega}$ be a C-indiscernible sequence with $b_0=b$. For $i<\omega$ there is some $\sigma_i\in\mathrm{Aut}(\mathbb{M}/C)$ such that $\sigma_i(b)=b_i$. By C-invariance, "$\varphi(x,b_i)=0\in p$. Therefore $\{\varphi(x,b_i)=0 : i<\omega\}\subseteq p$ is satisfiable. $\square$

The natural thing to do next is define forking. Many sources use the "implies a disjunction of dividing formulas" definition, which we thought might be too syntactical and/or dependent on classical logic. So instead, we use the definition illustrating the motivation for forking independence as the natural attempt to "force" the extension axiom on dividing independence (see [1, Section 3] and Proposition 1.12).

Definition 1.6. *A partial type $\pi(x,b)$ **forks over** C if there is some $D\supseteq Cb$ such that any extension of $p(x,b)$ to a complete type over D divides over C.*

Proposition 1.7. *Let $C\subseteq B$ and $\pi(x)$ a partial type over B. If π does not fork over C then it can be extended to a complete type $p(x)\in S(B)$, which does not fork over C.*

Proof. Suppose $D\supseteq B$ witnesses that π does not fork over C. Let Σ be the collection of consistent partial types over B, which extend π and do not fork over C. Then $\pi\in\Sigma$, and we show that Σ satisfies the increasing chain condition of Zorn's Lemma. Indeed, suppose $\pi_0\subseteq\pi_1\subseteq\ldots$ are elements of Σ. Let $\pi'=\bigcup_{i<\omega}\pi_i$. Then π' is a consistent partial type over B extending π. It follows that D witnesses that π' does not fork over C, so $\pi'\in\Sigma$.

Let $p(x)$ be a maximal element of Σ. Then p does not fork over C, so we have left to show that p is complete. So let $\varphi(x)\in\mathcal{L}(B)$and suppose, towards a contradiction, that "$\varphi(x)=r$" $\notin p$ for all $r\in[0,1]$. By maximality of p, $p_r:=p\cup\{\varphi(x)=r\}$ forks over A for all $r\in[0,1]$. Let $D_r\supseteq B$ witness that p_r forks over

C, and set $D' = \bigcup_{r\in\mathbb{R}} D_r$. Suppse p^* is a complete extension of p to D'. Then there is some $r \in \mathbb{R}$ such that "$\varphi(x) = r \in p^*$. It follows that p^* is a complete extension of p_r to $D' \supseteq D_r$. By assumption $p^*|_{D_r}$ divides over C, and so p^* divides over C. Altogether, we have shown D' witnesses that p forks over C, which is a contradiction. □

Given a formula $\varphi(x)$ and $\epsilon \geq 0$, we use $\varphi(x) \leq \epsilon$ to denote the condition $\varphi(x) \dot{-} \epsilon = 0$. If $\pi(x)$ is a type and $\varphi(x)$ is a formula then we use the notation

$$\pi(x) \vdash \varphi(x) = 0$$

to mean that $\varphi(a) = 0$ for any realization $a \models \pi(x)$. Note that by compactness, if $\pi(x,b) \vdash \varphi(x) = 0$ then for all $\epsilon > 0$ there is a finite subset $\pi_0(x,b) \subseteq \pi(x,b)$ such that $\pi_0(x,b) \vdash \varphi(x) \leq \epsilon$.

Lemma 1.8. *If $\varphi(x,b)$ divides over C then there is some $\epsilon > 0$ such that $\varphi(x,b) \leq \epsilon$ divides over C.*

Proof. Suppose $\varphi(x,b)$ divides over C, as witnessed by a C-indiscernible sequence $(b_i)_{i<\omega}$. Then

$$\{\varphi(x,b_i) \leq \tfrac{1}{n} : i < \omega,\ n > 0\}$$

is unsatisfiable. By compactness there is some $N > 0$ such that

$$\{\varphi(x,b_i) \leq \tfrac{1}{n} : i < \omega,\ 0 < n \leq N\}$$

is unsatisfiable. Therefore $(b_i)_{i<\omega}$ witnesses that $\varphi(x,b) \leq \frac{1}{N}$ divides over C. □

We can now show that in continuous logic, the analog of forking as "implying a disjunction of dividing formulas" still works.

Theorem 1.9. *The following are equivalent:*

(i) $\pi(x,b)$ forks over C;

(ii) There are formulas $\varphi_1(x), \ldots, \varphi_k(x)$ such that $\varphi_j(x)$ divides over C for all j, and

$$\pi(x,b) \vdash \min_{1\leq j\leq k} \varphi_j(x) = 0.$$

Proof. First suppose $\pi(x,b)$ forks over C, witnessed by $D \supseteq Cb$. Define

$$\Sigma = \{\psi(x) \in \mathcal{L}(D) :\ \psi(x) \text{ divides over } C\}.$$

By Lemma 1.8, for each $\psi \in \Sigma$ we can find $\epsilon_\psi > 0$ such that $\psi(x) \leq \epsilon_\psi$ divides over C. Define

$$p_0(x) := \pi(x,b) \cup \{\psi(x) \geq \epsilon_\psi : \psi \in \Sigma\},$$

which is a (possibly inconsistent) type over D. Suppose, towards a contradiction, that $p_0(x)$ is satisfiable. Then we can extend $p_0(x)$ to $p(x) \in S(D)$. By assumption $p(x)$ divides over C. By Corollary 1.4, let $\psi(x) \in p(x)$ divide over C (and so $\psi(x) \in \Sigma$). If a realizes p then $\psi(a) = 0$, but "$\psi(x) \geq \epsilon_\psi$" $\in p_0(x) \subseteq p(x)$, which is a contradiction.

Therefore $p_0(x)$ is unsatisfiable. By compactness there are $\psi_1(x),\ldots,\psi_k(x)$ and $\epsilon_j := \epsilon_{\psi_j}$ such that $\psi_j(x)\leq\epsilon_j$ divides over C and

$$\pi(x,b)\cup\{\psi_j(x)\geq\epsilon_j : 1\leq j\leq k\}$$

is unsatisfiable. If $\varphi_j(x) := \psi_j(x)\dot{-}\epsilon_j$, then we have

$$\pi(x,b)\vdash\min_{1\leq j\leq k}\varphi_j(x)=0.$$

Conversely, suppose we have $\varphi_1(x),\ldots,\varphi_k(x)$ satisfying condition (ii) of the theorem. Let D be the union of Cb together with all parameters from the $\varphi_j(x)$, for $1\leq j\leq k$. Suppose $p(x)\in S(D)$ is an extension of $\pi(x,b)$ and let $a\models p$. Then $a\models\pi(x,b)$ so, by assumption, $\varphi_j(a)=0$ for some j. But then "$\varphi_j(x)=0$" $\in\operatorname{tp}(a/D)=p$, which means p divides over C. Therefore $\pi(x,b)$ forks over C. □

As with dividing, we can now prove basic results about forking with little to no change from their classical counterparts. Many of these are exercises in [5, Chapter 7].

Exercise 1.10. *If $\pi(x)$ is finitely satisfiable in C then $\pi(x)$ does not fork over C.*

Proof. If $\pi(x,b)$ forks over C then let

$$\pi(x,b)\vdash\min_{1\leq j\leq k}\varphi_j(x,b)=0$$

with $\varphi_j(x,b)$ dividing over C for all $1\leq j\leq k$. By Lemma 1.8, there is some $\epsilon>0$ such that $\varphi_j(x,b)\leq\epsilon$ divides over C for all $1\leq j\leq k$. Let $\pi_0(x,b)\subseteq\pi(x,b)$ be finite such that

$$\pi_0(x,b)\vdash\min_{1\leq j\leq k}\varphi_j(x,b)\leq\epsilon.$$

Since $\pi(x,b)$ is finitely satisfiable in C, it follows that there is $a\in C$ and some $1\leq j\leq k$ such that $\varphi_j(a,b)\leq\epsilon$. If $(b_i)_{i<\omega}$ is C-indiscernible, with $b_0\equiv_C b$, then a satisfies $\{\varphi_j(a,b_i)\leq\epsilon : i<\omega\}$. Altogether, it follows that $\varphi_j(x,b)\leq\epsilon$ does not divide over C, which is a contradiction. □

Proposition 1.11. *Suppose M is a model and $C\subseteq M$ such that M is $|C|^+$-saturated. Then for $p\in S(M)$, if p forks over C then p divides over C.*

Proof. Suppose $p\vdash\min_{1\leq j\leq k}\varphi_j(x,b)=0$, with $\varphi_j(x,b)$ dividing over C for all $1\leq j\leq k$. By Lemma 1.8 there is $\epsilon>0$ such that $\varphi_j(x,b)\leq\epsilon$ divides over C for all $1\leq j\leq k$. Let p_0 be finite such that $p_0\vdash\min_{1\leq j\leq k}\varphi_j(x,b)\leq\epsilon$. Let m be the finite tuple of parameters mentioned in p_0. By assumption, there is some $d\in M$ realizing $\operatorname{tp}(b/Cm)$. Then $p\vdash p_0\vdash\min_{1\leq j\leq k}\varphi_j(x,d)\leq\epsilon$, which implies there is some j such that "$\varphi_j(x,d)\leq\epsilon$" $\in p$. But $d\equiv_A b$ implies $\varphi_j(x,d)\leq\epsilon$ divides over C, and so p divides over C. □

Proposition 1.12. *Non-forking satisfies extension, i.e. if* $\operatorname{tp}(a/BC)$ *does not fork over* C *and* $B' \supseteq B$ *then there is some* $a' \equiv_{BC} a$ *such that* $\operatorname{tp}(a'/B'C)$ *does not fork over* C.

Proof. Let M be a $|C|^+$-saturated model containing $B'C$. Since $p := \operatorname{tp}(a/BC)$ does not fork over C, there is some $q \in S(M)$ extending p such that q does not divide over C. By Proposition 1.11, q does not fork over C, and so $q|_{B'C}$ does not fork over C. If $a' \models q|_{B'C}$, then $a' \equiv_{BC} a$ and $\operatorname{tp}(a'/B'C) = q|_{B'C}$ does not fork over C. □

2. Erdös-Rado Results and Indiscernibles

In this section, we provide proofs of two fundamental theorems about indiscernibles. These are used frequently in classical model theory, and will be just as useful in the continuous setting. The main results are Theorem 2.3 and Theorem 2.4. We include complete proofs for continuous logic, but they are only slightly modified from the classical proofs. Both can be found in [5] (Lemma 5.1.3 and Lemma 7.2.12, respectively).

Definition 2.1. *Given a sequence* $\mathcal{I} = (a_i)_{i\in I}$ *and a set* A, *the* ***EM-type*** *of* $\mathcal{I}$ *over* A, *denoted* $EM(\mathcal{I}/A)$, *is the following partial type in the variables* $(x_n)_{n<\omega}$ *over* A

$$EM(\mathcal{I}/A) = \{\varphi(x_1,\dots,x_n) = 0 : \varphi(\bar{x}) \in \mathcal{L}(A),\ \forall\ i_1 < \dots < i_n,\ \varphi(a_{i_1},\dots,a_{i_n}) = 0\}.$$

Lemma 2.2. *Given a sequence* $(a_i)_{i\in I}$, *a set* A, *and a linear order* J, *there is a sequence* $(b_j)_{j\in J}$ *such that* $EM((a_i)_{i\in I}/A) \subseteq EM((b_j)_{j\in J}/A)$.

Proof. We want to satisfy

$$\Delta = \{\varphi(x_{j_1},\dots,x_{j_n}) = 0 : \text{“}\varphi(x_1,\dots,x_n) = 0\text{”} \in EM((a_i)_{i\in I}/A),\ j_1 < \dots < j_n \in J\}$$

Given a finite subset $\Delta_0 \subseteq \Delta$, let $j_1 < \dots < j_k \in J$ be such that $(x_{j_r})_{r=1}^k$ are all the variables occuring in the formulas in Δ_0. Fix $i_1 < \dots < i_k$ from I. If $\varphi(x_{j_{r_1}},\dots,x_{j_{r_n}}) \in \Delta_0$ then $1 \le r_l \le k$ for all $1 \le l \le n$ and $\varphi(x_1,\dots,x_n) \in EM((a_i)_{i\in I}/A)$, so by assumption, $\varphi(a_{i_{r_1}},\dots,a_{i_{r_n}}) = 0$. Therefore $(a_{i_r})_{r=1}^k$ satisfies Δ_0. By compactness, Δ is satisfiable, and if $(b_j)_{j\in J}$ is a realization, then $EM((a_i)_{i\in I}/A) \subseteq EM((b_j)_{j\in J}/A)$. □

Theorem 2.3. *Given a sequence* $(a_i)_{i\in I}$, *a set* A, *and a linear order* J, *there is an* A*-indiscernible sequence* $(b_j)_{j\in J}$ *realizing* $EM((a_i)_{i\in I}/A)$.

Proof. First, given a formula $\varphi(x_1,\dots,x_n)$, we define the following connective:

$$u_\varphi(x_1,\dots,x_n,y_1,\dots,y_n) := \max(\varphi(x_1,\dots,x_n) \dot{-} \varphi(y_1,\dots,y_n), \varphi(y_1,\dots,y_n) \dot{-} \varphi(x_1,\dots,x_n)).$$

So $u_\varphi(\bar{x},\bar{y})=0$ if and only if $\varphi(\bar{x})=\varphi(\bar{y})$.

Let $\Delta(A)$ be the elementary diagram of A. Let $\mathcal{L}^*=\mathcal{L}\cup A\cup\{b_j : b\in J\}$ and define the sets

$$\Gamma_1=\{u_\varphi(b_{j_1},\ldots,b_{j_n},b_{k_1},\ldots,b_{k_n})=0 : \varphi(\bar{x})\in\mathcal{L}_A,\ j_1<\ldots<j_n \text{ and } k_1<\ldots<k_n \text{ in } J\} \text{ and}$$

$$\Gamma_2=\{\varphi(b_{j_1},\ldots,b_{j_n})=0 : \text{“}\varphi(x_1,\ldots,x_n)=0\text{”}\in EM((a_i)_{i\in I}/A),\ j_1<\ldots<j_n \text{ in } J\}.$$

Set $\Sigma=T\cup\Delta(A)\cup\Gamma_1\cup\Gamma_2$. Suppose $\Sigma_0\subseteq\Sigma$ is finite. Let $\varphi_1(\bar{x}),\ldots,\varphi_m(\bar{x})$ be the formulas in $\Sigma_0\cap\Gamma_1$, and suppose $x_1,\ldots,x_n$ are the free variables occurring in the $\varphi_k(\bar{x})$. Set $\kappa=\beth_n(2^{\aleph_0})^+$. By Lemma 2.2, we may replace I by an index set of size κ. Define $F:[I]^n\longrightarrow[0,1]^m$ such that if $i_1<\ldots<i_n$ then

$$F(\{i_1,\ldots,i_n\})=(\varphi_1(x_{i_1},\ldots,x_{i_n}),\ldots,\varphi_m(x_{i_1},\ldots,x_{i_n})).$$

By the Erdös-Rado Theorem, there is an infinite subset $Y\subseteq I$ that is homogeneous for F; say $\eta\in[0,1]^m$ such that $F(I_0)=\eta$ for all $I_0\in[Y]^n$.

Let J_0 be the finite set of $j\in J$ such that b_j appears in a sentence of Σ_0. Let $\{i_j : j\in J_0\}\subseteq Y$ such that $i_j<i_k$ if $j<k$. Then if $j_1<\ldots<j_n$ and $k_1<\ldots<k_n$ are in J_0, we have

$$\varphi_k(a_{i_{j_1}},\ldots,a_{i_{j_n}})=r\in[0,1] \quad\Leftrightarrow\quad \eta(k)=r \quad\Leftrightarrow\quad \varphi(a_{i_{k_1}},\ldots,a_{i_{k_n}})=r.$$

Therefore if we interpret b_j as a_{i_j} for $j\in J_0$, we have $u_\varphi(b_{j_1},\ldots,b_{j_n},b_{k_1},\ldots,b_{k_n})=0$ for all $j_1<\ldots<j_n$ and $k_1<\ldots<k_n$ in J. It follows that $\mathbb{M}\models\Sigma_0\cap(T\cup\Delta(A)\cup\Gamma_1)$. If $\varphi(x_1,\ldots,x_l)\in\mathcal{L}_A$ and $\varphi(b_{j_1},\ldots,b_{j_l})\in\Sigma_0\cap\Gamma_2$, then $j_1<\ldots<j_l$ are in J_0 and $\mathbb{M}\models\varphi(a_{i_{j_1}},\ldots,a_{i_{j_l}})$. Therefore with these interpretations, $\mathbb{M}\models\Sigma_0$.

By compactness, there is $(b_j)_{j\in J}$ satisfying Σ. The sentences in Γ_1 ensure that $(b_j)_{j\in J}$ is indiscernible over A, while the sentences in Γ_2 ensure that $(b_j)_{j\in J}$ satisfies $EM((a_i)_{i\in I}/A)$. $\square$

Theorem 2.4. *For any $A\subset\mathbb{M}$, there is some λ such that for any linear order I of cardinality λ and any sequence $(a_i)_{i\in I}$, there is a sequence $(b_i)_{i<\omega}$, indiscernible over A, such that for all $j_1<\ldots<j_n<\omega$ there are $i_1<\ldots<i_n$ in I with $a_{i_1}\ldots a_{i_n}\equiv_A b_{j_1}\ldots b_{j_n}$.*

Proof. Let $\tau=\sup_{n<\omega}|S_n(A)|$. Let $\lambda=\beth_{\tau^+}$. By the Erdös-Rado Theorem we have

(i) $\mathrm{cof}(\lambda)>\tau$ and

(ii) for all $\kappa<\lambda$ and all $n<\omega$ there is some $\kappa'<\lambda$ with $\kappa'\to(\kappa)^n_\tau$.

Given $\kappa<\lambda$, since $\lambda\to(\kappa)^1_\tau$ and by considering $a_i\mapsto\mathrm{tp}(a_i/A)\in S_1(A)$, there is some $I_0\subseteq I$, with $|I_0|=\kappa$, and $p_1^\kappa(x_1)\in S_1(A)$ such that $\mathrm{tp}(a_i/A)=p_1^\kappa(x_1)$ for all $i\in I_0$. This gives a map from λ to $S_1(A)$ such that $\kappa\mapsto p_1^\kappa(x_1)$. Since $\mathrm{cof}(\lambda)>\tau$, it follows that there is some $p_1(x_1)$ such that $p_1(x_1)=p_1^\kappa(x_1)$ for cofinally many κ. Therefore we have $p_1(x_1)$ such that for all $\kappa<\lambda$ there is some $I_0\subseteq I$, with $|I_0|=\kappa$, such that $\mathrm{tp}(a_i/A)=p_1(x_1)$ for all $i\in I_0$.

Now suppose we have $p_1(x_1)\subseteq p_2(x_1,x_2)\subseteq\ldots\subseteq p_n(x_1,\ldots,x_n)$ in $S(A)$ such that for all $\kappa<\lambda$ there is some $I_0 \subseteq I$, with $|I_0| = \kappa$, such that $\text{tp}(a_{i_1}, \ldots, a_{i_n}/A) = p_n(x_1, \ldots, x_n)$ for all $i_1 < \ldots < i_n$ in I_0.

Given $\kappa < \lambda$, let $\kappa' < \lambda$ be such that $\kappa' \to (\kappa)^{n+1}_\tau$. Let $I' \subseteq I$, with $|I'| = \kappa'$ such that $\text{tp}(a_{i_1}, \ldots, a_{i_n}/A) = p_1(x_1, \ldots, x_n)$ for all $i_1 < \ldots < i_n$ in I'. Consider the map from $[I']^{n+1}$ to $S_{n+1}(A)$ where $i_1 < \ldots < i_{n+1} \mapsto \text{tp}(a_{i_1}, \ldots, a_{i_{n+1}}/A)$. By assumption on κ', there is $I_0 \subseteq I'$, with $|I_0| = n+1$, and $p^\kappa_{n+1}(x_1, \ldots, x_{n+1}) \in S_{n+1}(A)$ such that $\text{tp}(a_{i_1}, \ldots, a_{i_{n+1}}/A) = p^\kappa_{n+1}(x_1, \ldots, x_{n+1})$ for all $i_1 < \ldots < i_{n+1}$ in I_0. Note that $p_n(x_1, \ldots, x_n) \subseteq p^\kappa_{n+1}(x_1, \ldots, x_{n+1})$. Again, $\kappa \mapsto p^\kappa_{n+1}$ gives a map from λ to $S_{n+1}(A)$ and so there is $p_{n+1}(x_1, \ldots, x_{n+1})$ such that $p^\kappa_{n+1} = p_{n+1}$ for cofinally many κ. It follows that for all $\kappa < \lambda$ there is $I_0 \subseteq I$, with $|I_0| = \kappa$, such that $\text{tp}(a_{i_1}, \ldots, a_{i_{n+1}}/A) = p_{n+1}(x_1, \ldots, x_{n+1})$ for all $i_1 < \ldots < i_{n+1}$ in I_0.

Since $p_1 \subseteq p_2 \subseteq \ldots$, there is a realization $(b_i)_{i<\omega}$ of $\bigcup_{i<\omega} p_i$. Suppose $\varphi(v_1, \ldots, v_n) \in \mathcal{L}_A$ and $\mathbb{M} \models \varphi(b_0, \ldots, b_{n-1}) = 0$. If $j_1 < \ldots < j_n$ are in ω then $p_n \subseteq p_{j_n}$. Suppose, towards a contradiction, that $\mathbb{M} \models \varphi(b_{j_1}, \ldots, b_{j_n}) = r$ for some $r > 0$. By assumption, there is $I_0 \subseteq I$ infinite such that $\text{tp}(a_{i_1}, \ldots, a_{i_{j_n}}/A) = p_{j_n}$ for all $i_1 < \ldots < i_{j_n}$ in I_0. But "$\max(\varphi(x_1, \ldots, x_n), \varphi(x_{j_1}, \ldots, x_{j_n}) \dot{-} r) = 0$" $\in p_{j_n}$. So if we set $N = j_n - n$ and pick $i_1 < \ldots < i_{j_n+N}$ in I_0 then we have

$$\varphi(a_{i_1}, \ldots, a_{i_n}) = 0,\ \varphi(a_{i_{j_1}}, \ldots, a_{i_{j_n}}) = r \ \text{ and } \ \varphi(a_{i_{j_1}}, \ldots, a_{i_{j_n}}) = 0,\ \varphi(a_{i_{j_1+N}}, \ldots, a_{i_{j_n+N}}) = r.$$

which is a contradiction. Therefore $\mathbb{M} \models \varphi(b_{j_1}, \ldots, b_{j_n}) = 0$, which implies $(b_i)_{i<\omega}$ is indiscernible over A.

Suppose $j_1 < \ldots < j_n$ are in ω. Then there is an infinite set $I_0 \subseteq I$ such that $\text{tp}(a_{i_1}, \ldots, a_{i_n}/A) = p_n$ for all $i_1 < \ldots < i_n$ in I_0. Therefore, for any $i_1 < \ldots < i_n$ in I_0, we have $a_{i_1} \ldots a_{i_n} \equiv_A b_0 \ldots b_{n-1}$. By A-indiscernibility, it follows that $a_{i_1} \ldots a_{i_n} \equiv_A b_{j_1} \ldots b_{j_n}$. □

3. Dividing and Algebraic Closure

In many texts (e.g. [5]), dividing is defined by k-inconsistency of sequences, rather than inconsistency of indiscernible sequences. We can formulate and prove the analogous result for continuous logic.

Theorem 3.1. *Let $\varphi(x, b)$ be a formula and $C \subset \mathbb{M}$. The following are equivalent:*

(i) *$\varphi(x, b)$ divides over C;*

(ii) *there is an $\epsilon > 0$, an integer $k > 0$, and a sequence $(b_i)_{i<\omega}$, with $b_i \equiv_C b$ for all $i < \omega$, such that $\{\varphi(x, b_i) \leq \epsilon : i < \omega\}$ is k-unsatisfiable.*

Proof. Suppose $\varphi(x, b)$ divides over C, witnessed by the C-indiscernible sequence $(b_i)_{i<\omega}$. Then

$$\{\varphi(x, b_i) \leq \tfrac{1}{n} : i < \omega,\ n > 0\}$$

is unsatisfiable, so by compactness there is some $N > 0$ and some $i_1 < \ldots < i_k < \omega$ such that

$$\{\varphi(x, b_{i_s}) \leq \tfrac{1}{N} : 1 \leq s \leq k\}$$

is unsatisfiable. We claim that for all $j_1 < \ldots < j_k < \omega$, $\{\varphi(x, b_{j_s}) \leq \frac{1}{N} : 1 \leq s \leq k\}$ is unsatisfiable. Indeed, otherwise

$$\models \inf_x \max_{1 \leq s \leq k} \varphi(x, b_{j_s}) \leq \tfrac{1}{N},$$

and so $\models \inf_x \max_{1 \leq s \leq k} \varphi(x, b_{i_s}) \leq \frac{1}{N}$ by C-indiscernibility. A realization of this sentence contradicts that $\{\varphi(x, b_{i_s}) \leq \frac{1}{N} : 1 \leq s \leq k\}$ is unsatisfiable. Altogether we have shown that

$$\{\varphi(x, b_i) \leq \tfrac{1}{N} : i < \omega\}$$

is k-unsatisfiable.

Conversely, suppose we have a sequence $(b_i)_{i<\omega}$, with $b_i \equiv_C b$ for all $i < \omega$, such that for some $\epsilon > 0$ and some integer $k > 0$, $\{\varphi(x, b_i) \leq \epsilon : i < \omega\}$ is k-unsatisfiable. Given $i_1 < \ldots < i_k$, we have that for all $x \in \mathbb{M}$, $\varphi(x, b_{i_s}) > \epsilon$ for some s. Therefore

$$\models \sup_x \min_{1 \leq s \leq k} \epsilon \dot{-} \varphi(x, b_{i_s}) = 0.$$

It follows that "$\sup_x \min_{1 \leq s \leq k} \epsilon \dot{-} \varphi(x, y_s) = 0$" $\in EM((b_i)_{i<\omega}/C)$. Let $(b'_i)_{i<\omega}$ be a C-indiscernible sequence realizing $EM((b_i)_{i<\omega}/A)$. After conjugating by an automorphism, we may assume $b'_0 = b$. By the above,

$$\models \sup_x \min_{1 \leq s \leq k} \epsilon \dot{-} \varphi(x, b'_s) = 0.$$

Then for any $d \in \mathbb{M}$, there must be some $1 \leq s \leq k$ such that $\varphi(d, b'_s) \geq \epsilon$, and so $\{\varphi(x, b'_i) = 0 : i < \omega\}$ is unsatisfiable. Therefore $\varphi(x, b)$ divides over C. $\square$

Lemma 3.2. *Suppose $C \subset \mathbb{M}$ and $a \in \mathbb{M}$. The following are equivalent:*

(i) *$a \in \operatorname{acl}(C)$;*
(ii) *the set of realizations in $\mathbb{M}$ of $\operatorname{tp}(a/C)$ is compact;*
(iii) *the set of realizations in $\mathbb{M}$ of $\operatorname{tp}(a/C)$ has bounded density character;*
(iv) *for all $\epsilon > 0$ there is some $\varphi(x) \in \mathcal{L}(C)$ and $\delta > 0$ such that $\varphi(a) = 0$ and $\{b \in \mathbb{M} : \varphi(b) < \delta\}$ has a finite ϵ-net.*

Proof. This is [2, Exercise 10.8]. $\square$

Definition 3.3. *We define the ternary relations*

$$a \mathop{\smile\!\!\!\!\!\mid}^{f}_{C} B \quad \Leftrightarrow \quad \mathrm{tp}(a/BC) \text{ does not fork over } C$$

$$a \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} B \quad \Leftrightarrow \quad \mathrm{tp}(a/BC) \text{ does not divide over } C$$

$$a \mathop{\smile\!\!\!\!\!\mid}^{a}_{C} B \quad \Leftrightarrow \quad \mathrm{acl}(aC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$$

Note that dividing implies forking for partial types, and so $\mathop{\smile\!\!\!\!\!\mid}^{f} \Rightarrow \mathop{\smile\!\!\!\!\!\mid}^{d}$.

Lemma 3.4. *Suppose $b \in \mathrm{acl}(C)$ and $(b_i)_{i<\omega}$ is a C-indiscernible sequence, with $b_0 = b$. Then $b_i = b$ for all $i < \omega$.*

Proof. The set of realizations of $\mathrm{tp}(b/C)$ is compact. Since $(b_i)_{i<\omega}$ is a sequence of realizations of $\mathrm{tp}(b/C)$, it follows that $(b_i)_{i<\omega}$ contains a convergent subsequence. If there are $i < j < \omega$ such that $d(b_i, b_j) = \epsilon > 0$, then $d(b_i, b_j) = \epsilon$ for all $i < j < \omega$ by indiscernibility. This contradicts the existence of a convergent subsequence. Therefore $d(b_i, b_j) = 0$ for all $i < j < \omega$, and so $b_i = b$ for all $i < \omega$. □

Lemma 3.5. *Suppose $b \notin \mathrm{acl}(C)$. Then there is an $\epsilon > 0$ and a sequence $(b_i)_{i<\omega}$ such that $b_i \equiv_C b$ for all $i < \omega$ and $d(b_i, b_j) \geq \epsilon$ for all $i < j < \omega$.*

Proof. We will prove the contrapositive. Let $X = \{b' : b' \equiv_C b\}$. Suppose that for all $\epsilon > 0$ and all sequences $(b_i)_{i<\omega}$ from X, there are $i < j < \omega$ such that $d(b_i, b_j) < \epsilon$. Fix $\epsilon > 0$ and let $b_0 \in X$ be arbitrary. Suppose we have chosen $b_0, \ldots, b_{n-1} \in X$ such that for all $i \neq j$, $d(b_i, b_j) \geq \epsilon$. Let $Y = X \cap \bigcup_{i<n} B_\epsilon(b_i)$. If $X \backslash Y$ is nonempty, pick $b_n \in X \backslash Y$. If this process continues infinitely then we have a sequence $(b_i)_{i<\omega}$ such that $d(b_i, b_j) \geq \epsilon$ for all $i < j < \omega$, which is a contradiction. Therefore there is some n such that $X \subseteq \bigcup_{i<n} B_\epsilon(b_i)$.

We have shown that X is totally bounded. Since X is closed, it follows that X is compact, and so $b \in \mathrm{acl}(C)$. □

Proposition 3.6. *For any tuple $a \in \mathbb{M}$ and $C \subset \mathbb{M}$, $a \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} a$ if and only if $a \in \mathrm{acl}(C)$.*

Proof. Suppose $a \notin \mathrm{acl}(C)$. By Lemma 3.5, there is a sequence $(a_i)_{i<\omega}$ of realizations of $\mathrm{tp}(a/C)$ and an $\epsilon > 0$ such that $d(a_i, a_j) \geq \epsilon$ for all $i < j < \omega$. Then "$d(x, a) \leq \frac{\epsilon}{3}$" $\in \mathrm{tp}(a/Ca)$ and $\{d(x, a_i) \leq \frac{\epsilon}{3} : i < \omega\}$ is 2-unsatisfiable. By Theorem 3.1, $\mathrm{tp}(a/Ca)$ divides over C.

Conversely suppose $a \in \mathrm{acl}(C)$ and $\varphi(x, y)$ is an $\mathcal{L}(C)$-formula with "$\varphi(x, a) = 0$" $\in \mathrm{tp}(a/Ca)$. Suppose $(a_i)_{i<\omega}$ is indiscernible over C, with $a_0 = a$. By Lemma 3.4, $a_i = a$ for all $i < \omega$, and so $\{\varphi(x, a_i) = 0 : i < \omega\}$ is satisfied by a. Therefore $\varphi(x, a)$ does not divide over C, which implies $a \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} a$. □

The proof of the following theorem is adapted from that of [3, Proposition 3.9].

Theorem 3.7. $a \mathop{\smile\!\!\!\!\!\mid}^{d}_{C} \mathrm{acl}(BC) \;\Rightarrow\; a \mathop{\smile\!\!\!\!\!\mid}^{a}_{C} B$

Proof. Suppose $a \not\perp^a_C B$. Then $(\operatorname{acl}(aC) \cap \operatorname{acl}(BC)) \backslash \operatorname{acl}(C) \neq \emptyset$, so fix $b \in \operatorname{acl}(BC)$ such that $b \in \operatorname{acl}(aC) \backslash \operatorname{acl}(C)$. Let $p(x, y) = \operatorname{tp}(a, b/C)$ and $q(y) = \operatorname{tp}(b/C)$. Suppose, towards a contradiction, that for all formulas $\varphi(x, y)$, with "$\varphi(x, b) = 0$" $\in p(x, b)$, $\varphi(x, b)$ does not divide over C.

Since $b \notin \operatorname{acl}(C)$, we can fix $\epsilon > 0$ and $(b_i)_{i<\omega}$ as in Lemma 3.5. Then this sequence satisfies the type

$$\Gamma(y_i)_{i<\omega} := \bigcup_{i<\omega} q(y_i) \ \cup \ \{d(y_i, y_j) \geq \epsilon : i < j < \omega\}.$$

Claim: $\bigcup_{i<\omega} p(x, y_i) \ \cup \ \Gamma(y_i)_{i<\omega}$ is satisfiable.

Proof: By compactness, it suffices to satisfy

$$\{\max_{i \leq n} \varphi(x, y_i) = 0\} \ \cup \ \Gamma(y_i)_{i<\omega},$$

for $n < \omega$ and any formula $\varphi(x, y)$ such that "$\varphi(x, b) = 0$" $\in p(x, b)$. But $\varphi(x, b)$ does not divide over C by assumption. So by Theorem 3.1, it follows that $\{\varphi(x, b_i) \leq \epsilon : i < \omega\}$ is n-satisfiable.$_{/\!/}$

Let $(a', (b'_i)_{i<\omega})$ realize $\bigcup_{i<\omega} p(x, y_i) \ \cup \ \Gamma(y_i)_{i<\omega}$. In particular $(a', b'_0) \equiv_C (a, b)$, so by conjugating by an automorphism over C, we may assume $a' = a$. But then $b'_i \models \operatorname{tp}(b/aC)$ for all $i < \omega$. Since $(b'_i)_{i<\omega}$ does not contain a convergent subsequence, it follows that the set of realizations of $\operatorname{tp}(b/aC)$ is not compact. This contradicts that $b \in \operatorname{acl}(aC)$.

From this contradiction, we conclude that $\operatorname{tp}(a/bC)$ divides over C. Since $b \in \operatorname{acl}(BC)$, it follows that $a \not\perp^d_C \operatorname{acl}(BC)$. □

4. An Aside on Independence Relations

The study of abstract independence relations in first-order theories has become a popular combinatorial way to understand complexity. These independence relations usually take the form of a ternary relation $\perp$, together with various axioms of "good behavior". We list a few examples of such axioms, organized into categories which will be explained shortly.

Category 1

(automorphism invariance) If $A \perp_C B$ and $\sigma \in \operatorname{Aut}(\mathbb{M})$ then $\sigma(A) \perp_{\sigma(C)} \sigma(B)$.
(monotonicity) If $D \subseteq C \subseteq B$ and $A \perp_D B$ then $A \perp_D C$ and $A \perp_C B$.
(normality) $A \perp_C B$ if and only if $A \perp_C BC$.

Category 2

(transitivity) If $D \subseteq C \subseteq B$, $A \mathop{\smile\!\!\!\!\!\!|}_D C$, and $A \mathop{\smile\!\!\!\!\!\!|}_C B$, then $A \mathop{\smile\!\!\!\!\!\!|}_D B$.

(symmetry) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ then $B \mathop{\smile\!\!\!\!\!\!|}_C A$.

(local character) There is a cardinal κ such that if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ then there is some $C_0 \subseteq C$, with $|C_0| \leq \kappa$, such that $A \mathop{\smile\!\!\!\!\!\!|}_{C_0} B$.

Category 3

(extension) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $B' \supseteq B$ then there is some $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B'$.

(existence) For all A and C, $A \mathop{\smile\!\!\!\!\!\!|}_C C$.

(full existence) For all A, B, and C, there is some $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B$.

(irreflexivity) $a \mathop{\smile\!\!\!\!\!\!|}_C a$ if and only if $a \in \operatorname{acl}(C)$.

The properties in Category 1 are relatively weak and usually hold of any reasonable notion of independence, including $\mathop{\smile\!\!\!\!\!\!|}^f$ and $\mathop{\smile\!\!\!\!\!\!|}^d$.

The properties in Category 2 are desirable in order for a ternary relation to capture the feeling of an independence relation. Having $\mathop{\smile\!\!\!\!\!\!|}^d$ or $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfy even one of the properties in Category 2 is equivalent to having both $\mathop{\smile\!\!\!\!\!\!|}^d$ and $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfy all of them (see [4]). A theory in which this happens is called *simple.*

Category 3 is slightly more miscellaneous. We have shown $\mathop{\smile\!\!\!\!\!\!|}^d$ satisfies irreflexivity in Proposition 3.6. By Proposition 1.12, $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies extension, but *a priori* can fail existence (there are classical examples). On the other hand $\mathop{\smile\!\!\!\!\!\!|}^d$ satisfies existence and can fail extension.

Proposition 4.1. *$\mathop{\smile\!\!\!\!\!\!|}^d$ satisfies extension if and only if $\mathop{\smile\!\!\!\!\!\!|}^f = \mathop{\smile\!\!\!\!\!\!|}^d$.*

Proof. Suppose $\mathop{\smile\!\!\!\!\!\!|}^d$ satisfies extension. We always have $\mathop{\smile\!\!\!\!\!\!|}^f \Rightarrow \mathop{\smile\!\!\!\!\!\!|}^d$, so suppose $a \not\mathop{\smile\!\!\!\!\!\!|}^f_C B$, witnessed by some $D \supseteq BC$ (i.e. every extension of $\operatorname{tp}(a/BC)$ to $S(D)$ divides over C). Then $a \not\mathop{\smile\!\!\!\!\!\!|}^d_C B$, since otherwise the extension axiom would give some $a' \equiv_{BC} a$ such that $a \mathop{\smile\!\!\!\!\!\!|}^d_C D$, contradicting the choice of D.

Conversely, if $\mathop{\smile\!\!\!\!\!\!|}^f = \mathop{\smile\!\!\!\!\!\!|}^f$ then $\mathop{\smile\!\!\!\!\!\!|}^d$ satisfies extension by Proposition 1.12. $\square$

The reader should see [1] for more on abstract notions of independence.

In this section, we will take a slight detour with a discussion of the full existence axiom. In [1, Proposition 1.5], Adler gives a short and fairly self-contained proof that $\mathop{\smile\!\!\!\!\!\!|}^a$ satisfies full existence in any classical theory. Goldbring has asked if the same is true in continuous logic. We have not been successful in adapting Adler's proof to the continuous setting so, for now, we take an alternate route and answer the question for certain classes of theories.

To show that $\mathop{\smile\!\!\!\!\!\!|}^a$ satisfies full existence, it would suffice to find a *stronger* independence relation that satisfies it. In particular, we will want to see if $\mathop{\smile\!\!\!\!\!\!|}^f$ or $\mathop{\smile\!\!\!\!\!\!|}^d$ is stronger than $\mathop{\smile\!\!\!\!\!\!|}^a$. It would then follow that

$\mathop{\smile\!\!\!\!\!|}^a$ satisfies full existence whenever $\mathop{\smile\!\!\!\!\!|}^f$ or $\mathop{\smile\!\!\!\!\!|}^d$ does. This happens quite often, as seen from the following easy exercise in $\mathop{\smile\!\!\!\!\!|}$-calculus.

Proposition 4.2. *Suppose $\mathop{\smile\!\!\!\!\!|}$ is a ternary relation satisfying normality, existence and extension. Then $\mathop{\smile\!\!\!\!\!|}$ satisfies full existence.*

Proof. Fix A, B, and C. By existence, $A \mathop{\smile\!\!\!\!\!|}_C C$. By extension, there is some $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!|}_C BC$. By normality, $A' \mathop{\smile\!\!\!\!\!|}_C B$. □

In particular, if $\mathop{\smile\!\!\!\!\!|}^f$ satisfies existence then $\mathop{\smile\!\!\!\!\!|}^f$ satisfies full existence (normality for $\mathop{\smile\!\!\!\!\!|}^f$ is trivial). Alternatively if $\mathop{\smile\!\!\!\!\!|}^d$ satisfies extension (i.e. if $\mathop{\smile\!\!\!\!\!|}^d = \mathop{\smile\!\!\!\!\!|}^f$) then $\mathop{\smile\!\!\!\!\!|}^d$ satisfies full existence (since $\mathop{\smile\!\!\!\!\!|}^d$ always satisfies existence and normality). Of course in this situation we would then have that $\mathop{\smile\!\!\!\!\!|}^f$ satisfies existence anyway.

Proposition 4.3. *$\mathop{\smile\!\!\!\!\!|}^d$ satisfies full existence if and only if $\mathop{\smile\!\!\!\!\!|}^f$ satisfies existence.*

Proof. Suppose $\mathop{\smile\!\!\!\!\!|}^d$ satisfies full existence, and fix sets A and C. For any $D \supseteq C$, there is some $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!|}^d_C D$. Therefore $\mathrm{tp}(a/C)$ does not fork over C, i.e. $a \mathop{\smile\!\!\!\!\!|}^f_C C$.

Conversely suppose $\mathop{\smile\!\!\!\!\!|}^f$ satisfies existence. Then it satisfies full existence by Proposition 1.12 and Proposition 4.2. Therefore $\mathop{\smile\!\!\!\!\!|}^d$ satisfies full existence, since $\mathop{\smile\!\!\!\!\!|}^f \Rightarrow \mathop{\smile\!\!\!\!\!|}^d$. □

So now we try to see if $\mathop{\smile\!\!\!\!\!|}^d$ or $\mathop{\smile\!\!\!\!\!|}^f$ is stronger than $\mathop{\smile\!\!\!\!\!|}^a$. Of course, $\mathop{\smile\!\!\!\!\!|}^f \Rightarrow \mathop{\smile\!\!\!\!\!|}^d$, so $\mathop{\smile\!\!\!\!\!|}^d \Rightarrow \mathop{\smile\!\!\!\!\!|}^a$ would be the strongest result. However, this implication is stronger than what we need for now, and has a more difficult proof (see the appendix for further discussion). So we prove an easier fact (Lemma 4.4).

Lemma 4.4. $a \mathop{\smile\!\!\!\!\!|}^f_C B \;\Rightarrow\; a \mathop{\smile\!\!\!\!\!|}^d_C \mathrm{acl}(BC)$

Proof. Suppose $\mathrm{tp}(a/\mathrm{acl}(BC))$ divides over C, witnessed by some formula $\varphi(x,d)$, where "$\varphi(x,d) = 0$" $\in \mathrm{tp}(a/BC)$ and $d \in \mathrm{acl}(BC)$. Let $D := B \cup \{d' : d' \equiv_{BC} d\} \subseteq \mathrm{acl}(BC)$. Suppose $q \in S(D)$ is an extension of $\mathrm{tp}(a/BC)$. If $a' \models q$ then $a' \equiv_{BC} a$, and so $\varphi(a',d') = 0$ for some $d' \equiv_{BC} d$. Then $d' \in D$, so "$\varphi(x,d') = 0$" $\in q$. Therefore q divides over C, and so we have shown that $\mathrm{tp}(a/BC)$ forks over C. □

Corollary 4.5. $\mathop{\smile\!\!\!\!\!|}^f \Rightarrow \mathop{\smile\!\!\!\!\!|}^a$

Proof. Combine Theorem 3.7 and Lemma 4.4. □

Theorem 4.6. *If $\mathop{\smile\!\!\!\!\!|}^f$ satisfies existence then $\mathop{\smile\!\!\!\!\!|}^a$ satisfies full existence.*

Proof. Combine Proposition 1.12, Proposition 4.2, and Corollary 4.5. □

The class of theories where $\mathop{\smile\!\!\!\!\!\!|}^{f}$ satisfies existence includes simple theories and, more generally, any theory where forking equals dividing. In classical logic this includes most NIP theories with a notion of minimality (o-minimal, VC-minimal, etc...).

Appendix

Authors' Note (October 2021). In the original draft of these notes, this appendix detailed a proof that $\mathop{\smile\!\!\!\!\!\!|}^{d} \Rightarrow \mathop{\smile\!\!\!\!\!\!|}^{a}$ in continuous logic. This proof relied on Theorem A.2 (in the original draft), which claimed to prove $a \mathop{\smile\!\!\!\!\!\!|}^{d}_{C} B$ implies $a \mathop{\smile\!\!\!\!\!\!|}^{d}_{C} \operatorname{acl}(BC)$. In discrete logic, this statement appears as Remark 5.4(3) in [1], as well as Exercise 1.25(iii) in Adler's thesis. Our original argument was a continuous adaptation of a discrete proof of this exercise given to us by Adler. However, a gap in that proof was recently found by A. Kruckman, leading to the construction of a (discrete) counterexample, which will appear in forthcoming work. In the mean time, a direct proof of $\mathop{\smile\!\!\!\!\!\!|}^{d} \Rightarrow \mathop{\smile\!\!\!\!\!\!|}^{a}$ for continuous logic can be found in joint work of the first author and J. Hanson (see arXiv preprint 2110.07763). [**Added November 2023:** The preprint with Hanson is now published in *Fundamenta Mathematicae* vol. 259 (2022) pp. 97-109. I also neglected to mention in 2021 that the main purpose of that paper is to give a direct proof that $\mathop{\smile\!\!\!\!\!\!|}^{a}$ satisfies existence in any continuous theory (c.f., the discussion above at the bottom of page 11, and subsequent results). Finally, the "forthcoming work" with Kruckman alluded to above is now available; see arXiv preprint 2311.00609.]

Thus we have revised this appendix to remove the wrong result. But we have left the following lemma, which is correct and still useful for other purposes. On the other hand, the proof is entirely based on indiscernibles, automorphism, and Erdős-Rado, and thus looks identical to the discrete proof.

Lemma A.1. *The following are equivalent:*

(i) $a \mathop{\smile\!\!\!\!\!\!|}^{d}_{C} B$*;*

(ii) *for any C-indiscernible sequence $I = (b_i)_{i<\omega}$, with b_0 an enumeration of BC, there is some aC-indiscernible sequence I' such that $I' \equiv_{BC} I$.*

Proof. Assume (i), and let $(b_i)_{i<\omega}$ be C-indiscernible, with $b := b_0$ an enumeration of BC. By assumption, $\bigcup_{i<\omega} p(x, b_i)$ is consistent, where $p(x, b) = \operatorname{tp}(a/BC)$. Let a' be a realization of $\bigcup_{i<\omega} p(x, b_i)$. In particular, there is $\sigma \in \operatorname{Aut}(\mathbb{M}/BC)$ such that $\sigma(a') = a$. Let $I'' = (b''_i)_{i<\omega} := \sigma(I)$. Then $I'' \equiv_{BC} I$ and a realizes $\bigcup_{i<\omega} p(x, b''_i)$. Let $I^* = (b^*_i)_{i<\omega}$ be an aC-indiscernible sequence realizing $EM(I''/aC)$. Then $a \models p(x, b^*_0)$ so there is $\tau \in \operatorname{Aut}(\mathbb{M}/aC)$ such that $\tau(b^*_0) = b_0$. If $I' = (b'_i)_{i<\omega} := \tau(I^*)$, then I' is still aC-indiscernible. Moreover, $b'_0 = b_0$, so it follows that $I' \equiv_{BC} I$.

Now assume (ii), and let $p(x, b) = \operatorname{tp}(a/BC)$, with b an enumeration of BC. Given a C-indiscernible sequence $(b_i)_{i<\omega}$, with $b_0 = b$, let $I' = (b'_i)_{i<\omega} \equiv_{BC} I$ such that I' is aC-indiscernible. Since $b_0 \equiv_{BC} b'_0$ and $b_0 = BC$, it follows that $b'_0 = b_0 = b$. Then $a \models p(x, b'_0)$ and so a satisfies $\bigcup_{i<\omega} p(x, b'_i)$ by aC-indiscernibility. It follows that $\bigcup_{i<\omega} p(x, b_i)$ is satisfiable as well. $\square$

References

[1] H. Adler, *A geometric introduction to forking and thorn-forking*, Journal of Mathematical Logic **9** (2009) 1-20.

[2] I. Ben Yaacov, A. Berenstein, C. W. Henson, A. Usvyatsov, *Model theory for metric structures*, Model Theory with Applications to Algebra and Analysis, vol. 2, Lecture Notes series of the London Math. Soc. no. 350 (2008) 315-427.

[3] C. Ealy, & I. Goldbring, *Thorn-forking in continuous logic*, The Journal of Symbolic Logic **77** (2012) 63-91.

[4] B. Kim, *Simplicity, and stability in there*, Journal of Symbolic Logic **66** (2001) 822-836.

[5] K. Tent & M. Ziegler, *A Course in Model Theory*, Cambridge, 2012.

(GC) Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 S Morgan St, 322 SEO, Chicago, IL, 60607, USA;

Email address: gconan2@uic.edu

(CT) Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 S Morgan St, 322 SEO, Chicago, IL, 60607, USA;

Email address: cterry3@uic.edu