INDEPENDENCE, EXCHANGE, AND STRONGLY MINIMAL THEORIES

GABRIEL CONANT

ABSTRACT. We define five notions of independence, which can be used to measure the complexity of an arbitrary first order theory. After assuming the theory in question has the property that algebraic closure satisfies exchange, we add a sixth notion of dimensional independence, and see how it fits in with the others. Altogether, this gives a simple proof of the characterization of independence in strongly minimal theories.

T denotes a complete first order theory and $\mathbb{M}$ a sufficiently saturated monster model of T.

Definition 1. *Let $C \subset \mathbb{M}$ and $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}(C)$. Given $\bar{b} \in \mathbb{M}$, $\varphi(\bar{x}, \bar{b})$* ***divides over*** *$C$ if there is a sequence $(\bar{b}^i)_{i<\omega}$, with $\bar{b}^i \equiv_C \bar{b}$ for all $i < \omega$, such that $\{\varphi(\bar{x}, \bar{b}^i) : i < \omega\}$ is k-inconsistent for some $k > 0$.*

A formula ***forks over*** *C if it proves a finite disjunction of formulas that divide over C. A type* ***forks*** *(resp.* ***divides)*** ***over*** *C if it proves a formula that forks (resp. divides) over C.*

Definition 2.

$$\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{ucl}_{C} B \quad \Leftrightarrow \quad \text{tp}(\bar{a}/BC) \textit{ is finitely satisfiable in } \text{acl}(C)$$

$$\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{f}_{C} B \quad \Leftrightarrow \quad \text{tp}(\bar{a}/BC) \textit{ does not fork over } C$$

$$\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{d}_{C} B \quad \Leftrightarrow \quad \text{tp}(\bar{a}/BC) \textit{ does not divide over } C$$

Theorem 3. $\mathop{\smile\!\!\!\!\!\!|}^{ucl} \Rightarrow \mathop{\smile\!\!\!\!\!\!|}^{f} \Rightarrow \mathop{\smile\!\!\!\!\!\!|}^{d}$

Proof. The second implication is trivial since dividing implies forking by definition. Next, suppose $\text{tp}(\bar{a}/BC)$ forks over C. Then there are tuples $\bar{d} \in \mathbb{M}$, $\bar{b} \in B$, some and $\mathcal{L}(C)$-formulas $\psi(\bar{x}, \bar{y}), \varphi_1(\bar{x}, \bar{y}), \dots \varphi_n(\bar{x}, \bar{y})$ such that $\psi(\bar{x}, \bar{b}) \in \text{tp}(\bar{a}/BC)$, $\varphi_i(\bar{x}, \bar{d})$ divides over C for all i, and

$$\psi(\bar{x}, \bar{b}) \vdash \bigvee_{i=1}^{n} \varphi_i(\bar{x}, \bar{d}).$$

Suppose, $\bar{a}_* \models \psi(\bar{x}, \bar{b})$. Then there is some i such that $\mathbb{M} \models \varphi_i(\bar{a}_*, \bar{d})$. Fix $(\bar{d}^l)_{l<\omega}$, with $\bar{d}^l \equiv_C \bar{d}$ for all $l < \omega$, such that $\{\varphi(\bar{x}, \bar{d}^l) : l < \omega\}$ is finitely inconsistent. If $\sigma_l \in \text{Aut}(\mathbb{M}/C)$ is such that $\sigma_l(\bar{d}) = \bar{d}^l$, then

Date: May 2, 2013. *Revised:* November 2, 2023.

set $\bar{a}^l = \sigma_l(\bar{a}_*)$. Since $\bar{a}_* \models \varphi(\bar{x}, \bar{d})$ and $\{\varphi(\bar{x}, \bar{d}^l) : l < \omega\}$ is finitely inconsistent, it follows that $(\bar{a}^l)_{l<\omega}$ is an infinite sequence. Therefore $\bar{a}_* \notin \operatorname{acl}(C)$, and so $\psi(\bar{x}, \bar{b})$ is not satisfiable in $\operatorname{acl}(C)$. So $\bar{a} \not\downarrow^{ucl}_C B$. $\square$

Definition 4.

$$\bar{a} \downarrow^a_C B \quad\Leftrightarrow\quad \operatorname{acl}(\bar{a}C) \cap \operatorname{acl}(BC) = \operatorname{acl}(C)$$
$$\bar{a} \downarrow^M_C B \quad\Leftrightarrow\quad \bar{a} \downarrow^a_D B \text{ for all } C \subseteq D \subseteq \operatorname{acl}(BC)$$

Author's note (November 2023): At this point in the original draft of these notes, it was stated that $\downarrow^d$ implies $\downarrow^M$ in any theory. The proof quoted [1, Remark 5.4], which has since been found to be false. Several counterexamples are given in the arXiv preprint 2311.00609 (with Kruckman). On the other hand, the impact on these notes is minimal because we are about to assume that algebraic closure in T satisfies the exchange property (defined below). In the same arXiv preprint, we prove the following result (see Proposition 2.18 there).

Theorem 5. *If algebraic closure in T satisfies the exchange property, then $\downarrow^d \Rightarrow \downarrow^M$.*

Now we assume that in T, algebraic closure satisfies the **exchange property**, i.e., if $a \in \operatorname{acl}(bC) \setminus \operatorname{acl}(C)$ then $b \in \operatorname{acl}(aC)$.

Definition 6. *Fix sets $A, C \subset \mathbb{M}$. Then A is* ***independent over*** *C if $a \notin \operatorname{acl}(C \cup A \setminus \{a\})$ for all $a \in A$. A* ***basis for*** *A* ***over*** *C is a subset $A_0 \subseteq A$ such that A_0 is independent over C and $\operatorname{acl}(A_0C) = \operatorname{acl}(AC)$.*

Theorem 7. *Let $A, C \subset \mathbb{M}$. If $A' \subseteq A$ is such that $\operatorname{acl}(A'C) = \operatorname{acl}(AC)$, then A' contains a basis for A over C. Moreover, any two bases for A over C have the same cardinality.*

Proof. See [3]. $\square$

Definition 8. *Given $C \subset \mathbb{M}$ and $\bar{a} \in \mathbb{M}$, $\dim(\bar{a}/C)$ is the cardinality of a basis for $\bar{a}$ over C. We define*

$$\bar{a} \downarrow^{dim}_C B \quad\Leftrightarrow\quad \dim(\bar{a}/BC) = \dim(\bar{a}/C).$$

Proposition 9.

(a) *$\downarrow^{dim}$ is symmetric.*

(b) *$\downarrow^M = \downarrow^{dim}$*

Proof. This proof is essentially the same as that of [2, Proposition 2.2].

Part (a). Suppose $\bar{a} \not\downarrow^{dim}_C B$. Without loss of generality, assume $\bar{a}$ is independent over C. Then $\bar{a}$ is not independent over BC so there is some $a \in \bar{a}$ such that $a \in \operatorname{acl}(\bar{a}'BC)$, where $\bar{a}' := \bar{a} \setminus \{a\}$. Let $\bar{a}_0 \subseteq \bar{a}'$ and $\bar{b}_0 \subseteq B$ be such that $\bar{a}_0\bar{b}_0$ is a basis for $\bar{a}'B$ over C. Then $a \in \operatorname{acl}(\bar{a}_0\bar{b}_0C)$. Let $\bar{b} \subseteq \bar{b}_0$ be independent over

C and finite such that $a \in \operatorname{acl}(\bar{a}_0\bar{b}C)$. Note that $\bar{b} \neq \emptyset$ since $\bar{a}$ is independent over C. For $b \in \bar{b}$, we have $a \in \operatorname{acl}(\bar{a}_0\bar{b}C)\setminus\operatorname{acl}(\bar{a}_0\bar{b}\setminus\{b\}C)$, and so $b \in \operatorname{acl}(a\bar{a}_0\bar{b}\setminus\{b\}C)$. It follows that $\bar{b}$ is not independent over $\bar{a}C$. In particular, $B \not\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim}_C \bar{a}$.

Part (b). Suppose $\bar{a} \not\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim}_C B$. By part (a), $B \not\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim}_C \bar{a}$. Then there are $B_0 \subseteq B$ and $b \in B$ such that B_0b is independent over C, but $b \in \operatorname{acl}(\bar{a}B_0C)$. Let $D := \operatorname{acl}(B_0C)$. Then $C \subseteq D \subseteq \operatorname{acl}(BC)$, $b \in \operatorname{acl}(\bar{a}B_0C) \subseteq \operatorname{acl}(\bar{a}D)$, $b \in B \subseteq \operatorname{acl}(BD)$ and $b \notin D = \operatorname{acl}(D)$. Therefore D witnesses $\bar{a} \not\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{M}_C B$.

Conversely, suppose $\bar{a} \not\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{M}_C B$. Say $C \subseteq D \subseteq \operatorname{acl}(BC)$ such that $\bar{a} \not\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{a}_D B$. So there is some $d \in (\operatorname{acl}(\bar{a}D) \cap \operatorname{acl}(BD))\setminus\operatorname{acl}(D)$. Without loss of generality we may assume $D\setminus C$ is finite and independent over C. By exchange, follows that Dd is independent over C. But Dd is not independent over $\bar{a}C$, and so $Dd \not\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim}_C \bar{a}$. By symmetry, $\bar{a} \not\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim}_C \operatorname{acl}(BC)$. Therefore $\dim(\bar{a}/BC) = \dim(\bar{a}/\operatorname{acl}(BC)) < \dim(\bar{a}/C)$, and so $\bar{a} \not\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim}_C B$. □

Remark 10. Note that $\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{M} \Rightarrow \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{a}$ is always true, and so we have $\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim} \Rightarrow \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{a}$ by Proposition 9. Moreover, the following are equivalent:

- *(i)* $\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim} = \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{a}$;
- *(ii)* T is **modular**, i.e., $\dim(A) + \dim(B) = \dim(A \cup B) + \dim(A \cap B)$ for all closed sets A and B;
- *(iii)* $\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{a}$ satisfies **base monotonicity**, i.e., $\bar{a} \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{a}_D B$ and $D \subseteq C \subseteq B$ implies $\bar{a} \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{a}_C B$;
- *(iv)* $B \cap \operatorname{acl}(A \cup C) = \operatorname{acl}((B \cap A) \cup C)$ for all closed sets A, B, and C, with $C \subseteq B$.

See [3] for *(i)* ⇔ *(ii)*, and [1] for *(iii)* ⇔ *(iv)*. In [1], *(iv)* is used to define a **modular** theory without the assumption of acl satisfying exchange.

To finish the chain of equivalences, it suffices to show *(i)* ⇔ *(iii)*. But *(i)* implies *(iii)* since $\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim}$ clearly satisfies base monotonicity. Conversely, *(iii)* implies $\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{M} = \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{a}$, which implies *(i)* by Proposition 9.

Given a ternary relation $\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}$ and $C \subset \mathbb{M}$, let $\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}_C$ be the induced binary relation.

Theorem 11. *Suppose T is strongly minimal and $C \subset \mathbb{M}$. If $\operatorname{acl}(C)$ is infinite then*

$$\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{ucl}_C = \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{f}_C = \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{d}_C = \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{M}_C = \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim}_C .$$

Proof. By previous results, it suffices to show $\mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim}_C \Rightarrow \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{ucl}_C$. So assume $\bar{a} \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim}_C B$. We prove $\bar{a} \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{ucl}_C B$ by induction on $|\bar{a}|$. For the base case, let $\varphi(x, \bar{b}) \in \operatorname{tp}(a/BC)$. If $\varphi(\mathbb{M}, \bar{b})$ is finite then $\dim(a/C) = \dim(a/BC) = 0$, and so $a \in \operatorname{acl}(C)$. Therefore $a \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{ucl}_C B$ since $\operatorname{tp}(a/BC)$ is realized in $\operatorname{acl}(C)$. Otherwise $\varphi(\mathbb{M}, \bar{b})$ is cofinite and therefore intersects $\operatorname{acl}(C)$.

Assume the result for tuples of length strictly less than $l(\bar{a})$, and let $\bar{a} = (a_1, \ldots, a_n)$. Fix $\varphi(\bar{x}, \bar{b}) \in \operatorname{tp}(\bar{a}/BC)$. We want to show $\varphi(\bar{x}, \bar{b})$ has a solution in $\operatorname{acl}(C)$.

Suppose first that $\varphi(\mathbb{M}, a_2, \ldots, a_n, \bar{b})$ is infinite. Then there is some $c \in \operatorname{acl}(C)$ such that $\mathbb{M} \models \varphi(c, a_2, \ldots, a_2, \bar{b})$. Let $\psi(x) \in \mathcal{L}(C)$ such that $\mathbb{M} \models \psi(c)$ and $\psi(\mathbb{M})$ is finite. Then $\exists x(\varphi(x, x_2, \ldots, x_n) \wedge \psi(x)) \in \operatorname{tp}(a_2, \ldots, a_n/BC)$. Note that $a_2, \ldots, a_n \mathop{\smile\hspace{-0.9em}|\hspace{0.7em}}^{dim}_C B$, and so by induction there are $c_1, \ldots, c_n \in \mathbb{M}$ such that $c_2, \ldots, c_n \in \operatorname{acl}(C)$ and

$\mathbb{M} \models \varphi(\bar{c}, \bar{b}) \wedge \psi(c_1)$. Then $c_1 \in \mathrm{acl}(C)$ as well, and so $\varphi(\bar{x}, \bar{b})$ has a solution in $\mathrm{acl}(C)$.

Now suppose $\varphi(\mathbb{M}, a_2, \ldots, a_n, \bar{b})$ is finite. It follows that $\dim(\bar{a}/BC) < n$ and so $\dim(\bar{a}/C) < n$, since $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{dim}_{C} B$. In particular there is some a_i such that $a_i \in \mathrm{acl}(a_1 \ldots a_{i-1} a_{i+1} \ldots a_n C)$. Without loss of generality, assume $a_1 \in \mathrm{acl}(a_2 \ldots a_n C)$. Let $\psi(x, y_2, \ldots, y_n) \in \mathcal{L}(C)$ such that $\psi(\mathbb{M}, a_2, \ldots, a_n)$ is finite and $\mathbb{M} \models \psi(a_1, \ldots, a_n)$. Say $|\psi(\mathbb{M}, a_2, \ldots, a_n)| = m$. Then

$$\exists x(\varphi(x, x_2, \ldots, x_n, \bar{b}) \wedge \psi(x, x_2, \ldots, x_n)) \wedge \exists!^m x \psi(x, x_2, \ldots, x_n) \in \mathrm{tp}(a_2, \ldots, a_n/BC).$$

By induction there are $c_1, \ldots, c_n \in \mathbb{M}$ such that $c_2, \ldots, c_n \in \mathrm{acl}(C)$, $\psi(\mathbb{M}, c_2, \ldots, c_n)$ is finite, and

$$\mathbb{M} \models \varphi(c_1, \ldots, c_n, \bar{b}) \wedge \psi(c_1, \ldots, c_n).$$

Therefore $c_1 \in \mathrm{acl}(c_2 \ldots c_n C) = \mathrm{acl}(C)$, and so $\varphi(\bar{x}, \bar{b})$ has a solution in $\mathrm{acl}(C)$. □

References

[1] Adler H., *A geometric introduction to forking and thorn-forking*, Journal of Mathematical Logic 9 (2009) 1-20.

[2] Adler H., *Around pregeometric theories*, unpublished notes, 2007.

[3] Casanovas E., *Pregeometries and minimal types*, Model Theory Seminar notes, 2008. *http://www.ub.edu/modeltheory/documentos/pregeometries.pdf*