Model Theory and Combinatorics of Homogeneous Metric Spaces

by

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To my family and friends.

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Contents

List of Tables

List of Figures

List of Abbreviations

FFSOP	fully finite strong order property
FSOP	finitary strong order property
IP	independence property
RUS	class of $\mathcal{R}$-Urysohn spaces, where $\mathcal{R}$ is a Urysohn monoid
SOP	strict order property
SOP_n	strong order property
TP_1	tree property of the first kind
TP_2	tree property of the second kind

Summary

The work in this thesis focuses on the model theory of homogeneous structures produced as infinite "generic" limits of finite combinatorial objects. Examples of such structures include the random graph, the generic K_n-free graph, and the rational Urysohn space. These structures arise as motivational examples in many fields, including descriptive set theory (e.g. [91], [92]), topological dynamics of automorphism groups (e.g. [6], [38], [41], [46], [69]), infinite combinatorics (e.g. [12], [18], [27], [54], [76]), and finite combinatorics (e.g. [43], [85]).

We will consider these structures from the perspective of first-order model theory, which seeks to understand and classify the "definable behavior" of general mathematical structures. This endeavor finds its greatest success in the study of *stable theories* (see Definition 1.1.1), in which there is a well-defined and unique notion of "independence" used to understand local and global definable behavior. However, the kinds of homogeneous structures discussed above frequently yield unstable theories. Therefore, we turn to a field of study, sometimes called "neostability", which focuses on the extent to which dependable tools from stability can be applied to the landscape of unstable theories. The exploration and navigation of this landscape has resulted in an army of combinatorial "dividing lines", which carve the universe of first-order theories into varying regions of complexity. Notable examples include the *simple* theories and the NIP theories (see Definitions 1.1.8 and 1.1.6), in which a significant amount of progress has been made in understanding both global and local model theoretic behavior. However, the aforementioned homogeneous structures frequently escape these more general dividing lines as well. In particular, these structures are often found in the region of unstable theories without the *strict order property*, which, in some sense, can be seen as orthogonal to the unstable NIP theories (see Fact 1.1.7). Furthermore, these structures are frequently not simple. Altogether, a starting point for this thesis is the tension between the following observations.

1. Many interesting homogeneous structures give rise to first-order theories that are non-simple and without the strict order property.

2. Non-simple theories without the strict order property are not well understood, as a general region in the classification of first-order theories.

Concerning the classification of theories without the strict order property, many navigational tools were invented in a 1996 paper, *Toward classifying unstable theories*, by Shelah [83]. In particular, Shelah defines a hierarchy of *strong order properties* (see Section 1.4), which stratify the region of non-simple theories without the strict order property. In the twenty years since this paper, a considerable amount of work has been done with the dividing lines at the lower end of the hierarchy (e.g from simple to SOP_3, see [19], [28], [35], [50], [51], [60], [61], [84]). However, the rest of this hierarchy has yet to gain real traction as a meaningful system of dividing lines. Part of the problem has been a general lack of examples. In particular, most "natural" examples of non-simple theories without the strict order property congregate in the region between simple and SOP_3. Examples of theories further up the hierarchy are often artificially constructed for the purposes of exemplifying a desired place among the dividing lines.

In this thesis, we will develop the model theory of a large class of generalized metric spaces, denoted **RUS** (for "$\mathcal{R}$-Urysohn spaces"). This class will include many well-known and important examples of homogeneous structures (e.g. the random graph and the rational Urysohn space), and we will show that, moreover, this class exhibits the following behavior, which is motivated by the previous discussion.

(*a*) The model theoretic complexity of the structures in **RUS** can be determined in an explicit and meaningful way, which is directly related to the natural mathematical behavior of the structure.

(*b*) The structures in **RUS** exemplify model theoretic complexity throughout the region of theories without the strict order property, including the entirety of the strong order property hierarchy.

We will also find regions of complexity that are not detected by **RUS** (see Section 3.7.2), and discuss the connection to some well-known open questions in classification theory. Finally, we will explore the combinatorial behavior of the structures in **RUS**. In addition to new results and open questions, we will discover several surprising connections to areas of research outside of model theory.

We now give a broad outline of the following work. In Chapter 1, we record the tools from classification theory that will be used for our results. We also give a careful treatment of the SOP_n-hierarchy, which includes a new formulation in terms of a rank on first-order theories. In Chapter 2, we begin the model theoretic study of generalized metric spaces as combinatorial structures in first-order relational languages. By "generalized metric space", we will mean a metric space taking distances in an arbitrary ordered commutative structure $\mathcal{R} = (R, \oplus, \leq, 0)$, called a *distance magma*. This level of generality is naturally motivated by the fact that, when working with metric spaces in discrete logic, one may easily construct saturated models containing points of nonstandard "distance" (e.g. infinitesimals). To solve this problem, we first fix a distance magma $\mathcal{R}$ and a subset $S \subseteq R$ of distinguished distances (e.g. $S = \mathbb{Q}$ in $\mathcal{R} = (\mathbb{R}^{\geq 0}, +, \leq, 0)$. We then construct a canonical distance magma extension $\mathcal{S}^*$ such that, given an $\mathcal{R}$-metric space M with distances in S, any model of the theory of M (in a specified language) can be equipped with an $\mathcal{S}^*$-metric in a way coherent with the theory. We then work toward defining the class **RUS**. In particular, we consider generalizations of the rational Urysohn space obtained by constructing a countable, homogeneous, and universal $\mathcal{R}$-metric space $\mathcal{U}^S_\mathcal{R}$, with distances in some distinguished countable subset $S \subseteq \mathcal{R}$. The existence of $\mathcal{U}^S_\mathcal{R}$ can be completely characterized via a combinatorial property of S, which we obtain by generalizing previous work of Delhommé, Laflamme, Pouzet, and Sauer [27]. We then focus on subsets $S \subseteq R$ closed under the operation $u \oplus_S v := \sup\{x \in S : x \leq u \oplus v\}$, in which case the existence of $\mathcal{U}^S_\mathcal{R}$ is equivalent to associativity of $\oplus_S$. Without loss of generality, we may then restrict to the case when $\mathcal{R}$ is a countable *distance monoid* (i.e. $\oplus$ is associative) and $S = R$. The final main result of Chapter 2 is a characterization of quantifier elimination for $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ (where $\mathcal{U}_\mathcal{R} = \mathcal{U}^R_\mathcal{R}$) via a natural continuity property in $\mathcal{R}^*$. Altogether, we can finally define **RUS** to be the class of structures $\mathcal{U}_\mathcal{R}$, where $\mathcal{R}$ is a countable distance monoid such that $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ has quantifier elimination.

In Chapter 3, we analyze "neostability" properties of the metric spaces in **RUS**. We focus especially on instances when, given some property P (e.g. stability), there is a first order sentence φ_P, in the language of ordered monoids, such that a metric space $\mathcal{U}_\mathcal{R}$ in **RUS** has the property P if and only if $\mathcal{R} \models \varphi_P$. In this case, we say that P is *axiomatizable*. As first examples of this behavior, we show that stability and simplicity are axiomatizable properties of **RUS**. We then characterize superstability and supersimplicity, and show that these properties are not axiomatizable. Next, we give a uniform upper bound for the complexity of $\mathrm{Th}(\mathcal{U}_\mathcal{R})$, when $\mathcal{U}_\mathcal{R}$ is in **RUS**. As a corollary, it will follow that $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ never has the strict order property. We then use an analysis of indiscernible sequences to prove that the position of $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ in the SOP_n-hierarchy is axiomatizable by straightforward properties of $\mathcal{R}$. Finally, we generalize previous work of Casanovas and Wagner [15] to give necessary conditions for weak elimination of imaginaries and elimination of hyperimaginaries in $\mathrm{Th}(\mathcal{U}_\mathcal{R})$.

The final two chapters shift in focus from model theory to combinatorics. In Chapter 4, we consider the group of isometries, $\mathrm{Isom}(\mathcal{U}_\mathcal{R})$, where $\mathcal{R}$ is a countable distance monoid. We address the question of extending partial isometries of finite $\mathcal{R}$-metric spaces, which continues a line of investigation begun by Hrushovski and developed by many authors (e.g. [37], [38], [39], [41], [43], [85]). We translate work of Solecki [85] to the setting of generalized metric spaces, in order to give a metric analog of a theorem of Herwig and Lascar [39] on extending automorphisms in classes of relational structures. This immediately obtains an isometry extension result for $\mathrm{Isom}(\mathcal{U}_\mathcal{R})$, when $\mathcal{R}$ is an archimedean monoid. Combined with work of Kechris and Rosendal [46], we conclude that, when $\mathcal{R}$ is archimedean, $\mathrm{Isom}(\mathcal{U}_\mathcal{R})$ has ample generics, automatic continuity, and the small index property. We then prove the same isometry extension results for a larger class of $\mathcal{R}$-Urysohn spaces, which includes the case when $\mathcal{U}_\mathcal{R}$ is an ultrametric space. This establishes an interesting open question as to whether these results hold for $\mathrm{Isom}(\mathcal{U}_\mathcal{R})$, when $\mathcal{R}$ is arbitrary. In Chapter 5, we continue a line of investigation begun by Nguyen Van Thé [69] into the combinatorial behavior of finite distance monoids. We give upper and lower bounds for the asymptotic behavior of the number $\mathrm{DM}(n)$ of distance monoids with n nontrivial elements. We also classify certain families of finite distance monoids, which arise naturally from well-structured sets of integers. The chapter ends with a classification of distance monoids with $n \leq 6$ nontrivial elements.

Chapter 1

Classification Theory

The purpose of this chapter is to state the fundamental model theoretic background we will need in the subsequent work. We will endeavor to provide citations to original sources, as well as to standard texts in model theory, stability theory, and mathematical logic. In particular, we often provide references to Tent and Ziegler's recent text, *A Course in Model Theory* [86].

We assume familiarity with the basic concepts of first-order mathematical logic, including languages, structures, theories, quantifier elimination, compactness, spaces of types, indiscernibles, and saturated and homogeneous models (including the notion of a monster model). In this chapter, we will define many fundamental notions in classification theory, including stability, simplicity, and forking. However, some prior familiarity with these concepts may be helpful. There are many wonderful introductory texts, which the reader may consult to supplement the material presented here (e.g. [7], [62], [86], [93]).

Next, we set notation and conventions, which will be used throughout the thesis. Suppose T is a complete first-order theory and $\mathbb{M}$ is a sufficiently saturated monster model of T. We use $A, B, C, \ldots$ to denote subsets of $\mathbb{M}$. We write "$A \subset \mathbb{M}$" to mean $A \subseteq \mathbb{M}$ and $\mathbb{M}$ is $|A|^+$-saturated. Given subsets $A, B \subset \mathbb{M}$, we use AB to denote $A \cup B$. We use tuples $\bar{a}, \bar{b}, \bar{c}, \ldots$ to denote tuples of elements of $\mathbb{M}$, and $\bar{x}, \bar{y}, \bar{x}, \ldots$ to denote tuples of variables. Unless otherwise specified, these tuples may be infinite in length, but always smaller in cardinality than $\mathbb{M}$. We let $\ell(\bar{x})$ denote the index set or length of a tuple. Singleton letters $a, b, c, x, y, z, \ldots$ will always denote singleton elements or variables. This convention will be temporarily suspended in Section 1.1 in which, to ease notation, we allow singleton letters to denote tuples of elements. We denote sequences of tuples as $(\bar{a}^l)_{l<\lambda}$, where λ is an ordinal. We use superscripts $l, m, n, \ldots$ to denote the index of the tuple and subscripts $i, j, k, \ldots$ for coordinates in a specific tuple (e.g a^l_i is a coordinate in the tuple $\bar{a}^l$). When working with sequences of tuples, "indiscernible" always means indiscernible as a *sequence* (i.e. *order* indiscernible). If a sequence is also indiscernible as a *set*, then this will be explicitly stated. We will primarily work with sequences $(\bar{a}^l)_{l<\lambda}$ indexed by infinite ordinals λ. Such a sequence may contain repeated tuples. Therefore, we will say $(\bar{a}^l)_{l<\omega}$ is an *infinite sequence* if it contains infinitely many pairwise distinct tuples. All other conventions are standard, or established as new definitions are introduced.

For the rest of the chapter, we fix a complete first-order theory T, in a first-order language $\mathcal{L}$, and a monster model $\mathbb{M}$ of T.

A word of caution. Throughout this chapter, we will define several syntactic properties of formulas in T (e.g. the *order property*). Most of these properties were originally defined by Shelah in [82] and [83]. Since then, generalizations, specifications, and alternate definitions have appeared throughout the literature. In most cases, the various definitions all become equivalent when viewed as a property of the theory T. However, as local properties of a formula they are sometimes inequivalent. Our interest will exclusively be in the global properties of the theory. Therefore, we warn the reader that our versions of the definitions may differ slightly from what they are used to, or from what they have seen in other sources.

Concerning acronyms. By the end of this chapter, we will have defined several "dividing lines" in first-order theories, which are thereafter referenced via host of acronyms (e.g. IP, SOP, TP_2, etc...). These properties describe complexity in theories, and so we are often interested in situations when T does *not* have the said property. Therefore, to denote the negation of a certain property, we place the letter "N" before the acronym for the property (e.g. NIP, NSOP, NTP_2, etc...).

1.1 Dividing Lines in First-Order Theories

The purpose of this section is to introduce and define common model theoretic "dividing lines", which are used to organize first-order theories by relative complexity of definable sets.

In this section, we allow singleton letters $a, b, c, x, y, z, \ldots$ to denote tuples of parameters in $\mathbb{M}$ or variables in $\mathcal{L}$-formulas.

Definition 1.1.1.

1. Given an infinite cardinal λ, T is **λ-stable** if $|S_1(A)| \leq \lambda$ for all $A \subset \mathbb{M}$ such that $|A| \leq \lambda$.
2. T is **stable** if it is λ-stable for some infinite cardinal λ.
3. T is **superstable** if there exists some infinite cardinal κ such that T is λ-stable for all $\lambda \geq \kappa$.

Clearly, superstable theories are stable. A nontrivial fact is that ω-stable[1] the-

[1] It is a common practice to use ω as a cardinal in the context of stability (in place of $\aleph_0$). Some sources use "$\aleph_0$-stable", which is more consistent with the definition. There is no difference between the two concepts.

ories are superstable. We state this result for $\mathcal{L}$ countable, and refer the reader to [82, Section II.3] for more general statements.

Fact 1.1.2. [86, Theorem 5.2.6] *Assume $\mathcal{L}$ is countable. If T is ω-stable then T is λ-stable for all $\lambda \geq \aleph_0$, and therefore superstable.*

The following is a major result of Shelah, which shows that the collection of infinite cardinals, for which a theory is stable, is quite well-behaved. We again state the result for $\mathcal{L}$ countable, and refer the reader to [82, Chapter III] for details on the more general situation.

Fact 1.1.3 (Stability Spectrum)**.** [86, Theorem 8.6.5] *Assume $\mathcal{L}$ is countable. Let* $\mathrm{Spec}(T) = \{\lambda : T$ *is* λ*-stable*$\}$*. Exactly one of the following holds.*

(*i*) $\mathrm{Spec}(T) = \{\lambda : \lambda \geq \aleph_0\}$ *(i.e. T is ω-stable).*

(*ii*) $\mathrm{Spec}(T) = \{\lambda : \lambda \geq 2^{\aleph_0}\}$ *(i.e. T is **strictly superstable**).*

(*iii*) $\mathrm{Spec}(T) = \{\lambda : \lambda^{\aleph_0} = \lambda\}$ *(i.e. T is **strictly stable**).*

(*iv*) $\mathrm{Spec}(T) = \emptyset$ *(i.e. T is **unstable**).*

Next, we give a local syntactic characterization of stability.

Definition 1.1.4. A formula $\varphi(x, y)$, with $\ell(x) = \ell(y)$, has the **order property in** T if there is a sequence $(a_i)_{i<\omega}$ such that $\mathbb{M} \models \varphi(a_i, a_j)$ if and only if $i \leq j$. The theory T has the **order property** if some formula has the order property in T.

Fact 1.1.5. [82, Theorem II.2.2] *T is stable if and only if it does not have the order property.*

The order property is the first of many syntactic dividing lines we will introduce in this chapter. It also the weakest in the sense that every other syntactic property we define will imply the order property (see Figure 1, Section 1.4). Here are two essential examples.

Definition 1.1.6.

1. A formula $\varphi(x, y)$ has the **strict order property in** T if there is a sequence $(b_i)_{i<\omega}$ such that

$$\mathbb{M} \models \forall x(\varphi(x, b_i) \to \varphi(x, b_j)) \quad \Leftrightarrow \quad i \leq j.$$

T has the **strict order property**, SOP, if some formula has the strict order property in T.

2. A formula $\varphi(x,y)$ has the **independence property in** T if there are sequences $(a_i)_{i<\omega}$, $(b_I)_{I\subseteq\omega}$ such that

$$\mathbb{M} \models \varphi(a_i, b_I) \quad \Leftrightarrow \quad i \in I.$$

   T has the **independence property**, IP, if some formula has the independence property in T.

Each of these properties is strictly stronger than the order property, and neither implies the other.[2] A beautiful theorem, due to Shelah, is that any theory with the order property must have one of the two strengthenings.

Fact 1.1.7. [82, Theorem II.4.7] *T is stable if and only if it is* NIP *and* NSOP*.*

Next, we turn to simple theories, which were first studied in Shelah's *Simple unstable theories* [81], to push the tools of stability theory into the class of unstable theories. The word "simple", which is a somewhat misleading choice of terminology, is due to the abstract of [81], which begins, "We point out a class of unstable theories, which are simple, ..."

Definition 1.1.8.

1. A formula $\varphi(x,y)$ has the **tree property in** T if there is a sequence $(a_\eta)_{\eta\in\omega^{<\omega}}$ of parameters from $\mathbb{M}$ such that

   (*i*) for all $\sigma \in \omega^\omega$, $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent, and

   (*ii*) for all $\eta \in \omega^{<\omega}$, $\{\varphi(x, a_{\eta\frown n}) : n < \omega\}$ is 2-inconsistent.

2. T has the **tree property** if there is a formula with the tree property in T.

3. T is **simple** if no formula has the tree property in T.

Fact 1.1.9. [86, Corollary 8.3.6] *Every stable theory is simple.*

Similar to Fact 1.1.7, theories with the tree property have one of two stronger tree properties. To motivate the definitions, we set the following terminology, which takes an approach similar to [60].

Definition 1.1.10. Let $\varphi(x,y)$ be a formula and $A = (a_\eta)_{\eta\in\omega^{<\omega}}$ a sequence of parameters from $\mathbb{M}$. Let $I \subseteq \omega^{<\omega}$ be any subset.

1. I is **(φ, A)-inconsistent** if $\{\varphi(x, a_\eta) : \eta \in I\}$ is inconsistent.

2. I is **incomparable** if there are distinct $\mu, \nu \in I$, which do not lie on the same branch.

[2] For example, the theory of the random graph and the theory of dense linear orders each have the order property. The former is NSOP and IP, while the latter is NIP and SOP.

3. I is **strongly incomparable** if there are distinct $\mu, \nu \in I$, which are immediate successors of the same node.

Observe that, with this terminology, a formula $\varphi(x, y)$ has the tree property in T if and only if there is a sequence of parameters A from $\mathbb{M}$ such that, for all $I \subseteq \omega^{<\omega}$, if I is (φ, A)-inconsistent then I is incomparable, and if I is strongly incomparable then I is (φ, A)-inconsistent. The next two properties are defined by extending the tree property to two opposite extremes.

Definition 1.1.11.

1. A formula $\varphi(x, y)$ has the **tree property of the first kind in** T if there is a sequence of parameters $A = (a_\eta)_{\eta \in \omega^{<\omega}}$ from $\mathbb{M}$ such that, for all $I \subseteq \omega^{<\omega}$, I is (φ, A)-inconsistent if and only if I is incomparable.
2. A formula $\varphi(x, y)$ has the **tree property of the second kind in** T if there is a sequence of parameters $A = (a_\eta)_{\eta \in \omega^{<\omega}}$ from $\mathbb{M}$ such that, for all $I \subseteq \omega^{<\omega}$, I is (φ, A)-inconsistent if and only if I is strongly incomparable.
3. T is TP_1 if some formula has the tree property of the first kind in T.
4. T is TP_2 if some formula has the tree property of the second kind in T.

In short, TP_1 is obtained from tree property by adding the most inconsistency possible; and TP_2 is obtained from the tree property by adding the most consistency possible. The following result is Theorem III.7.11 of [82]. Other versions of the proof, with further detail, can be found in [2] and [51].

Fact 1.1.12. *T is simple if and only if it is NTP_1 and NTP_2.*

The above definitions of TP_1 and TP_2 have a pleasing duality. However, TP_2 is usually phrased with an array-indexed set of parameters (see e.g [20]).

Exercise 1.1.13. *A formula $\varphi(x, y)$ has TP_2 in T if and only if there is an array of parameters $(a_{i,j})_{i,j<\omega}$ such that*

(i) for all $n < \omega$, $\{\varphi(x, a_{n,i}) : i < \omega\}$ is 2-inconsistent, and

(ii) for all functions $\sigma : \omega \longrightarrow \omega$, $\{\varphi(x, a_{n,\sigma(n)}) : n < \omega\}$ is consistent.

Remark 1.1.14. Despite the blanket warning given at the beginning of the chapter, it is worth emphasizing the variety of existing definitions surrounding these notions of tree properties. In particular, we focus on generalizing the occurrences of "2-inconsistency" in the previous definitions.

Fix $I \subseteq \omega^{<\omega}$ and $k \geq 2$. Let us say that I is **k-incomparable** (resp. **strongly k-incomparable**) if I contains k distinct nodes, no two of which lie on the same branch (resp. are immediate successors of the same node). Note that incomparable and 2-incomparable are equivalent. We now apply this generalization to each of the three previously defined tree properties.

1. Given $k \geq 2$, we say $\varphi(x,y)$ has the tree property with respect to k, denoted k-TP, in T if there is a sequence of parameters A such that, for all $I \subseteq \omega^{<\omega}$,

   (i) if I is (φ, A)-inconsistent then I is incomparable, and

   (ii) if I is strongly k-incomparable then I is (φ, A)-inconsistent.

   If $\varphi(x,y)$ has k-TP in T, for some $k \geq 2$, then it can be shown that some conjunction $\varphi(x,y_1) \wedge \ldots \wedge \varphi(x,y_n)$ has the tree property in T. This result follows from Theorem III.7.7 of [82] (see also [49, Proposition 2.3.10]).

2. Given $k \geq 2$, we say a formula has k-TP_1 in T if it satisfies the property obtained from k-TP by removing "strongly" in condition (ii).

   In [50, Theorem 4.1], it is shown that if a formula $\varphi(x,y)$ has k-TP_1 in T, for some $k \geq 2$, then some conjunction $\varphi(x,y_1) \wedge \ldots \wedge \varphi(x,y_n)$ has TP_1 in T.

3. Given $k \geq 2$, we say a formula has k-TP_2 in T if it satisfies the property obtained from k-TP by inserting "strongly" in condition (i).

   As in the case when $k = 2$, this is usually phrased in terms of an array. In particular, a formula has k-TP_2 in T if and only if it satisfies the property obtained from Exercise 1.1.13 by replacing "2-inconsistent" with "k-inconsistent". Using this version of the definition, it is shown in [2, Proposition 13] that if $\varphi(x,y)$ has k-TP_2 in T, for some $k \geq 2$, then some conjunction $\varphi(x,y_1) \wedge \ldots \wedge \varphi(x,y_n)$ has TP_2 in T. The proof is also given in [51, Proposition 5.7].

In each case, we see that varying the parameter k makes no difference when viewing these tree properties as properties of the theory T.

1.2 Forking and Dividing

Later results of this thesis will make extensive use of ternary relations on subsets of $\mathbb{M}$. The leading examples are those given by forking and dividing, which are indispensable tools in classification theory. In this section, we define these notions and state some basic facts.

The motivation for the definition of dividing is to capture a notion of what it means for a definable subset of $\mathbb{M}$ to be "independent" or "generic" over some set of parameters C. The notion of dividing addresses the negation of this idea, i.e., when a set is "dependent" or "non-generic", which should be regarded as a notion of smallness. Roughly speaking, a definable set A *divides* over C if there are infinitely many C-conjugates of A (i.e. images of A under automorphisms fixing C pointwise), which have relatively little intersection with each other. Since this idea is meant as a notion of smallness for sets, we define the forking sets to be those contained in the ideal generated by the dividing sets.

We now give the precise definitions, which are stated in terms of types.

Definition 1.2.1. Fix $C \subset \mathbb{M}$.

1. A partial type $\pi(\bar{x}, \bar{b})$ **divides** over C if there is a sequence $(\bar{b}^l)_{l<\omega}$ and some $k < \omega$ such that $\bar{b}^l \equiv_C \bar{b}$ for all $l < \omega$ and $\bigcup_{l<\omega} \pi(\bar{x}, \bar{b}^l)$ is k-inconsistent. A formula $\varphi(\bar{x}, \bar{b})$ **divides over** C if the partial type $\{\varphi(\bar{x}, \bar{b})\}$ divides over C.

2. A partial type $\pi(\bar{x}, \bar{b})$ **forks over** C if there are formulas $\phi_1(\bar{x}, \bar{b}^1), \ldots, \phi_n(\bar{x}, \bar{b}^n)$, each of which divides over C, such that $\pi(\bar{x}, \bar{b}) \vdash \bigvee_{i=1}^n \varphi_i(\bar{x}, \bar{b}^i)$. A formula $\varphi(\bar{x}, \bar{b})$ **forks over** C if the partial type $\{\varphi(\bar{x}, \bar{b})\}$ forks over C.

3. Define ternary relations on $A, B \subset \mathbb{M}$,

$$A \mathop{\smile\!\!\!\!\!\!|}\nolimits^d_C B \text{ if and only if } \operatorname{tp}(A/BC) \text{ does not divide over } C,$$
$$A \mathop{\smile\!\!\!\!\!\!|}\nolimits^f_C B \text{ if and only if } \operatorname{tp}(A/BC) \text{ does not fork over } C.$$

To clarify the previous notation, we remark that $\operatorname{tp}(A/BC)$ denotes $\operatorname{tp}(\bar{a}/BC)$, where $\bar{a}$ is some fixed enumeration of the set A.

The following facts are standard exercises (see e.g. [4], [82], [86]).

Fact 1.2.2.

(*a*) *A partial type $\pi(\bar{x}, \bar{b})$ divides over C if and only if there is a C-indiscernible sequence $(\bar{b}^l)_{l<\omega}$ such that $\bar{b}^0 = \bar{b}$ and $\bigcup_{l<\omega} \pi(\bar{x}, \bar{b}^l)$ is inconsistent.*

(*b*) *A partial type $\pi(\bar{x}, \bar{b})$ forks over C if and only if there some $D \supseteq \bar{b}C$ such that any extension of $\pi(\bar{x}, \bar{b})$ to a complete type over D divides over C.*

Fact 1.2.3. *Suppose $A, B, C \subset \mathbb{M}$. Then $A \mathop{\smile\!\!\!\!\!\!|}\nolimits^d_C B$ if and only if $A_0 \mathop{\smile\!\!\!\!\!\!|}\nolimits^d_C B_0$ for all finite $A_0 \subseteq A$ and $B_0 \subseteq B$.*

Fact 1.2.4. *The following are equivalent.*

(*i*) *For all $A, B, C, \subset \mathbb{M}$, $A \mathop{\smile\!\!\!\!\!\!|}\nolimits^d_C B$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}\nolimits^f_C B$.*

(*ii*) *Nondividing satisfies* ***extension****, i.e., for all $A, B, C \subset \mathbb{M}$, if $A \mathop{\smile\!\!\!\!\!\!|}\nolimits^d_C B$ and $D \supseteq BC$ then there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}\nolimits^d_C D$.*

(*iii*) *For all $A, B, C \subset \mathbb{M}$, if $A \mathop{\smile\!\!\!\!\!\!|}\nolimits^d_C B$ and $b_* \in \mathbb{M}$ is a singleton then there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}\nolimits^d_C Bb_*$.*

Proof. The equivalence of (*i*) and (*ii*) is a standard fact (see e.g. [4, Section 5]). Condition (*iii*) weakens the extension axiom by considering only extensions obtained by adding a single element. By Fact 1.2.3, and induction, this is sufficient to obtain the full extension axiom. □

1.3 Abstract Independence Relations

Continuing in the direction established by the previous section, we now consider the axiomatic treatment of arbitrary ternary relations on subsets of $\mathbb{M}$.

Definition 1.3.1.

1. A **ternary relation on** $\mathbb{M}$ is a relation $\mathop{\smile\!\!\!\!\!\!\;|}$ on triples (A,B,C), where $A,B,C \subset \mathbb{M}$. We write $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ to denote that $\mathop{\smile\!\!\!\!\!\!\;|}$ holds on the triple (A,B,C).
2. Let $\mathop{\smile\!\!\!\!\!\!\;|}^1$ and $\mathop{\smile\!\!\!\!\!\!\;|}^2$ be ternary relations on $\mathbb{M}$. We say $\mathop{\smile\!\!\!\!\!\!\;|}^1$ **implies** $\mathop{\smile\!\!\!\!\!\!\;|}^2$ if, for all $A,B,C \subset \mathbb{M}$, $A \mathop{\smile\!\!\!\!\!\!\;|}^1_C B$ implies $A \mathop{\smile\!\!\!\!\!\!\;|}^2_C B$. We say $\mathop{\smile\!\!\!\!\!\!\;|}^1$ **coincides with** $\mathop{\smile\!\!\!\!\!\!\;|}^2$ if $\mathop{\smile\!\!\!\!\!\!\;|}^1$ implies $\mathop{\smile\!\!\!\!\!\!\;|}^2$ and $\mathop{\smile\!\!\!\!\!\!\;|}^2$ implies $\mathop{\smile\!\!\!\!\!\!\;|}^1$.

Definition 1.3.2. Define the following properties of a ternary relation $\mathop{\smile\!\!\!\!\!\!\;|}$ on $\mathbb{M}$.

(i) **(invariance)** For all $A,B,C \subset \mathbb{M}$ and $\sigma \in \mathrm{Aut}(\mathbb{M})$, $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ if and only if $\sigma(A) \mathop{\smile\!\!\!\!\!\!\;|}_{\sigma(C)} \sigma(B)$.

(ii) **(symmetry)** For all $A,B,C \subset \mathbb{M}$, $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ if and only if $B \mathop{\smile\!\!\!\!\!\!\;|}_C A$.

(iii) **(full transitivity)** For all $A,B,C,D \subset \mathbb{M}$, $A \mathop{\smile\!\!\!\!\!\!\;|}_C BD$ if and only if $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!\;|}_{BC} D$.

(iv) **(finite character)** For all $A,B,C \subset \mathbb{M}$, $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ if and only if $A_0 \mathop{\smile\!\!\!\!\!\!\;|}_C B_0$ for all finite $A_0 \subseteq A$ and $B_0 \subseteq B$.

(v) **(full existence)** For all $A,B,C \subset \mathbb{M}$ there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!\;|}_C B$.

(vi) **(local character)** For all $A \subset \mathbb{M}$ there is a cardinal $\kappa(A)$ such that, for all $B \subset \mathbb{M}$, there is $C \subseteq B$, with $|C| < \kappa(A)$, such that $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$.

(vii) **(extension)** For all $A,B,C,D \subset \mathbb{M}$, with $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$, there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!\;|}_C BD$.

(viii) **(stationarity over models)** For all $A, A', B \subset \mathbb{M}$ and models $M \subset \mathbb{M}$, if $A \mathop{\smile\!\!\!\!\!\!\;|}_M B$, $A' \mathop{\smile\!\!\!\!\!\!\;|}_M B$, and $A \equiv_M A'$, then $A \equiv_{BM} A'$.

(ix) **(amalgamation over models)** For all $A,B,A',B' \subset \mathbb{M}$ and models $M \subset \mathbb{M}$, if $A \mathop{\smile\!\!\!\!\!\!\;|}_M B$, $A' \mathop{\smile\!\!\!\!\!\!\;|}_M A$, $B' \mathop{\smile\!\!\!\!\!\!\;|}_M B$, and $A' \equiv_M B'$, then there is some $D \subset \mathbb{M}$ such that $D \equiv_{AM} A'$, $D \equiv_{BM} B'$, and $D \mathop{\smile\!\!\!\!\!\!\;|}_M AB$.

Concerning axioms *(viii)* and *(ix)* in the previous definition, we will also be interested in the strengthenings obtained by replacing M with an arbitrary small subset of $\mathbb{M}$. We will refer to the resulting axioms as, respectively, *stationarity over sets* and *amalgamation over sets*. When there is no possibility for confusion, "over sets" is omitted.

It is also common to localize these two axioms to fixed base sets. In particular, given a ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ and a subset $C \subset \mathbb{M}$, we have the induced binary relation $\mathop{\smile\!\!\!\!\!\!|}_C$ on subsets of $\mathbb{M}$. We say $\mathop{\smile\!\!\!\!\!\!|}_C$ satisfies *stationarity* (resp. *amalgamation*) if the statement of axiom (*viii*) (resp. axiom (*ix*)) holds, where we remove the clause "for all models $M \subset \mathbb{M}$" and replace all other instances of M with C.

Next, we state well-known and important facts concerning stability, simplicity, and axioms of ternary relations. The overall theme of these results is that, in many ways, the notion of simplicity precisely captures when the ternary relation $\mathop{\smile\!\!\!\!\!\!|}^f$ is well-behaved, in the sense that it satisfies the axioms desirable for a "reasonable" notion of independence. Moreover, stability and simplicity can each be characterized by the existence of any ternary relation satisfying a certain collection of the axioms in Definition 1.3.2 (see Theorems 1.3.5 and 1.3.6 below).

Fact 1.3.3.

(*a*) *The following are equivalent.*

- (*i*) *T is simple.*
- (*ii*) *$\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies symmetry.*
- (*iii*) *$\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies transitivity.*
- (*iv*) *$\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies local character.*

(*b*) *Suppose T is simple. Then T is stable if and only if $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies stationarity over models.*

Proof. Part (*a*) is a result of Kim (see [48]). Part (*b*) is a standard exercise (see [93, Remark 2.6.9]). □

Definition 1.3.4.

1. A ternary relation on $\mathbb{M}$ is a **stable independence relation** if it satisfies invariance, symmetry, full transitivity, finite character, full existence, local character, and stationarity over models.

2. A ternary relation on $\mathbb{M}$ is a **simple independence relation** if it satisfies invariance, symmetry, full transitivity, finite character, full existence, local character, and amalgamation over models.

The next fact is a beautiful result of Kim and Pillay, which characterizes simplicity via the existence of any ternary relation satisfying a certain list of axioms.

Theorem 1.3.5. [52] *T is simple if and only if it has a simple independence relation. Moreover, if T has a simple independence relation $\mathop{\smile\!\!\!\!\!\!|}$, then $\mathop{\smile\!\!\!\!\!\!|}$, $\mathop{\smile\!\!\!\!\!\!|}^f$, and $\mathop{\smile\!\!\!\!\!\!|}^d$ all coincide.*

Using this result, we recover a similar characterization for stability, which was first proved by Harnik and Harrington [34].

Theorem 1.3.6. *T is stable if and only if T has a stable independence relation. Moreover, if T has a stable independence relation $\mathop{\smile\!\!\!\!\!\!|}\;$, then $\mathop{\smile\!\!\!\!\!\!|}\;$, $\mathop{\smile\!\!\!\!\!\!|}^f$, and $\mathop{\smile\!\!\!\!\!\!|}^d$ all coincide.*

Proof. This follows from Theorem 1.3.5, Fact 1.3.3(b), and the fact, which is a good exercise, that any ternary relation $\mathop{\smile\!\!\!\!\!\!|}\;$ satisfying extension and stationarity over models also satisfies amalgamation over models. □

We can also use Fact 1.3.3 to motivate the definition of supersimplicity.

Definition 1.3.7. T is **supersimple** if, for all $A, B \subset \mathbb{M}$, with A finite, there is some finite subset $C \subseteq B$ such that $A \mathop{\smile\!\!\!\!\!\!|}^f_C B$.

By Fact 1.3.3(a)[(i) $\Leftrightarrow$ (iv)], it is clear that supersimple theories are simple. We also have the following characterization.

Fact 1.3.8. [14, Theorem 13.25][3] *T is superstable if and only if it is stable and supersimple.*

The inclusion of local character in the definitions of stable and simple independence relations has serious ramifications. In particular, the following variations of the previous notions are obtained by removing the local character axiom from Definition 1.3.4.

Definition 1.3.9. Fix a ternary relation $\mathop{\smile\!\!\!\!\!\!|}\;$ on $\mathbb{M}$.

1. Given $C \subset \mathbb{M}$, we say $\mathop{\smile\!\!\!\!\!\!|}\;$ is a **stationary independence relation over** C if $\mathop{\smile\!\!\!\!\!\!|}\;$ satisfies invariance, symmetry, full transitivity, finite character, and full existence, and, moreover, $\mathop{\smile\!\!\!\!\!\!|}_C$ satisfies stationarity.

   We say $\mathop{\smile\!\!\!\!\!\!|}\;$ is a **stationary independence relation** if it is a stationary independence relation over C for all $C \subset \mathbb{M}$.

2. Given $C \subset \mathbb{M}$, we say $\mathop{\smile\!\!\!\!\!\!|}\;$ is an **amalgamation independence relation over** C if $\mathop{\smile\!\!\!\!\!\!|}\;$ satisfies invariance, symmetry, full transitivity, finite character, and full existence, and, moreover, $\mathop{\smile\!\!\!\!\!\!|}_C$ satisfies amalgamation.

   We say $\mathop{\smile\!\!\!\!\!\!|}\;$ is an **amalgamation independence relation** if it is an amalgamation independence relation over C for all $C \subset \mathbb{M}$.

[3] In [14], superstability is defined as "stable and supersimple". Theorem 13.25 proves the equivalence with our definition.

The notion of a stationary independence relation was formulated in [88] by Tent and Ziegler, who consider ternary relations defined on finite subsets of a countable structure. In particular, Tent and Ziegler do not include the finite character axiom.

As it turns out, there are many unstable (and even non-simple) theories with a stationary independence relation. Standard examples include the random graph, the Henson graphs, and the rational Urysohn space (which is the motivating example for Tent and Ziegler in [88]). However, in contrast to Theorem 1.3.6, we have the following easy observation.

Proposition 1.3.10. *If $\mathop{\smile\!\!\!\!\!\!|}^f$ is a stationary independence relation on $\mathbb{M}$ then T is stable.*

Proof. By assumption, $\mathop{\smile\!\!\!\!\!\!|}^f$ is symmetric and so T is simple by Fact 1.3.3(a). Therefore T is stable by Fact 1.3.3(b). □

In other words, if T is unstable and $\mathop{\smile\!\!\!\!\!\!|}$ is a stationary independence relation, then $\mathop{\smile\!\!\!\!\!\!|}$ must disagree with $\mathop{\smile\!\!\!\!\!\!|}^f$. However, we can still recover some relationship to nonforking via the following fact, which was observed in joint work with Terry [26]. The proof is a nice exercise, in the style of Adler [4], involving the manipulation of axioms of ternary relations.

Proposition 1.3.11. *Suppose $C \subset \mathbb{M}$ and $\mathop{\smile\!\!\!\!\!\!|}$ is a stationary independence relation over C. Then, for any $A, B \subset \mathbb{M}$, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ implies $A \mathop{\smile\!\!\!\!\!\!|}^f_C B$ and $B \mathop{\smile\!\!\!\!\!\!|}^f_C A$.*

Proof. Since $\mathop{\smile\!\!\!\!\!\!|}$ satisfies symmetry it suffices to show $\mathop{\smile\!\!\!\!\!\!|}_C$ implies $\mathop{\smile\!\!\!\!\!\!|}^f_C$. We first show $\mathop{\smile\!\!\!\!\!\!|}_C$ satisfies extension (which is essentially the content of [4, Remark 1.2(3)]). Fix A, B, D such that $A \mathop{\smile\!\!\!\!\!\!|}_C B$. We want to find $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C BD$. By full existence, there is $A'' \equiv_C A$ such that $A'' \mathop{\smile\!\!\!\!\!\!|}_C BC$, and so $A'' \mathop{\smile\!\!\!\!\!\!|}_C B$ by monotonicity. By stationarity over C, we have $A''B \equiv_C AB$, and so $A \mathop{\smile\!\!\!\!\!\!|}_C BC$ by invariance. Next, by full existence, there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_{BC} BD$. By invariance, we also have $A' \mathop{\smile\!\!\!\!\!\!|}_C BC$, and so $A' \mathop{\smile\!\!\!\!\!\!|}_C BD$ by transitivity.

Finally, since $\mathop{\smile\!\!\!\!\!\!|}_C$ satisfies extension, it suffices to show $\mathop{\smile\!\!\!\!\!\!|}_C$ implies $\mathop{\smile\!\!\!\!\!\!|}^d_C$. For this, suppose $A \mathop{\smile\!\!\!\!\!\!|}_C B$. Fix a C-indiscernible sequence $(B_i)_{i<\omega}$, with $B_0 = B$. We want to find A' such that $A'B_i \equiv_C AB$ for all $i < \omega$. By full existence there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C \bigcup_{i<\omega} B_i$. By monotonicity, we have $A' \mathop{\smile\!\!\!\!\!\!|}_C B_i$ for all $i < \omega$. Given $i < \omega$, fix A_i such that $A_i B_i \equiv_C AB$. For any $i < \omega$, we have $A_i \equiv_C A \equiv_C A'$ and, by invariance, $A_i \mathop{\smile\!\!\!\!\!\!|}_C B_i$. By stationarity over C, we have $A'B_i \equiv_C A_i B_i \equiv_C AB$ for all $i < \omega$, as desired. □

An interesting question, which we have not been able to answer, is the following weakening of Proposition 1.3.11.

Question 1.3.12. Suppose $C \subset \mathbb{M}$ and $\mathop{\smile\!\!\!\!\!\!|}$ is an amalgamation independence relation over C. Is it true that $\mathop{\smile\!\!\!\!\!\!|}_C$ is stronger than $\mathop{\smile\!\!\!\!\!\!|}^f_C$?

We also note that Adler [3] defines a theory T to be *mock stable* if there is a ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ on $\mathbb{M}^{\mathrm{eq}}$ such that $\mathop{\smile\!\!\!\!\!\mid}_M$ is a stationary independence relation for all models M. Similarly, T is *mock simple* if there is a ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ on $\mathbb{M}^{\mathrm{eq}}$ such that $\mathop{\smile\!\!\!\!\!\mid}_M$ is an amalgamation independence relation for all models M (see also [50]).

1.4 The Strong Order Property

In this section, we examine the *strong order properties*, which were first defined in [83]. Our formulations are also influenced by [3]. To motivate these definitions, consider a partial type $p(\bar{x},\bar{y})$ (possibly over parameters), with $\ell(\bar{x}) = \ell(\bar{y})$. Then p induces a directed graph structure on $\mathbb{M}^{\ell(\bar{x})}$, consisting of pairs $(\bar{a},\bar{b})$ such that $\mathbb{M} \models p(\bar{a},\bar{b})$. The following hierarchy of strong order properties is defined from combinatorial complexity arising in this directed graph structure.

Definition 1.4.1. Suppose $p(\bar{x},\bar{y})$ is a partial type (possibly over parameters), with $\ell(\bar{x}) = \ell(\bar{y})$.

1. $p(\bar{x},\bar{y})$ **admits infinite chains** if there is a sequence $(\bar{a}^l)_{l<\omega}$ such that $\mathbb{M} \models p(\bar{a}^l,\bar{a}^m)$ for all $l < m$.

2. Given $n > 0$, $p(\bar{x},\bar{y})$ is n**-cyclic** if

$$p(\bar{x}^1,\bar{x}^2) \cup p(\bar{x}^2,\bar{x}^3) \cup \ldots \cup p(\bar{x}^{n-1},\bar{x}^n) \cup p(\bar{x}^n,\bar{x}^1)$$

   is consistent.

Definition 1.4.2.

1. Given $n \geq 3$, a type $p(\bar{x},\bar{y})$ has the n**-strong order property in** T if it admits infinite chains and is not n-cyclic.

   T has the n**-strong order property**, SOP_n, if there is a type with the n-strong order property in T.

2. T has the **strong order property**, SOP_ω, if there is a type $p(\bar{x},\bar{y})$ with the n-strong order property in T for all $n > 0$. If $\ell(\bar{x})$ is finite then we say **finitary strong order property**, FSOP. If $p(\bar{x},\bar{y})$ is a formula then we say **fully finite strong order property**, FFSOP.

Note that, if $n < \omega$ and T has SOP_n then, by compactness, there is a formula $\varphi(\bar{x},\bar{y})$ with the n-strong order property in T. However, we emphasize that this same compactness argument does not work to glean FFSOP from SOP_ω, and these two properties are indeed inequivalent (we will see examples of this in Chapter 3). We also mention that Adler [3] has formulated many other variations of the strict and strong order properties (for example, using partial orders).

Remark 1.4.3.

1. The acronym SOP_ω is not standard for the strong order property. Many sources use SOP for the strong order property, while others use this for the strict order property (as we have done here). We have chosen the subscript ω for the strong order property because a straightforward exercise shows that T has the strong order property if and only if it has SOP_n for all $n \geq 3$.

2. In [28], the acronyms SOP_1 and SOP_2 are assigned to two properties, which are not defined in the same way as SOP_n above for $n \geq 3$, but rather as stronger versions of the tree property. It is a straightforward exercise to show that, as a property of formulas, SOP_2 is equivalent to TP_1, and moreover, SOP_1 implies the tree property (see e.g. [50]).

The following reformulation of the strict order property is a standard exercise.

Exercise 1.4.4. *T has the strict order property if and only if there is a formula $\varphi(\bar{x}, \bar{y})$, with $\ell(\bar{x}) = \ell(\bar{y})$, such that*

(i) $\varphi(\bar{x}, \bar{y})$ is a partial order on $\mathbb{M}^{\ell(\bar{x})}$, and

(ii) there is an infinite sequence $(\bar{a}^l)_{l<\omega}$ such that $\mathbb{M} \models \varphi(\bar{a}^l, \bar{a}^m)$ for all $l \leq m$.

Comparing this fact to the definition of FFSOP, we easily obtain the following corollary.

Corollary 1.4.5. *If T has the strict order property then it is* FFSOP*.*

This places the strict order property above the SOP_n-hierarchy, in terms of complexity. On the other hand if a formula $\varphi(\bar{x}, \bar{y})$ omits n-cycles for all $n > 0$ then, by taking the transitive closure of φ, one obtains an $\bigvee$-definable partial order. Therefore, the SOP_n-hierarchy can be viewed as a yardstick measuring how close T comes to having the strict order property.

Below the hierarchy, we have that SOP_3 implies TP_1. This is shown in [28] (modulo the equivalence of SOP_2 and TP_1).

We have now defined (or at least referenced) every syntactic dividing line of present interest. Therefore, we include a diagram of implications (Figure 1).

$$
\begin{array}{ccccccccccccc}
\text{SOP} & \Rightarrow & \text{FFSOP} & \Rightarrow & \text{FSOP} & \Rightarrow & \text{SOP}_\omega & \Rightarrow & \text{SOP}_{n+1} & \Rightarrow & \text{SOP}_n & \Rightarrow & \dots \\
 & & \dots & \Rightarrow & \text{SOP}_3 & \Rightarrow & \text{TP}_1 & \Rightarrow & \text{SOP}_1 & \Rightarrow & \text{TP} & \Rightarrow & \text{OP} \\
 & & & & & & & & & & \Uparrow & & \Uparrow \\
 & & & & & & & & & & \text{TP}_2 & \Rightarrow & \text{IP}
\end{array}
$$

Figure 1: Implications between dividing lines.

In Figure 1, TP is for *tree property*, OP is for *order property*, and $n \geq 3$. All of the implications in Figure 1 are known to be strict, except for each of the implications in the chain $\mathrm{SOP}_3 \Rightarrow \mathrm{TP}_1 \Rightarrow \mathrm{SOP}_1$.[4]

We now return to the SOP_n-hierarchy. An easy observation is that, given $n = 1$ or $n = 2$, if one were to define properties analogous to SOP_n for $n \geq 3$ then, as properties of T, those definitions would yield, respectively, "T has an infinite model" and "T has the order property". Therefore, to streamline the notation and attempt to resolve the confusion regarding acronyms, we will use the following definition.

Definition 1.4.6. Let T be a complete first-order theory. We define $\mathrm{SO}(T)$, the **strong order rank of** T, as follows:

(i) $\mathrm{SO}(T) = 0$ if T has finite models;

(ii) $\mathrm{SO}(T) = 1$ if T has infinite models, but does not have the order property;

(iii) $\mathrm{SO}(T) = 2$ if T has the order property, but does not have SOP_3;

(iv) given $n \geq 3$, $\mathrm{SO}(T) = n$ if T has SOP_n, but does not have SOP_{n+1};

(v) $\mathrm{SO}(T) = \omega$ if T has SOP_ω, but does not have FSOP;

(vi) $\mathrm{SO}(T) = \infty$ if T has FSOP.

At first glance, it might seem as though we are simply cherry-picking other definitions to artificially force a rank on first-order theories. However, by reformulating previous notions in terms of indiscernible sequences, we can smooth out the definition of strong order rank.

Definition 1.4.7. Suppose $\mathcal{I} = (\bar{a}^l)_{l<\omega}$ is an indiscernible sequence in $\mathbb{M}$.

1. Given $n > 0$ and $C \subset \mathbb{U}$, $\mathcal{I}$ is n-**cyclic over** C if $\mathrm{tp}(\bar{a}^0, \bar{a}^1/C)$ is n-cyclic.

   If $C = \emptyset$ then we say $\mathcal{I}$ is n-**cyclic**.

2. Define the set of **non-parameter indices of** $\mathcal{I}$

$$\mathrm{NP}(\mathcal{I}) := \{i \in \ell(\bar{a}^0) : a_i^0 \neq a_i^1\}.$$

The following proposition gives a uniformization of strong order rank in terms of cyclic indiscernible sequences. For clarity, we extend the ordering on $\mathbb{N}$ to $\mathbb{N} \cup \{\omega, \infty\}$ by setting $\omega < \infty$, and $n < \omega$ for all $n \in \mathbb{N}$.

[4] The implication $\mathrm{SOP}_1 \Rightarrow \mathrm{TP}$ was also, until very recently, not known to be strict. In particular, consider T^*_{feq}, the model completion of the theory of parameterized equivalence relations. This theory was shown in [83] to be non-simple and NSOP_3. In [84], it is claimed to be NSOP_1, although errors were found in the proof. The strategy of the proof was recovered in [35] to show NTP_1. Very recently, a new proof of NSOP_1 for T^*_{feq}, as well as for other theories long claimed to be NSOP_1 (see e.g. [50]), was given in [19].

Proposition 1.4.8.

(a) *Given $n > 0$, $\mathrm{SO}(T) < n$ if and only if every indiscernible sequence in $\mathbb{M}$ is n-cyclic.*

(b) *The following are equivalent.*

- (i) *T is NFSOP (i.e. $\mathrm{SO}(T) \leq \omega$).*
- (ii) *For any indiscernible sequence $\mathcal{I} = (\bar{a}^l)_{l<\omega}$, if $\mathrm{NP}(\mathcal{I})$ finite then $\mathcal{I}$ is n-cyclic for some $n > 0$.*
- (iii) *For any $C \subset \mathbb{M}$, and any C-indiscernible sequence $\mathcal{I} = (\bar{a}^l)_{l<\omega}$, if $\ell(\bar{a}^0)$ finite then $\mathcal{I}$ is n-cyclic over C for some $n > 0$.*

Proof. Part (a). Suppose $\mathrm{SO}(T) \geq n$. Then T has SOP_n, witnessed by a type $p(\bar{x}, \bar{y})$ and a sequence $(\bar{a}^l)_{l<\omega}$. By a standard application of Ramsey's theorem and the Ehrenfeucht-Mostowski type (see e.g. [86, Lemma 7.1.1]), we may assume $(\bar{a}^l)_{l<\omega}$ is indiscernible. We have $p(\bar{x}, \bar{y}) \subseteq \mathrm{tp}(\bar{a}^0, \bar{a}^1)$, and so $(\bar{a}^l)_{l<\omega}$ is not n-cyclic.

Conversely, if there is an indiscernible sequence $(\bar{a}^l)_{l<\omega}$, which is not n-cyclic, then this sequence, together with the type $\mathrm{tp}(\bar{a}^0, \bar{a}^1)$ witnesses SOP_n for T. Therefore $\mathrm{SO}(T) \geq n$.

Part (b). $(i) \Rightarrow (ii)$: Suppose (ii) fails. Then there is an indiscernible sequence $\mathcal{I} = (\bar{a}^l)_{l<\omega}$ such that $\mathrm{NP}(\mathcal{I})$ is finite and $\mathcal{I}$ is not n-cyclic for any $n > 0$. Suppose $\ell(\bar{a}^0) = \lambda$. Without loss of generality there is some $k < \omega$ such that $i \notin \mathrm{NP}(\mathcal{I})$ for all $k < i < \lambda$. Let $C = (a^0_i : k < i < \lambda)$, $\bar{b}^l = (a^l_0, \ldots, a^l_k)$, and $p(\bar{x}, \bar{y}) = \mathrm{tp}(\bar{b}^0, \bar{b}^1/C)$. Note that $(\bar{b}^l)_{l<\omega}$ is C-indiscernible, and so $p(\bar{x}, \bar{y})$ has the n-strong order property for all $n > 0$. Since $\ell(\bar{x})$ is finite, T has FSOP.

$(ii) \Rightarrow (iii)$: Assume (ii) and suppose we have $C \subset \mathbb{M}$ and a C-indiscernible sequence $(\bar{a}^l)_{l<\omega}$, with $\ell(\bar{a}^0)$ finite. Let $\bar{c}$ be an enumeration of C and set $\bar{b}^l = (\bar{a}^l, \bar{c})$. Then $\mathcal{I} = (\bar{b}^l)_{l<\omega}$ is an indiscernible sequence and $\mathrm{NP}(\mathcal{I})$ is finite. Therefore $\mathcal{I}$ is n-cyclic for some $n > 0$. Let $(\bar{z}^1, \ldots, \bar{z}^n)$ witness that $\mathcal{I}$ is n-cyclic. Let $k = \ell(\bar{a}^0)$ and set $\bar{c}' = \bar{z}^0 \backslash \{z^0_1, \ldots, z^0_k\}$. Then $\bar{c}' = \bar{z}^l \backslash \{z^l_1, \ldots, z^l_k\}$ for all $l < \omega$ and $\bar{c}' \equiv \bar{c}$. If $\sigma \in \mathrm{Aut}(\mathbb{M})$ is such that $\sigma(\bar{c}') = \bar{c}$, then $(\sigma(\bar{z}^0) \backslash \bar{c}, \ldots, \sigma(\bar{z}^n) \backslash \bar{c})$ witnesses that $(\bar{a}^l)_{l<\omega}$ is n-cyclic over C.

$(iii) \Rightarrow (i)$: Suppose T has FSOP, witnessed by a type $p(\bar{x}, \bar{y})$ and a sequence $(\bar{a}^l)_{l<\omega}$, with $\ell(\bar{a}^0)$ finite. Let C be the set of parameters appearing in $p(\bar{x}, \bar{y})$. As in part (a), we may assume $(\bar{a}^l)_{l<\omega}$ is C-indiscernible. Then $p(\bar{x}, \bar{y}) \subseteq \mathrm{tp}(\bar{a}^0, \bar{a}^1/C)$, and so $(\bar{a}^l)_{l<\omega}$ is not n-cyclic over C. □

The use of non-parameter indices $\mathrm{NP}(\mathcal{I})$ in indiscernible sequences is simply a technical way to avoid worrying about types over parameters. This will be used in Chapter 3, where we give a large class of NFSOP theories.

For later reference, we reiterate the positions of stability and simplicity with respect to strong order rank.

Fact 1.4.9.

(a) [82, Theorem II.2.2] *T is stable if and only if* $\mathrm{SO}(T) \leq 1$.

(b) [83, Claim 2.7] *If T is simple then* $\mathrm{SO}(T) \leq 2$.

Combining this fact with the $n = 2$ case of Proposition 1.4.8(a), we recover the standard fact that a theory is stable if and only if every indiscernible sequence is an indiscernible set (see e.g. [70, Exercise 7.41]).

1.5 Imaginaries and Hyperimaginaries

In this section, we briefly summarize the basic notions surrounding imaginaries and hyperimaginaries.

Suppose $E(\bar{x}, \bar{y})$ is an equivalence relation on $\mathbb{M}^{\ell(\bar{x})}$. We say E is 0-*invariant* if, for any $\sigma \in \mathrm{Aut}(\mathbb{M})$ and $\bar{a}, \bar{b} \in \mathbb{M}^{\ell(\bar{x})}$, $E(\bar{a}, \bar{b})$ implies $E(\sigma(\bar{a}), \sigma(\bar{b}))$. We say E is 0-*type-definable* (resp. 0-*definable*) if $E(\bar{x}, \bar{y})$ is equivalent (in $\mathbb{M}^{\ell(\bar{x})}$) to a type (resp. formula) without parameters. Given $\bar{a} \in \mathbb{M}^{\ell(\bar{x})}$, we let $[\bar{a}]_E$ denote the equivalence class of $\bar{a}$ modulo E.

$\mathbb{M}^{\mathrm{heq}}$ denotes the class consisting of equivalence classes $[\bar{a}]_E$, where E is any 0-type-definable equivalence relation and $\bar{a} \in \mathbb{M}^{\ell(\bar{x})}$. When viewing equivalence classes $[\bar{a}]_E$ as elements of M^{heq} (versus subsets of $\mathbb{M}^{\ell(\bar{x})}$), we use the notation $\bar{a}_E$. Elements of $\mathbb{M}^{\mathrm{heq}}$ are *hyperimaginaries*. If E is 0-definable, then $\bar{a}_E$ is an *imaginary*. We let $\mathbb{M}^{\mathrm{eq}}$ denote the subclass of imaginaries in $\mathbb{M}^{\mathrm{heq}}$. Note that any singleton $a \in \mathbb{M}$ can be identified with the imaginary a_E, where E is equality. Therefore, $\mathbb{M}$ is naturally a subclass of $\mathbb{M}^{\mathrm{eq}}$.

Given a hyperimaginary $\bar{a}_E \in \mathbb{M}^{\mathrm{heq}}$ and an automorphism $\sigma \in \mathrm{Aut}(\mathbb{M})$, we define $\sigma(\bar{a}_E) = \sigma(\bar{a})_E$. Note that this is well-defined by 0-invariance of E. With this notation, we may extend many common notions to $\mathbb{M}^{\mathrm{heq}}$.

Definition 1.5.1. Fix $C \subset \mathbb{M}^{\mathrm{heq}}$.

1. Let $\mathrm{Aut}(\mathbb{M}/C) = \{\sigma \in \mathrm{Aut}(\mathbb{M}) : \sigma(e) = e$ for all $c \in C\}$.

2. Given $e \in \mathbb{M}^{\mathrm{heq}}$, define $\mathcal{O}(e/C) = \{\sigma(e) : \sigma \in \mathrm{Aut}(\mathbb{M}/C)$.

3. Define

$$\mathrm{acl}^{\mathrm{heq}}(C) = \{e \in \mathbb{M}^{\mathrm{heq}} : \mathcal{O}(e/C) \text{ is finite}\},$$
$$\mathrm{dcl}^{\mathrm{heq}}(C) = \{e \in \mathbb{M}^{\mathrm{heq}} : \mathcal{O}(e/C) = \{e\}\}.$$

4. If $C \subset \mathbb{M}^{\mathrm{eq}}$ then we define

$$\mathrm{acl}^{\mathrm{eq}}(C) = \mathrm{acl}^{\mathrm{heq}}(C) \cap \mathbb{M}^{\mathrm{eq}} \text{ and } \mathrm{dcl}^{\mathrm{eq}}(C) = \mathrm{dcl}^{\mathrm{heq}}(C) \cap \mathbb{M}^{\mathrm{eq}}.$$

We now define elimination of hyperimaginaries for the theory T.

Definition 1.5.2. T has **elimination of hyperimaginaries** if, for any hyperimaginary e, there is a sequence $(e_i)_{i\in I}$ if imaginaries such that

$$\operatorname{dcl}^{\mathrm{heq}}(e) = \operatorname{dcl}^{\mathrm{heq}}(\{e_i : i \in I\}).$$

We will use the following reformulation of this notion.

Proposition 1.5.3. [14, Corollary 18.3] *The following are equivalent.*

(*i*) *T has elimination of hyperimaginaries.*

(*ii*) *Let $E(\bar{x}, \bar{y})$ be a 0-type-definable equivalence relation, with $\bar{x} = (x_i)_{i<\mu}$, and fix a real tuple $\bar{a} = (a_i)_{i<\mu}$. Then there is a sequence $(E_i(\bar{x}^i, \bar{x}^i))_{i<\lambda}$ of 0-definable n_i-ary equivalence relations, with $\bar{x}^i = (x^i_{j_1}, \ldots, x^i_{j_{n_i}})$ and $\bar{y}^i = (y^i_{j_1}, \ldots, y^i_{j_{n_i}})$ for some $j_1 < \ldots < j_{n_i} < \mu$, such that, for all $\bar{b}, \bar{b}' \models \operatorname{tp}(\bar{a})$, $E(\bar{b}, \bar{b}')$ holds if and only if $E_i(\bar{b}, \bar{b}')$ holds for all $i < \lambda$.*

Next, we define elimination of imaginaries and weak elimination of imaginaries.

Definition 1.5.4.

1. Given an imaginary $e \in \mathbb{M}^{\mathrm{eq}}$, a **canonical parameter** (resp. **weak canonical parameter**) for e is a finite real tuple $\bar{c} \in \mathbb{M}^{\ell(\bar{c})}$ such that $\bar{c} \in \operatorname{dcl}^{\mathrm{eq}}(e)$ (resp. $\bar{c} \in \operatorname{acl}^{\mathrm{eq}}(e)$) and $e \in \operatorname{dcl}^{\mathrm{eq}}(\bar{c})$.

2. T has **elimination of imaginaries** (resp. **weak elimination of imaginaries**) if every imaginary has a canonical parameter (resp. weak canonical parameter).

One reason to distinguish elimination of imaginaries and weak elimination of imaginaries is that many nice theories fail elimination of imaginaries simply because finite sets do not have canonical parameters. This is especially true when considering homogeneous combinatorial structures with symmetric relations, such as a countably infinite set (in the empty language) or the countable random graph (in the graph language). In both cases, the failure of elimination of imaginaries is a consequence of the following general observation.

Lemma 1.5.5. *Let $\mathbb{M}$ be a monster model of a complete first-order theory T. Assume $\operatorname{acl}(C) = C$ for all $C \subset \mathbb{M}$. Fix $n > 1$. Given $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{M}^n$ and $f \in \operatorname{Sym}(1, \ldots, n)$, let $\bar{a}^f = (a_{f(1)}, \ldots, a_{f(n)})$. Let E_n be the 0-definable equivalence relation on $\mathbb{M}^n$ such that, given $\bar{a}, \bar{b} \in \mathbb{M}^n$,*

$$E_n(\bar{a}, \bar{b}) \;\;\Leftrightarrow\;\; \bar{b} = \bar{a}^f \text{ for some } f \in \operatorname{Sym}(1, \ldots, n).$$

Suppose $\bar{a} \in \mathbb{M}^n$ is a tuple of pairwise distinct elements such that $\bar{a}^f \equiv \bar{a}$ for all $f \in \operatorname{Sym}(1, \ldots, n)$. Then $\bar{a}_{E_n}$ does not have a canonical parameter.

Proof. Fix $n > 1$ and E_n as in the statement. Let $\bar{a} \in \mathbb{M}^n$ be a tuple of distinct elements such that $\bar{a}^f \equiv \bar{a}$ for all $f \in \mathrm{Sym}(1, \ldots, n)$. Let $e = \bar{a}_{E_n}$. For any $f \in \mathrm{Sym}(1, \ldots, n)$, we may fix $\sigma_f \in \mathrm{Aut}(\mathbb{M})$ such that $\sigma_f(\bar{a}) = \bar{a}^f$. For any other $g \in \mathrm{Sym}(1, \ldots, n)$, we have $\sigma_f(\bar{a}^g) = \bar{a}^{fg} \sim_{E_n} a$ and $\sigma_f^{-1}(\bar{a}^g) = \bar{a}^{f^{-1}g} \sim_{E_n} a$. Therefore $\sigma_f(e) = e$.

Suppose, toward a contradiction, that $\bar{c}$ is a canonical parameter for e. Then $\bar{c} \in \mathrm{dcl}^{\mathrm{eq}}(e)$ and so, since $\sigma_f(e) = e$ for all f, we have $\sigma_f(\bar{c}) = \bar{c}$ for all f.

Case 1: $\bar{c}$ contains a_i for some $1 \leq i \leq n$.

Since $n > 1$ we may fix $j \neq i$ and let f be a permutation of $\{1, \ldots, n\}$ such that $f(i) = j$. Then $\sigma_f(\bar{c}) = \bar{c}$ implies $a_i = a_j$, which is a contradiction.

Case 2: $\bar{c}$ is disjoint from $\bar{a}$.

Since $\mathrm{acl}(\bar{c}) = \bar{c}$, it follows that $\mathrm{tp}(\bar{a}/\bar{c})$ has infinitely many realizations, and so we may fix $\bar{a}' \equiv_{\bar{c}} \bar{a}$ such that $\bar{a}' \backslash \bar{a} \neq \emptyset$. If $\sigma \in \mathrm{Aut}(\mathbb{M}/\bar{c})$ is such that $\sigma(\bar{a}) = \bar{a}'$ then, by assumption, $\sigma(e) = e$ and so $E_n(\bar{a}, \bar{a}')$ holds, which is a contradiction. $\square$

Chapter 2

Distance Structures for Generalized Metric Spaces

2.1 Introduction

The fundamental objects of interest in this thesis are metric spaces. Specifically, we study the behavior of metric spaces as combinatorial structures in relational languages. This is the setting of a vast body of literature (e.g. [15], [27], [69], [85], [87], [88]) focusing on topological dynamics of automorphism groups and Ramsey properties of countable homogeneous structures. Our goal is to develop the model theory of metric spaces in this setting. We face the immediate obstacle that the notion of "metric space" is not very well controlled by classical first-order logic, in the sense that models of the theory of a metric space need not be metric spaces. Indeed, this is a major motivation for working in continuous logic and model theory for *metric structures*, which are always complete metric spaces with the metric built into the logic (see [8]). However, we wish to study the model theory of (possibly incomplete) metric spaces treated as combinatorial structures (specifically, labeled graphs where complexity is governed by the triangle inequality). In some sense, we will sacrifice the global topological structure of metric spaces for the sake of understanding local combinatorial complexity. Moreover, our results will uncover and exploit the relationship between this complexity and the algebraic structure of distance sets.

Another benefit of our framework is that it is flexible enough to encompass generalized metric spaces with distances in arbitrary ordered additive structures. This setting appears often in the literature, with an obvious example of extracting a metric from a valuation. Other examples include [67], where Narens considers topological spaces "metrizable" by a generalized metric over an ordered abelian group, as well as [66], where Morgan and Shalen use metric spaces over ordered abelian groups to generalize the notion of an $\mathbb{R}$-tree. Finally, in [15], Casanovas and Wagner use the phenomenon of "infinitesimal distance" to construct a theory without the strict order property that does not eliminate hyperimaginaries. We will analyze this example at the end of Section 2.9.

We will consider metric spaces as first-order *relational* structures, with binary relations given by distance inequalities However, when working directly with metric spaces as mathematical objects outside of this first-order setting, it will usually be much more convenient to treat these spaces as consisting of a set of points together with a distance function into a set of distances. For example, this will be especially true when making definitions involving metric spaces or manipulating distances in a particular metric space. Moreover, most of our results will crucially depend on a careful analysis of a certain algebraic structure defined on sets of distances appearing in metric spaces. Altogether, there will many possible sources of confusion regarding our precise first-order context. Therefore, we will endeavor to explain this context in full detail. This explanation requires the following basic definitions. The reader should take these definitions at face value, and refrain from analyzing the first-order setting until the discussion following Definition 2.1.3.

Definition 2.1.1. Let $\mathcal{L}_{\text{om}} = \{\oplus, \leq, 0\}$ be a first-order language, where $\oplus$ is a binary function symbol, $\leq$ is a binary relation symbol, and 0 is a constant symbol. We refer to $\mathcal{L}_{\text{om}}$ as the **language of ordered monoids**. Fix an $\mathcal{L}_{\text{om}}$-structure $\mathcal{R} = (R, \oplus, \leq, 0)$.

1. $\mathcal{R}$ is a **distance magma** if

   (*i*) (*totality*) $\leq$ is a total order on R;

   (*ii*) (*positivity*) $r \leq r \oplus s$ for all $r, s \in R$;

   (*iii*) (*order*) for all $r, s, t, u \in R$, if $r \leq t$ and $s \leq u$ then $r \oplus s \leq t \oplus u$;

   (*iv*) (*commutativity*) $r \oplus s = s \oplus r$ for all $r, s \in R$;

   (*v*) (*unity*) $r \oplus 0 = r = 0 \oplus r$ for all $r \in R$.

2. $\mathcal{R}$ is a **distance monoid** if it is a distance magma and

   (*vi*) (*associativity*) $(r \oplus s) \oplus t = r \oplus (s \oplus t)$ for all $r, s, t \in R$.

Remark 2.1.2. Recall that, according to [10], a *magma* is simply a set together with a binary operation. After consulting standard literature on ordered algebraic structures (e.g. [21]), one might refer to a distance magma as a *totally and positively ordered commutative unital magma*, and a distance monoid as a *totally and positively ordered commutative monoid*. So our terminology is partly chosen for the sake of brevity. We are separating the associativity axiom because it is not required for our initial results and, moreover, associativity often characterizes some useful combinatorial property of metric spaces (see Propositions 2.6.3(e), 2.7.9, and 2.7.16).

Next, we observe that the notion of a distance magma allows for a reasonable definition of a generalized metric space. Definitions of a similar flavor can be found in [5], [66], and [67].

Definition 2.1.3. Suppose $\mathcal{R} = (R, \oplus, \leq, 0)$ is a distance magma. Fix a set A and a function $d : A \times A \longrightarrow R$. We call (A, d) an **$\mathcal{R}$-colored space**, and define the **spectrum of** (A, d), denoted $\text{Spec}(A, d)$, to be the image of d. Given an $\mathcal{R}$-colored space (A, d), we say d is an **$\mathcal{R}$-metric on** A if

(*i*) for all $x, y \in A$, $d(x, y) = 0$ if and only if $x = y$;

(*ii*) for all $x, y \in A$, $d(x, y) = d(y, x)$;

(*iii*) for all $x, y, z \in A$, $d(x, z) \leq d(x, y) \oplus d(y, z)$.

In this case, (A, d) is an **$\mathcal{R}$-metric space**.

We now detail the first-order setting of this thesis. Given a distance magma $\mathcal{R}$, we want to interpret $\mathcal{R}$-metric spaces as first-order structures in a relational language. Moreover, we will often want to restrict our attention to a specific set of distances in $\mathcal{R}$. Altogether, given a distance magma $\mathcal{R} = (R, \oplus, \leq, 0)$ and a fixed subset $S \subseteq R$, with $0 \in S$, we define a first-order language

$$\mathcal{L}_S = \{d(x, y) \leq s : s \in S\},$$

where, given $s \in S$, $d(x, y) \leq s$ is a binary relation symbol.

If $\mathcal{R}$ is a distance magma, and d is an $\mathcal{R}$-metric on a set A, we will use the notation $\mathcal{A}$ to refer to the $\mathcal{R}$-metric space (A, d). We use the phrase "generalized metric space" to refer to the class of all $\mathcal{R}$-metric spaces, where $\mathcal{R}$ is any distance magma. We will also frequently use the notation (A, d_A) in order to distinguish semantic statements about the metric on A from formulas in a relational language of the kind discussed above.

Finally, we describe the interpretation of $\mathcal{R}$-metric spaces as relational structures. For later purposes, this done in the more general setting of $\mathcal{R}$-colored spaces. Fix a distance magma $\mathcal{R} = (R, \oplus, \leq, 0)$ and a subset $S \subseteq R$, with $0 \in S$. Given an $\mathcal{R}$-colored space $\mathcal{A} = (A, d_A)$, we interpret $\mathcal{A}$ as an $\mathcal{L}_S$-structure by interpreting the symbol $d(x, y) \leq s$ as $\{(a, b) \in A^2 : d_A(a, b) \leq s\}$. We let $\text{Th}_{\mathcal{L}_S}(\mathcal{A})$ denote the complete $\mathcal{L}_S$-theory of the resulting $\mathcal{L}_S$-structure.

All of our model theoretic statements about metric spaces will be in this relational context. Our first main result (Theorem A below) applies in the setting where we consider an arbitrary expansion $\mathcal{L}$ of the language $\mathcal{L}_S$. In this case, the interpretation of any new symbols in $\mathcal{L}$ does not affect the statement of the theorem. In this particular theorem, we use $\text{Th}_{\mathcal{L}}(\mathcal{A})$ to denote the first-order $\mathcal{L}$-theory of this expanded structure.

Recall that we have yet another first-order language, namely, $\mathcal{L}_{\text{om}}$. This language is exclusively used when considering distance magmas and monoids. Moreover, our primary focus will be on model theoretic properties of generalized metric spaces as relational structures, which will not explicitly use the language $\mathcal{L}_{\text{om}}$. However, many of our results will associate model theoretic properties of generalized metric spaces with algebraic and combinatorial properties of distance magmas, and, in particular, these properties of magmas can often be expressed in a first-order way using $\mathcal{L}_{\text{om}}$. Therefore, the reader should consider $\mathcal{L}_{\text{om}}$ as an auxiliary language used mostly for convenience.

It is worth emphasizing that, throughout this thesis, we will be working with two different classes of structures. The primary class is the class of generalized metric spaces, and our main goal is to develop the model theory of these objects in the relational setting discussed above. The secondary class of structures is the class of distance magmas. We will not focus on this class from a model theoretic perspective, but rather use the algebraic and combinatorial behavior of these structures to analyze model theoretic properties of generalized metric spaces.

One motivation for generalized distance structures comes from the wide variety of examples this notion encompasses. The following are a few examples arising naturally in the literature.

Example 2.1.4.

1. If $\mathcal{R} = (\mathbb{R}^{\geq 0}, +, \leq, 0)$ then $\mathcal{R}$-metric spaces coincide with usual metric spaces. In this case, we refer to $\mathcal{R}$-metric spaces as *classical metric spaces*.

2. If $\mathcal{R} = (\mathbb{R}^{\geq 0}, \max, \leq, 0)$ then $\mathcal{R}$-metric spaces coincide with classical ultrametric spaces.

3. Given $S \subseteq \mathbb{R}^{\geq 0}$, with $0 \in S$, we consider classical metric spaces with distances restricted to S. This is the context of [27], which has inspired much of the following work (especially Section 2.7). If S satisfies the property that, for all $r, s \in S$, the subset $\{x \in S : x \leq r + s\}$ contains a maximal element, then we can endow S with the structure of a distance magma under the induced operation $r +_S s := \max\{x \in S : x \leq r + s\}$. This situation is closely studied by Sauer in [77] and [78]. In Section 2.5, we develop this example in full generality.

A more important motivation for considering distance structures and metric spaces at this level of generality is that we will eventually obtain a class of structures invariant under elementary equivalence. Roughly speaking, we will show that models of the $\mathcal{L}_S$-theory of an $\mathcal{R}$-metric space are still generalized metric spaces over a canonical distance magma, which depends only on S and $\mathcal{R}$, but may contain distances not in $\mathcal{R}$. For example, suppose $\mathcal{A}$ is a classical metric space over $(\mathbb{R}^{\geq 0}, +, \leq, 0)$, which contains points of arbitrarily small distances. Then we can use compactness to build models of the $\mathcal{L}_{\mathbb{Q}^{\geq 0}}$-theory of $\mathcal{A}$, which contain distinct points infinitesimally close together. Therefore, when analyzing these models, we must relax the notion of distance and consider a "nonstandard" extension of the distance set. The first main result of this chapter is that such an extension can always be found, even when starting in full generality.

Theorem A. *Let $\mathcal{R}$ be a distance magma and fix $S \subseteq R$, with $0 \in S$. Then there is an $\mathcal{L}_{\mathrm{om}}$-structure $\mathcal{S}^* = (S^*, \oplus_S^*, \leq^*, 0)$ satisfying the following properties.*

(*a*) *$\mathcal{S}^*$ is a distance magma.*

(*b*) *$(S^*, \leq^*)$ is an extension of $(S, \leq)$, and S is dense in S^* (with respect to the order topology).*

(*c*) *Given $r, s \in S$, if $r \oplus s \in S$ then $r \oplus_S^* s = r \oplus s$.*

(*d*) *Suppose $\mathcal{A} = (A, d_A)$ is an $\mathcal{R}$-metric space such that* $\mathrm{Spec}(\mathcal{A}) \subseteq S$*. Let $\mathcal{L}$ be a first-order language, with $\mathcal{L}_S \subseteq \mathcal{L}$. Fix $M \models \mathrm{Th}_{\mathcal{L}}(\mathcal{A})$.*

(*i*) *For all $a, b \in M$, there is a unique $\alpha = \alpha(a, b) \in S^*$ such that, given any $s \in S$, we have $M \models d(a, b) \leq s$ if and only if $\alpha \leq^* s$.*

(*ii*) *If $d_M : M \times M \longrightarrow S^*$ is defined such that $d_M(a, b) = \alpha(a, b)$, then (M, d_M) is an $\mathcal{S}^*$-metric space.*

The structure $\mathcal{S}^*$ from Theorem A is obtained by defining a distance magma structure on the space of quantifier-free 2-types consistent with a natural set of axioms for $\mathcal{R}$-metric spaces with distances in S. We will also give explicit combinatorial descriptions of the set S^* and the operation $\oplus_S^*$. Moreover, we will isolate conditions under which, in part (*d*) of this theorem, the requirement $\mathrm{Spec}(\mathcal{A}) \subseteq S$ can be weakened (for example, in order to keep $\mathcal{L}_S$ countable). Theorem A appears again in its final form as Theorem 2.4.3.

We then consider the existence of an *$\mathcal{R}$-Urysohn space over S*, denoted $\mathcal{U}_{\mathcal{R}}^S$, where S is a countable subset of some distance magma $\mathcal{R}$. When it exists, $\mathcal{U}_{\mathcal{R}}^S$ is a countable, homogeneous $\mathcal{R}$-metric space with spectrum S, which is universal for finite $\mathcal{R}$-metric spaces with distances in S. In [27], Delhommé, Laflamme, Pouzet, and Sauer characterize the existence of $\mathcal{U}_{\mathcal{R}}^S$ when $\mathcal{R} = (\mathbb{R}^{\geq 0}, +, \leq, 0)$. In Section 2.7 we show that, after appropriate translation, the same characterization goes through for any $\mathcal{R}$. A corollary is that, given a *countable* distance magma $\mathcal{R} = (R, \oplus, \leq, 0)$, the $\mathcal{R}$-Urysohn space $\mathcal{U}_{\mathcal{R}} := \mathcal{U}_{\mathcal{R}}^R$ exists if and only if $\oplus$ is associative. Therefore, in Section 2.8, we fix a countable distance *monoid* $\mathcal{R}$ and consider $\mathrm{Th}(\mathcal{U}_{\mathcal{R}}) := \mathrm{Th}_{\mathcal{L}_R}(\mathcal{U}_{\mathcal{R}})$, the first-order $\mathcal{L}_R$-theory of $\mathcal{U}_{\mathcal{R}}$. Our second main result characterizes quantifier elimination for $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ in terms of continuity in $\mathcal{R}^* = (R^*, \oplus_R^*, \leq^*, 0)$.

Theorem B. *If $\mathcal{R}$ is a countable distance monoid then* $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ *has quantifier elimination if and only if, for all $\alpha \in R^*$, if α is nonzero with no immediate predecessor in R^* then, for all $s \in R$,*

$$\alpha \oplus_R^* s = \sup\{x \oplus_R^* s : x <^* \alpha\}.$$

This theorem appears again as Theorem 2.8.11. A corollary of this result is the existence of an $\mathcal{L}_{\mathrm{om}}$-sentence φ such that, if $\mathcal{R}$ is a countable distance monoid, then

$\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ has quantifier elimination if and only if $\mathcal{R} \models \varphi$. When quantifier elimination holds, we also obtain an $\forall\exists$-axiomatization of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$. Finally, in Section 2.9, we consider several classes of natural examples, which occur frequently in the literature, and we verify that they all have quantifier elimination.

It is worth emphasizing that the significance of Theorem B lies in the case when $\mathcal{R}$ is infinite. Indeed, if $\mathcal{R}$ is finite then $\mathcal{L}_R$ is finite, and so quantifier elimination for $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ follows from standard results in Fraïssé theory in finite languages. However, if $\mathcal{R}$ is infinite then the language is infinite and the theory is not $\aleph_0$-categorical. In this situation, quantifier elimination for Fraïssé limits can fail (see Example 2.8.15). Therefore, Theorem B uncovers a natural class of non-$\aleph_0$-categorical Fraïssé limits in which quantifier elimination holds, and is characterized by analytic behavior of the structure.

The characterization of quantifier elimination for $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ also initiates a program of study concerning the relationship between model theoretic properties of $\mathcal{U}_{\mathcal{R}}$ and algebraic properties of $\mathcal{R}$. This is the subject of Chapter 3. The result is a rich class of first-order structures without the strict order property, which represent a wide range of complexity in examples both classical and exotic (e.g. stable theories of refining equivalence relations as ultrametric Urysohn spaces; the simple, unstable random graph as the Urysohn space over $\{0, 1, 2\}$; and the rational Urysohn space, which is SOP_n for all n). Moreover, each measure of complexity (e.g. stability, simplicity, and the strong order properties) is characterized in Chapter 3 by natural algebraic and combinatorial properties of the monoid $\mathcal{R}$.

Remark 2.1.5. This chapter has been rewritten in preparation for submission for publication. A preprint is available on the arXiv [23].

2.2 The First-Order Setting

Our first main goal is to construct the structure $(S^*, \oplus^*_S, \leq^*, 0)$ described in Theorem A, where S is some subset of a distance magma $\mathcal{R} = (R, \oplus, \leq, 0)$. Each step of the construction is motivated by an attempt to capture the first-order theory of $\mathcal{R}$-metric spaces, with distances in S.

Definition 2.2.1. Fix a distance magma $\mathcal{R} = (R, \oplus, \leq, 0)$ and a subset $S \subseteq R$, with $0 \in S$.

1. Define the first-order relational language $\mathcal{L}_S = \{d(x, y) \leq s : s \in S\}$ where, for each $s \in S$, $d(x, y) \leq s$ is a binary relation symbol in the variables x and y. Let $d(x, y) > s$ denote the negation $\neg(d(x, y) \leq s)$.

2. Define the binary relation $<_S$ on R such that, given $u, v \in R$,

$$u <_S v \quad \Leftrightarrow \quad u < s \leq v \text{ for some } s \in S.$$

3. Define the following schemes of $\mathcal{L}_S$-sentences:

   (MS1) $\forall x \forall y (d(x,y) \leq 0 \leftrightarrow x = y)$;

   (MS2) for all $s \in S$,
   $$\forall x \forall y (d(x,y) \leq s \leftrightarrow d(y,x) \leq s);$$

   (MS3) for all $r, s, t \in S$ such that $t \not<_S r \oplus s$,
   $$\forall x \forall y \forall z ((d(x,y) \leq r \wedge d(y,z) \leq s) \to d(x,z) \leq t);$$

   (MS4) if S has a maximal element s,
   $$\forall x \forall y \ d(x,y) \leq s.$$

4. Let $T^{\text{ms}}_{S,\mathcal{R}}$ be the union of the axiom schemes (MS1), (MS2), (MS3), and (MS4) (where (MS4) is only defined if S has a maximal element).

It is not difficult to see that $\mathcal{R}$-metric spaces, with distances in S, satisfy the axioms in $T^{\text{ms}}_{S,\mathcal{R}}$. However, we will prove a stronger statement concerning when an $\mathcal{R}$-metric space, with distances possibly outside of S, still satisfies theses axioms. We first define a notion of approximation, which captures the extent to which atomic $\mathcal{L}_S$-formulas can distinguish distances in R.

Definition 2.2.2. Suppose $\mathcal{R} = (R, \oplus, \leq, 0)$ is a distance magma. Fix $S \subseteq R$, with $0 \in S$.

1. Define
   $$I(S,\mathcal{R}) = \{\{0\}\} \cup \{(r,s] : r, s \in S, \ r < s\},$$
   where, given $r, s \in S$ with $r < s$, $(r,s]$ denotes the interval
   $$\{x \in R : r < x \leq s\}.$$
   (These sets are chosen to reflect $\mathcal{L}_S$-formulas of the form $r < d(x,y) \leq s$.)

2. Given $X \subseteq R$, a function $\Phi : X \longrightarrow I(S,\mathcal{R})$ is an $(S,\mathcal{R})$**-approximation of** X if $x \in \Phi(x)$ for all $x \in X$. When $\Phi(x) \neq \{0\}$, we use the notation $\Phi(x) = (\Phi^-(x), \Phi^+(x)]$.

3. Suppose $(x_1, \ldots, x_n) \in R^n$, Φ is an $(S,\mathcal{R})$-approximation of $\{x_1, \ldots, x_n\}$, and $(s_1, \ldots, s_n) \in S^n$. Then $(s_1, \ldots, s_n)$ Φ**-approximates** $(x_1, \ldots, x_n)$ if $s_i \in \Phi(x_i)$ for all $1 \leq i \leq n$.

Note that, if Φ is an $(S,\mathcal{R})$-approximation of $X \subseteq R$ and $0 \in X$, then we must have $\Phi(0) = \{0\}$. Therefore, whenever defining some specific $(S,\mathcal{R})$-approximation Φ, we will always tacitly define $\Phi(0) = \{0\}$.

Next, we define a condition on $\mathcal{R}$-metric spaces $\mathcal{A}$ and sets $S \subseteq R$, which will ensure $\mathcal{A} \models T^{\text{ms}}_{S,\mathcal{R}}$.

Definition 2.2.3. Suppose $\mathcal{R} = (R, \oplus, \leq, 0)$ is a distance magma.

1. A triple $(r, s, t) \in R^3$ is an **$\mathcal{R}$-triangle** if $r \leq s \oplus t$, $s \leq r \oplus t$, and $t \leq r \oplus s$.
2. Suppose $\mathcal{A}$ is an $\mathcal{R}$-metric space. A subset $S \subseteq R$ is **$\mathcal{R}$-metrically dense over $\mathcal{A}$** if
   - (i) for all $r \in \operatorname{Spec}(\mathcal{A})$ there is $s \in S$ such that $r \leq s$;
   - (ii) for any $\mathcal{R}$-triangle (r, s, t), if there are $a, b, c \in A$ such that $d_A(a, b) = r$, $d_A(b, c) = s$, and $d_A(a, c) = t$, then, for any $(S, \mathcal{R})$-approximation Φ of $\{r, s, t\}$, there is an $\mathcal{R}$-triangle in S that Φ-approximates (r, s, t).

Example 2.2.4.

1. For any $\mathcal{R}$-metric space $\mathcal{A}$, we trivially have that $\operatorname{Spec}(\mathcal{A})$ is $\mathcal{R}$-metrically dense over $\mathcal{A}$.
2. Let $\mathcal{R} = (\mathbb{R}^{\geq 0}, +, \leq, 0)$. Given $n > 0$, $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ is $\mathcal{R}$-metrically dense over any classical metric space $\mathcal{A}$ such that $\operatorname{Spec}(\mathcal{A}) \subseteq [0, 1]$. Similarly, $\mathbb{N}$ and $\mathbb{Q}^{\geq 0}$ are both $\mathcal{R}$-metrically dense over any classical metric space.

Proposition 2.2.5. *Let $\mathcal{R} = (R, \oplus, \leq, 0)$ be a distance magma. Suppose $\mathcal{A}$ is an $\mathcal{R}$-metric space and $S \subseteq R$ is such that $0 \in S$ and S is $\mathcal{R}$-metrically dense over $\mathcal{A}$. Then $\mathcal{A} \models T^{\mathrm{ms}}_{S,\mathcal{R}}$.*

Proof. The axiom schemes (MS1) and (MS2) are immediate. Axiom (MS4) follows from Definition 2.2.3(2i). So it remains to verify axiom scheme (MS3). Fix $r, s, t \in S$ such that $t \not\leq_S r \oplus s$, and suppose $a, b, c \in A$, with $\mathcal{A} \models d(a, b) \leq r \wedge d(b, c) \leq s$. Let $d_A(a, b) = u$, $d_A(b, c) = v$, and $d_A(a, c) = w$. Then we have $u \leq r$ and $v \leq s$, and we want to show $w \leq t$. Suppose, toward a contradiction, that $t < w$. Let Φ be an $(S, \mathcal{R})$-approximation of $\{u, v, w\}$ such that $\Phi^+(u) = r$, $\Phi^+(v) = s$, and $\Phi^-(w) = t$. Since S is $\mathcal{R}$-metrically dense over $\mathcal{A}$, there is an $\mathcal{R}$-triangle (r', s', t') in S, which realizes Φ. Then $t < t' \leq r' \oplus s' \leq r \oplus s$, which contradicts $t \not\leq_S r \oplus s$. □

Suppose $\mathcal{R}$ is a distance magma and $S \subseteq R$, with $0 \in S$. The distance magma $\mathcal{S}^*$ from Theorem A will have the property that any $\mathcal{L}_S$-structure satisfying $T^{\mathrm{ms}}_{S,\mathcal{R}}$ can be equipped with an $\mathcal{S}^*$-metric in a coherent and canonical way. In other words, $T^{\mathrm{ms}}_{S,\mathcal{R}}$ axiomatizes the class of $\mathcal{S}^*$-metric spaces (see Proposition 2.4.2 for the precise statement). Once $\mathcal{S}^*$ is defined, this result will follow quite easily. The work lies in the construction of $\mathcal{S}^*$, and the proof that $\mathcal{S}^*$ is a distance magma.

2.3 Construction of $\mathcal{S}^*$

Throughout all of Section 2.3, we fix a distance magma $\mathcal{R} = (R, \oplus, \leq, 0)$, and work with a fixed subset $S \subseteq R$, with $0 \in S$ (these assumptions may be repeated in the statements of the main results). The goal of this section is to construct $\mathcal{S}^*$ satisfying Theorem A. The essential idea is that we are defining a distance magma structure on the space of quantifier-free 2-types consistent with $T^{\text{ms}}_{S,\mathcal{R}}$. This statement is made precise by Proposition 2.3.8 and Definition 2.3.9.

2.3.1 Construction of $(S^*, \leq^*)$

Definition 2.3.1.

1. A subset $X \subseteq S$ is an **end segment** if, for all $r, s \in S$, if $r \in X$ and $r \leq s$ then $s \in X$.
2. An end segment is a **cut** if it does not have a greatest lower bound in S. Let $\kappa(S)$ denote the set of cuts in S.
3. An end segment is a **noncut** if it has a greatest lower bound in S.
4. A noncut is **proper** if it is nonempty and does not contain its greatest lower bound. Let $\dot{\nu}(S)$ denote the set of proper noncuts in S.
5. A noncut is **principal** if it contains its greatest lower bound. Let $\dot{S}$ denote the set of principal noncuts in S.
6. Define $(S^*, \leq^*)$ such that $S^* = \dot{S} \cup \dot{\nu}(S) \cup \kappa(S)$ and, given $X, Y \in S^*$, $X \leq^* Y$ if and only if $Y \subseteq X$.

Note that, if S has no maximal element, then S^* consists precisely of *all* end segments in S. On the other hand, if S has a maximal element, then S^* consists precisely of all nonempty end segments in S.

We may identify S with $\dot{S}$ via the map $s \mapsto \{x \in S : s \leq x\}$. Therefore, we may view $(S^*, \leq^*)$, $(\dot{S} \cup \kappa(S), \leq^*)$, and $(\dot{S} \cup \dot{\nu}(S), \leq^*)$ as extensions of $(S, \leq)$.

Remark 2.3.2. The reader may verify $(\dot{S} \cup \kappa(S), \leq^*)$ is precisely the *Dedekind-MacNeille completion* of $(S, \leq)$ (see [58, Section 11]), which is the smallest complete linear order containing $(S, \leq)$. Moreover, it is easy to see that no new cuts are added when extending $(S, \leq)$ to $(\dot{S} \cup \dot{\nu}(S), \leq^*)$. Therefore $(S^*, \leq^*)$ is the Dedekind-MacNeille completion of $(\dot{S} \cup \dot{\nu}(S), \leq^*)$. In particular, $(S^*, \leq^*)$ is a complete linear order. In fact, $(S^*, \leq^*)$ is the smallest complete linear order containing $(S, \leq)$, in which every non-maximal element of S has an immediate successor.

Definition 2.3.3. Given $r \in S$, define $\nu_S(r) = \{x \in S : r < x\}$. We say $r \in S$ is a **noncut** if $\nu_S(r)$ is a proper noncut, i.e., if r has no immediate successor in S and is not the maximal element of S. Define

$$\nu(S) = \{r \in S : r \text{ is a noncut}\} = \{r \in S : \nu_S(r) \in \dot{\nu}(S)\}.$$

Remark 2.3.4. For the rest of the thesis, we will use the following explicit description of $(S^*, \leq^*)$. Identify S^* with

$$S \cup \{r^+ : r \in \nu(S)\} \cup \{g_X : X \in \kappa(S)\},$$

where r^+ and g_X are distinct new symbols not in S. Then $\leq^*$ is described by the following rules (see Figure 2):

1. If $r \in \nu(S)$ then $r <^* r^+ <^* s$ for all $s \in \nu_S(r)$.

2. If $X \in \kappa(S)$ then $r <^* g_X <^* s$ for all $r \in S \backslash X$ and $s \in X$.

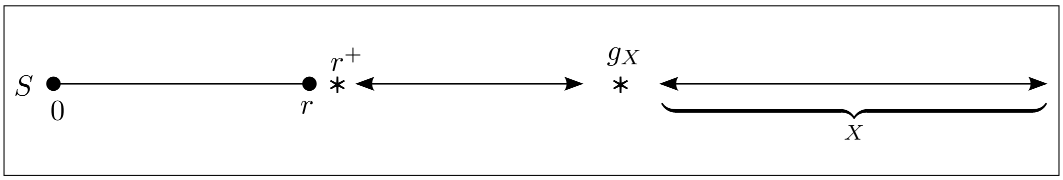

Figure 2: New elements of S^*.

Next, we make some useful observations about $(S^*, \leq^*)$. Since $(S^*, \leq^*)$ is a Dedekind complete linear order, we may calculate infima and suprema in S^*. Unless otherwise stated, the reader should assume these calculations are done with respect to S^*.

Proposition 2.3.5.

(a) *For all* $\alpha, \beta \in S^*$, *if* $\alpha <^* \beta$ *then there is some* $t \in S$ *such that* $\alpha \leq^* t <^* \beta$.

(b) *Suppose* $X \subseteq S^*$ *is nonempty. If* $\sup X = r^+$ *for some* $r \in \nu(S)$, *then* $\sup X \in X$. *If* $\inf X \in S$ *then* $\inf X \in X$.

Proof. Part (a). Fix $\alpha, \beta \in S^*$ with $\alpha <^* \beta$. We may clearly assume $\alpha \notin S$. Suppose first that $\alpha = r^+$ for some $r \in \nu(S)$. If $\beta = s \in S$ or $\beta = s^+$ for some $s \in \nu(S)$, then $r < s$ and so, since $r \in \nu(S)$, there is some $t \in S$ such that $r < t < s$. On the other hand, if $\beta = g_X$ for some $X \in \kappa(S)$, then $r \notin X$ and so there is $t \notin X$, with $r < t$. In either case, $\alpha <^* t <^* \beta$.

Finally, suppose $\alpha = g_X$ for some $X \in \kappa(S)$. If $\beta = s \in S$ or $\beta = s^+$ for some $s \in \nu(S)$, then $s \in X$ and so there is some $t \in X$, with $t < s$. On the other hand, if $\beta = g_Y$ for some $Y \in \kappa(S)$, then $Y \subsetneq X$ and so there is some $t \in X \backslash Y$. In either case, $\alpha <^* t <^* \beta$.

Part (b). For the first claim, note that any r^+ has an immediate predecessor in S^*, namely r. For the second claim, note that any non-maximal $r \in S$ has an immediate successor in S^*, namely either r^+ or an immediate successor in S. $\square$

Part (a) of the previous result will be used frequently throughout the entirety of the thesis. Therefore, for smoother exposition, we will say "by density of S" when using this fact.

Finally, we connect $(S^*, \leq^*)$ back to the first-order setting.

Notation 2.3.6.

1. Note that S^* has a maximal element, which occurs in one of two ways:

   (i) If S has a maximal element s, then s is also the maximal element of S^*.

   (ii) If S has no maximal element then $\emptyset \in \kappa(S)$, and so $g_\emptyset$ is the maximal element of S^*.

   We will use ω_S to denote the maximal element of S^*. We can distinguish between the two cases above by observing either $\omega_S \in S$ or $\omega_S \notin S$.

2. Note that, in Definition 2.2.2, the notion of an $(S, \mathcal{R})$-approximation does not depend on $\oplus$. Therefore, we may apply this definition with $S^*_< := (S^*, \leq^*, 0)$ in place of $\mathcal{R}$. In this case, we let $I(S)$ denote $I(S \cup \{\omega_S\}, S^*_<)$, and we say S**-approximation** in place of $(S \cup \{\omega_S\}, S^*_<)$-approximation.

Definition 2.3.7. Given $\alpha \in S^*$, define the set of $\mathcal{L}_S$-formulas

$$p_\alpha(x, y) = \{d(x, y) \leq s : s \in S,\ \alpha \leq^* s\} \cup \{d(x, y) > s : s \in S,\ s <^* \alpha\}.$$

Proposition 2.3.8. *Let $S_2^{\mathrm{qf}}(T^{\mathrm{ms}}_{S,\mathcal{R}})$ denote the space of complete quantifier-free 2-types $p(x, y)$ over $\mathcal{L}_S$, such that $p(x, y) \cup T^{\mathrm{ms}}_{S,\mathcal{R}}$ is consistent. Then the map $\alpha \mapsto p_\alpha(x, y)$ is a bijection from S^* to $S_2^{\mathrm{qf}}(T^{\mathrm{ms}}_{S,\mathcal{R}})$.*

Proof. We first show the map is well-defined. Fix $\alpha \in S^*$. Note that if $p_\alpha(x, y) \cup T^{\mathrm{ms}}_{S,\mathcal{R}}$ is consistent then, by axiom schemes (MS1) and (MS2), $p_\alpha(x, y)$ determines a complete quantifier-free type in $S_2^{\mathrm{qf}}(T^{\mathrm{ms}}_{S,\mathcal{R}})$. Moreover, for any $s \in S$, the space $\mathcal{A}$, where $A = \{a, b\}$ and $d_A(a, b) = s$, satisfies $T^{\mathrm{ms}}_{S,\mathcal{R}}$ by Proposition 2.2.5. Therefore, to show $p_\alpha(x, y) \in S_2^{\mathrm{qf}}(T^{\mathrm{ms}}_{S,\mathcal{R}})$, it suffices by compactness to fix $I \in I(S)$, with $\alpha \in I$, and show $I \cap S \neq \emptyset$. If $I = \{0\}$ or $I = (r, s]$ for some $s \in S$ then this is obvious. So we may assume $I = (r, \omega_S]$ and $\omega_S \notin S$. Then S has no maximal element, so there is $s \in S$ such that $r < s$. Therefore $s \in I \cap S$.

For injectivity, fix $\alpha, \beta \in S^*$, with $\alpha <^* \beta$. By density of S, there is $s \in S$ such that $\alpha \leq^* s <^* \beta$. Therefore $p_\alpha(x, y) \vdash d(x, y) > s$ and $p_\beta(x, y) \vdash d(x, y) \leq s$.

Finally, we show surjectivity. Given $p(x, y) \in S_2^{\mathrm{qf}}(T^{\mathrm{ms}}_{S,\mathcal{R}})$, define

$$X(p) = \{s \in S : p \vdash d(x, y) \leq s\}.$$

By axiom schemes (MS1) and (MS2), p is completely determined by $X(p)$. So it suffices to fix $p(x, y) \in S_2^{\mathrm{qf}}(T^{\mathrm{ms}}_{S,\mathcal{R}})$ and show there is some $\alpha \in S^*$ with $X(p) = X(p_\alpha)$.

Let $X = X(p)$, and note that X is an end segment by axiom scheme (MS3). Note also that if $X = \emptyset$ then S has no maximal element by axiom (MS4), and so $\emptyset$ is a cut. Therefore, one of the following cases must hold.
Case 1: X is a principal noncut. Then $X = X(p_s)$, where s is the greatest lower bound of X.
Case 2: X is a proper noncut. Then $X = X(p_{s^+})$, where s is the greatest lower bound of X.
Case 3: X is a cut. Then $X = X(p_{g_X})$. □

2.3.2 Definition of $\oplus^*_S$

The definition of $\oplus^*_S$ is motivated by the trivial observation that, given $r, s \in R$,

$$r \oplus s = \sup\{t \in R : (r, s, t) \text{ is an } \mathcal{R}\text{-triangle}\}.$$

Given $\alpha, \beta \in S^*$, we define $\alpha \oplus^*_S \beta$ in an analogous way.

Definition 2.3.9.

1. Fix $\alpha, \beta \in S^*$.
   - (a) Given $\gamma \in S^*$, the triple (α, β, γ) is a **logical S^*-triangle** if

     $$T^{\text{ms}}_{S,\mathcal{R}} \cup p_\alpha(x, y) \cup p_\beta(y, z) \cup p_\gamma(x, z)$$

     is consistent.
   - (b) Define $\Delta(\alpha, \beta) = \{\gamma \in S^* : (\alpha, \beta, \gamma) \text{ is a logical } S^*\text{-triangle}\}$.
   - (c) Define $\alpha \oplus^*_S \beta = \sup \Delta(\alpha, \beta)$.
2. Let $\mathcal{S}^*$ denote the $\mathcal{L}_{\text{om}}$-structure $(S^*, \oplus^*_S, \leq^*, 0)$.

2.3.3 Explicit reformulation of $\oplus^*_S$

This subsection is devoted to proving $\mathcal{S}^*$ is a distance magma. The main tool will be an explicit expression for $\oplus^*_S$ (Proposition 2.3.13). We start with some basic properties of logical S^*-triangles.

Proposition 2.3.10.

(a) *For any* $\alpha, \beta \in S^*$, $(\alpha, \beta, \max\{\alpha, \beta\})$ *is a logical* S^**-triangle, and so* $\max\{\alpha, \beta\} \leq^* \alpha \oplus^*_S \beta$.

(b) *Given* $\alpha, \beta, \gamma \in S^*$, (α, β, γ) *is a logical* S^**-triangle if and only if, for every* S*-approximation* Φ *of* $\{\alpha, \beta, \gamma\}$*, there is an* $\mathcal{R}$*-triangle in* S *that* Φ*-approximates* (α, β, γ).

Proof. Part (a). Suppose $\alpha, \beta \in S^*$, with $\alpha \leq^* \beta$. Fix an S-approximation Φ of $\{\alpha, \beta\}$. We may choose $r \in \Phi(\alpha) \cap S$ such that $r \leq s$ (if $\alpha = \beta$ let $r = s$ and if $\alpha <^* \beta$ use density of S). Then (r, s, s) is an $\mathcal{R}$-triangle, which Φ-approximates (α, β, β). By compactness, and Proposition 2.2.5, (α, β, β) is a logical S^*-triangle.

Part (b). The reverse direction follows from Proposition 2.2.5 and compactness. For the forward direction, let Φ be an S-approximation of $\{\alpha, \beta, \gamma\}$. We may assume $\alpha \leq^* \beta \leq^* \gamma$ and $\Phi^+(\alpha) \leq^* \Phi^+(\beta) \leq^* \Phi^+(\gamma)$. By the proof of part ($a$), we may also assume $\beta <^* \gamma$. Let $r = \Phi^+(\alpha)$ and $s = \Phi^+(\beta)$. We may assume $r, s \in S$, with $r \leq s$. Moreover, by density of S, we may assume $s \leq \Phi^-(\gamma)$. Since $\Phi^-(\gamma) <^* \gamma$, $\alpha \leq^* r$, and $\beta \leq^* s$, it follows from axiom scheme (MS3) that $\Phi^-(\gamma) <_S r \oplus s$. So we may fix $t \in \Phi(\gamma) \cap S$ such that $t \leq r \oplus s$. Then (r, s, t) is an $\mathcal{R}$-triangle, which Φ-approximates (α, β, γ). □

Definition 2.3.11.

1. Given $\alpha \in S^*$, define $\overline{\nu}_S(\alpha) = \{x \in S : \alpha \leq^* x\}$.
2. Given $\alpha, \beta \in S^*$, define

$$P_S(\alpha, \beta) = \{x \in S : x \leq r \oplus s \text{ for all } r \in \overline{\nu}_S(\alpha),\ s \in \overline{\nu}_S(\beta)\}.$$

In many cases, we will have $\alpha \oplus^*_S \beta = \sup P_S(\alpha, \beta)$. However, in the case that $\mu := \sup P_S(\alpha, \beta)$ is an element $\nu(S)$, $\alpha \oplus^*_S \beta$ may be equal to either μ or μ^+. Distinguishing between the two cases requires further analysis of the relationship between $P_S(\alpha, \beta)$ and $\Delta(\alpha, \beta)$.

Lemma 2.3.12. *Fix $\alpha, \beta \in S^*$ and let $\mu = \sup P_S(\alpha, \beta)$.*

(a) *$\mu \leq^* \alpha \oplus^*_S \beta$.*

(b) *If $t \in S$ and $t <^* \alpha \oplus^*_S \beta$ then $t <_S r \oplus s$ for all $r \in \overline{\nu}_S(\alpha)$ and $s \in \overline{\nu}_S(\beta)$. In particular, $t \leq^* \mu$.*

(c) *If $\mu <^* \alpha \oplus^*_S \beta$ then $\alpha \oplus^*_S \beta$ is the immediate successor of μ in S^*.*

(d) *If $\mu \in \nu(S)$ then $\mu^+ \in \Delta(\alpha, \beta)$ if and only if $\mu <_S r \oplus s$ for all $r \in \overline{\nu}_S(\alpha)$ and $s \in \overline{\nu}_S(\beta)$.*

Proof. Part (a). By Proposition 2.3.10(a), we may assume $\max\{\alpha, \beta\} <^* t$. Therefore, using density of S, in order show $t \in \Delta(\alpha, \beta)$ it suffices to fix $r \in \overline{\nu}_S(\alpha)$ and $s \in \overline{\nu}_S(\beta)$, with $\max\{r, s\} \leq t$, and show (r, s, t) is an $\mathcal{R}$-triangle. This is immediate from $\max\{r, s\} \leq t$ and $t \in P_S(\alpha, \beta)$.

Part (b). If $t <^* \alpha \oplus^*_S \beta$ then we may fix $\gamma \in \Delta(\alpha, \beta)$, with $t <^* \gamma$. Fix $r \in \overline{\nu}_S(\alpha)$, $s \in \overline{\nu}_S(\beta)$, and let Φ be an S-approximation of $\{\alpha, \beta, \gamma\}$ such that $\Phi^+(\alpha) = r$, $\Phi^+(\beta) = s$ and $\Phi^-(\gamma) = t$. By Proposition 2.3.10(b), there is an $\mathcal{R}$-triangle (v_1, v_2, v_3) in S, which Φ-approximates (α, β, γ). Then $t < v_3 \leq v_1 \oplus v_2 \leq r \oplus s$, and so $t <_S r \oplus s$.

Part (c). If $\mu <^* \alpha \oplus^*_S \beta$ and $\alpha \oplus^*_S \beta$ is not the immediate successor of u then, by density of S, there is some $t \in S$ with $\mu <^* t <^* \alpha \oplus^*_S \beta$. But then $t \in P_S(\alpha, \beta)$ by part (b), which contradicts $\mu <^* t$.

Part (d). The forward direction follows immediately from part (b). Conversely, suppose $\mu \in \nu(S)$ and $\mu <_S r \oplus s$ for all $r \in \overline{\nu}_S(\alpha)$ and $s \in \overline{\nu}_S(\beta)$. We want to show $\mu^+ \in \Delta(\alpha, \beta)$.

Case 1: There are $r \in \overline{\nu}_S(\alpha) \cap P_S(\beta, \mu^+)$ and $s \in \overline{\nu}_S(\beta) \cap P_S(\alpha, \mu^+)$.

Fix Φ, an S-approximation of $\{\alpha, \beta, \mu^+\}$. Without loss of generality, we may assume $\Phi^+(\alpha) \leq r$ and $\Phi^+(\beta) \leq s$. We have $\mu <_S \Phi^+(\alpha) \oplus \Phi^+(\beta)$ so there is some $v \in S$ such that $\mu < v \leq \Phi^+(\alpha) \oplus \Phi^+(\beta)$. Without loss of generality, we may assume $v \leq \Phi^+(\mu^+)$, and so $v \in \Phi(\mu^+)$. Note that $v \in \overline{\nu}_S(\mu^+)$, and so $\Phi^+(\alpha) \leq r \leq \Phi^+(\beta) \oplus v$ and $\Phi^+(\beta) \leq s \leq \Phi^+(\alpha) \oplus v$. Therefore $(\Phi^+(\alpha), \Phi^+(\beta), v)$ is an $\mathcal{R}$-triangle in S that Φ-approximates (α, β, μ^+), and so $\mu^+ \in \Delta(\alpha, \beta)$.

Case 2: Without loss of generality, assume $\overline{\nu}_S(\alpha) \cap P_S(\beta, \mu^+) = \emptyset$.

Note that $\mu \in P_S(\beta, \mu^+)$, so we must have $\mu <^* \alpha$. By part (c) and Proposition 2.3.10(a), it follows that $\alpha = \alpha \oplus^*_S \beta = \mu^+$. So we want to show $\mu^+ \in \Delta(\mu^+, \beta)$. By Proposition 2.3.10$(a)$, we may assume $\mu^+ <^* \beta$. By density of S, there is $t \in S$ such that $\mu^+ <^* t <^* \beta$. Then $\mu < t$ and so $t \notin P_S(\alpha, \beta)$. Therefore, there are $r \in \overline{\nu}_S(\alpha)$ and $s \in \overline{\nu}_S(\beta)$ such that $r \oplus s < t$. But this is a contradiction, since $t <^* \beta \leq^* s$. □

We can now give a completely explicit description of $\oplus^*_S$, along with some useful observations concerning the relationship between $\oplus^*_S$ and $\oplus$.

Proposition 2.3.13. *Let $\mathcal{R}$ be a distance magma and fix $S \subseteq R$, with $0 \in S$.*

(a) *If $\alpha, \beta \in S^*$ and $\mu = \sup P_S(\alpha, \beta)$ then*

$$\alpha \oplus^*_S \beta = \begin{cases} \mu^+ & \text{if } \mu \in \nu(S) \text{ and } \mu <_S r \oplus s \text{ for all } r \in \overline{\nu}_S(\alpha) \text{ and } s \in \overline{\nu}_S(\beta) \\ \mu & \text{otherwise.} \end{cases}$$

(b) *If $r, s \in S$ then $r \oplus^*_S s = \sup P_S(r, s)$.*

(c) *If $r, s, r \oplus s \in S$ then $r \oplus^*_S s = r \oplus s$.*

(d) *If $\alpha, \beta \in S^*$ then $\alpha \oplus^*_S \beta \in \Delta(\alpha, \beta)$.*

Proof. Part (a). By parts (a) and (d) of Lemma 2.3.12, it suffices to show that if $\mu <^* \alpha \oplus^*_S \beta$ then $\mu \in \nu(S)$ and $\alpha \oplus^*_S \beta = \mu^+$. So assume $\mu <^* \alpha \oplus^*_S \beta$. By Lemma 2.3.12$(c)$, $\alpha \oplus^*_S \beta$ is the immediate successor of μ in S^*. In particular, $\mu \in S$. So it remains to show $\mu \in \nu(S)$. If not, then $v := \alpha \oplus^*_S \beta$ is an element of S. Since $\mu < v$, there are $r \in \overline{\nu}_S(\alpha)$ and $s \in \overline{\nu}_S(\beta)$ such that $r \oplus s < v$. But then $\mu <_S r \oplus s < v$ by Lemma 2.3.12(b), which is a contradiction.

Part (b). Fix $r, s \in S$ and let $\mu = \sup P_S(r, s)$. Note that if $\mu \in \nu(S)$ then $\mu \not<_S r \oplus s$ by definition of μ. Therefore $r \oplus^*_S s = \mu$ by part (a).

Part (c). If $r, s, r \oplus s \in S$ then $r \oplus s = \sup P_S(r,s)$, so this follows from part (b).

Part (d). Let $\mu = \sup P_S(\alpha,\beta)$. By Proposition 2.3.5$(b)$, if $\alpha \oplus^*_S \beta = \mu^+$ then $\mu^+ \in \Delta(\alpha,\beta)$. So we assume $\alpha \oplus^*_S \beta = \mu$ and show $\mu \in \Delta(\alpha,\beta)$. If $\mu \leq^* \max\{\alpha,\beta\}$ then $\mu = \max\{\alpha,\beta\} \in \Delta(\alpha,\beta)$ by Proposition 2.3.10(a). Therefore, by density of S, we may fix $r, s \in S$ such that $\alpha \leq^* r <^* \mu$ and $\beta \leq^* s <^* \mu$. Suppose Φ is an S-approximation of $\{\alpha,\beta,\mu\}$. Without loss of generality, we assume $\Phi^+(\alpha) \leq r$, $\Phi^+(\beta) \leq s$, and $\max\{r,s\} \leq \Phi^-(\mu)$. Since $\Phi^-(\mu) <^* \mu$, we may find $v \in P_S(\alpha,\beta)$ such that $\Phi^-(\mu) <^* v$. Then $(\Phi^+(\alpha), \Phi^+(\beta), v)$ is an $\mathcal{R}$-triangle, which Φ-approximates (α,β,μ). □

Theorem 2.3.14. *If $\mathcal{R}$ is a distance magma and $S \subseteq R$, with $0 \in S$, then $\mathcal{S}^*$ is a distance magma.*

Proof. By construction, $(S^*, \leq^*, 0)$ is a linear order with least element 0, and $\oplus^*_S$ is commutative. By Propositions 2.3.8 and 2.3.10(a), 0 is the identity element of $\mathcal{S}^*$.

We have left to show that, for all $\alpha,\beta,\gamma,\delta \in S^*$, if $\alpha \leq^* \gamma$ and $\beta \leq^* \delta$ then $\alpha \oplus^*_S \beta \leq^* \gamma \oplus^*_S \delta$. Since $\oplus^*_S$ is commutative, it suffices to assume $\beta = \delta$. Let $\mu = \sup P_S(\alpha,\beta)$ and $\eta = \sup P_S(\gamma,\beta)$. Note first that, since $\alpha \leq^* \gamma$, we have $\mu \leq^* \nu$. Therefore, if $\alpha \oplus^*_S \beta = \mu$ then $\alpha \oplus^*_S \beta \leq^* \nu \leq^* \gamma \oplus^*_S \beta$ by Lemma 2.3.12(a). Therefore, by Proposition 2.3.13(a), we may assume $\alpha \oplus^*_S \beta = \mu^+$. If $\mu <^* \nu$ then $\mu^+ \leq^* \nu$, which, as before, implies $\alpha \oplus^*_S \beta \leq^* \gamma \oplus^*_S \beta$. Therefore, we may assume $\mu = \nu$. Since $\alpha \oplus^*_S \beta = \mu^+$, we have $\mu <_S r \oplus s$ for all $r \in \overline{\nu}_S(\alpha)$ and $s \in \overline{\nu}_S(\beta)$. Since $\alpha \leq^* \gamma$, it follows that $\mu <_S r \oplus s$ for all $r \in \overline{\nu}_S(\alpha)$ and $s \in \overline{\nu}_S(\gamma)$. By Proposition 2.3.13$(a)$, we have $\gamma \oplus^*_S \beta = \mu^+$. □

2.4 First-Order Theories of Metric Spaces

The purpose of this section is to collect the previous results and prove Theorem A. We first show that $T^{\mathrm{ms}}_{S,\mathcal{R}}$ can be thought of as a collection of axioms for the class of $\mathcal{S}^*$-metric spaces (as a subclass of $\mathcal{S}^*$-colored spaces).

Definition 2.4.1. Suppose $\mathcal{R}$ is a distance magma and $S \subseteq R$, with $0 \in S$. An $\mathcal{L}_S$-structure A is **$\mathcal{S}^*$-colorable** if, for all $a, b \in A$, there is a (unique) $\alpha = \alpha(a,b) \in S^*$ such that $A \models p_\alpha(a,b)$. In this case, we define $d_A : A \times A \longrightarrow S^*$ such that $d_A(a,b) = \alpha(a,b)$.

Proposition 2.4.2. *Suppose $\mathcal{R}$ is a distance magma and $S \subseteq R$, with $0 \in S$.*

(a) Let A be an $\mathcal{L}_S$-structure. If $A \models T^{\mathrm{ms}}_{S,\mathcal{R}}$ then A is $\mathcal{S}^*$-colorable.

(b) Let $\mathcal{A} = (A, d_A)$ be an $\mathcal{S}^*$-colored space. Then $\mathcal{A} \models T^{\mathrm{ms}}_{S,\mathcal{R}}$ if and only if $\mathcal{A}$ is an $\mathcal{S}^*$-metric space.

Proof. Part (a). By Proposition 2.3.8.

Part (b). If $\mathcal{A} \models T^{\mathrm{ms}}_{S,\mathcal{R}}$ then $\mathcal{A}$ is an $\mathcal{S}^*$-metric space by axioms schemes (MS1) and (MS2), and the definition of $\oplus^*_S$. Conversely, suppose $\mathcal{A}$ is an $\mathcal{S}^*$-metric space. Then $\mathcal{A}$ clearly satisfies axiom schemes (MS1), (MS2), and (MS4). From Proposition 2.3.10(b), we have that $S \cup \{\omega_S\}$ is $\mathcal{S}^*$-metrically dense over $\mathcal{A}$. Therefore, (MS3) follows as in the proof of Proposition 2.2.5. □

We can now state and prove an updated version of Theorem A.

Theorem 2.4.3. *Let $\mathcal{R}$ be a distance magma and fix $S \subseteq R$, with $0 \in S$. Then there is an $\mathcal{L}_{\mathrm{om}}$-structure $\mathcal{S}^* = (S^*, \oplus^*_S, \leq^*, 0)$ satisfying the following properties.*

(a) *$\mathcal{S}^*$ is a distance magma.*

(b) *$(S^*, \leq^*)$ is an extension of $(S, \leq)$, and S is dense in S^* (with respect to the order topology).*

(c) *For all $r, s \in S$, if $r \oplus s \in S$ then $r \oplus^*_S s = r \oplus s$.*

(d) *Suppose $\mathcal{A} = (A, d_A)$ is an $\mathcal{R}$-metric space such that S is $\mathcal{R}$-metrically dense over $\mathcal{A}$. Let $\mathcal{L}$ be a first-order language, with $\mathcal{L}_S \subseteq \mathcal{L}$. If $M \models \mathrm{Th}_{\mathcal{L}}(\mathcal{A})$ then, as an $\mathcal{L}_S$-structure, M is $\mathcal{S}^*$-colorable and (M, d_M) is an $\mathcal{S}^*$-metric space.*

Proof. Parts (a), (b), and (c) follow from Theorem 2.3.14, Proposition 2.3.5(a), and Proposition 2.3.13(c), respectively. For part (d), we have $T^{\mathrm{ms}}_{S,\mathcal{R}} \subseteq \mathrm{Th}_{\mathcal{L}}(\mathcal{A})$ by Proposition 2.2.5, and so the statements follow from Proposition 2.4.2. □

Recall that in Proposition 2.3.10(b), we showed that logical S^*-triangles are approximated by $\mathcal{R}$-triangles in S. Since logical S^*-triangles are in correspondence with $\mathcal{S}^*$-metric spaces with at most 3 points, we can naturally extend this notion of approximation to larger $\mathcal{S}^*$-metric spaces.

Definition 2.4.4. Let $\mathcal{R}$ be a distance magma and fix $S \subseteq R$, with $0 \in S$. An $\mathcal{S}^*$-colored space (A, d_A) is **approximately** $(S, \mathcal{R})$**-metric** if, for all finite $A_0 \subseteq A$ and all S-approximations Φ of $\mathrm{Spec}(A_0, d_A)$, there is an $\mathcal{R}$-metric d_Φ on A_0 such that $d_\Phi(a, b) \in \Phi(d_A(a, b)) \cap S$ for all $a, b \in A_0$.

Proposition 2.4.5. *Let $\mathcal{R}$ be a distance magma and fix $S \subseteq R$, with $0 \in S$. Suppose $\mathcal{A} = (A, d_A)$ is an $\mathcal{S}^*$-colored space. If $\mathcal{A}$ is approximately $(S, \mathcal{R})$-metric then $\mathcal{A}$ is an $\mathcal{S}^*$-metric space.*

Proof. Suppose $\mathcal{A}$ is approximately $(S, \mathcal{R})$-metric. Let $A = \{a_i : i < \lambda\}$. By compactness and Proposition 2.2.5,

$$T^{\mathrm{ms}}_{S,\mathcal{R}} \cup \bigcup_{i,j<\lambda} p_{d_A(a_i,a_j)}(x_i, x_j)$$

is consistent. Therefore, we may embed $\mathcal{A}$ as an $\mathcal{L}_S$-substructure of some model of $T^{\text{ms}}_{S,\mathcal{R}}$. Since $T^{\text{ms}}_{S,\mathcal{R}}$ is universal, it follows that $\mathcal{A} \models T^{\text{ms}}_{S,\mathcal{R}}$. Therefore $\mathcal{A}$ is an $\mathcal{S}^*$-metric space by Proposition 2.4.2(b). $\square$

Regarding the converse of this fact, Proposition 2.3.10(b) shows that $\mathcal{S}^*$-metric spaces of size at most 3 are approximately $(S, \mathcal{R})$-metric. For larger $\mathcal{S}^*$-metric spaces, this can fail.

Example 2.4.6. Let $\mathcal{R} = (\mathbb{R}^{\geq 0}, +, \leq, 0)$ and $S = [0, 2) \cup [3, \infty)$. Using Proposition 2.3.13, one may check $1 +^*_S 3 = 4$ and $1 +^*_S 1 = 3$. Define the $\mathcal{S}^*$-metric space $\mathcal{A}$, where $A = \{w, x, y, z\}$, $d_A(w, x) = d_A(x, z) = d_A(w, y) = 1$, $d_A(x, y) = d_A(w, z) = 3$, and $d_A(y, z) = 4$. Then the S-approximation of $\text{Spec}(\mathcal{A})$, given by $\Phi(1) = (0, 1]$, $\Phi(3) = (0, 3]$, and $\Phi(4) = (3, 4]$, witnesses that $\mathcal{A}$ is not approximately $(S, \mathcal{R})$-metric. To see this, suppose, toward a contradiction, there is an $\mathcal{R}$-metric d_Φ on A realizing Φ. Then

$$d_\Phi(x, y) \leq d_\Phi(x, w) + d_\Phi(w, y) \leq 2 \text{ and } d_\Phi(y, z) \leq d_\Phi(x, y) + d_\Phi(x, z) \leq d_\Phi(x, y) + 1.$$

Therefore $d_\Phi(y, z) \leq 3$, which contradicts $d_\Phi(y, z) \in \Phi(4)$.

In the next section, we will isolate a natural assumption on S under which the converse of Proposition 2.4.5 holds.

2.5 Magmal Sets of Distances

Until this point, we have made no assumptions on the set of distances S, other than $0 \in S$. In this section, we define a property of S under which S itself can be endowed with the structure of a distance magma. As a result, we will obtain the converse of Proposition 2.4.5. Throughout the section, we fix a distance magma $\mathcal{R} = (R, \oplus, \leq, 0)$. Given $S \subseteq R$ and $r, s \in S$, note that $P_S(r, s) = \{x \in S : x \leq r \oplus s\}$, and so the definition of $P_S(r, s)$ does not depend on $\mathcal{S}^*$.

Definition 2.5.1. A subset $S \subseteq R$ is **$\mathcal{R}$-magmal** if $0 \in S$ and, for all $r, s \in S$, $P_S(r, s)$ contains a maximal element, denoted $r \oplus_S s$. In this case, we let $\mathcal{S}$ denote the $\mathcal{L}_{\text{om}}$-structure $(S, \oplus_S, \leq, 0)$.

Canonical examples of $\mathcal{R}$-magmal subsets are those subsets $S \subseteq R$, which contain 0 and are closed under $\oplus$. However, there are many more examples. In particular, any finite subset $S \subseteq R$, with $0 \in S$, is $\mathcal{R}$-magmal. If the operation $\oplus$ is assumed to have some level of continuity (e.g. if $\mathcal{R} = (\mathbb{R}^{\geq 0}, +, \leq, 0)$), then any $S \subseteq R$, which contains 0 and is closed in the order topology on R, is $\mathcal{R}$-magmal.

Proposition 2.5.2. *$S \subseteq R$ is $\mathcal{R}$-magmal if and only if $0 \in S$ and, for all $r, s \in S$, $r \oplus^*_S s \in S$ and $r \oplus^*_S s \leq r \oplus s$. In this case, $r \oplus^*_S s = r \oplus_S s$ for all $r, s \in S$, and $\mathcal{S}$ is a distance magma.*

Proof. This follows easily from Proposition 2.3.13(b). □

Remark 2.5.3. Suppose $S \subseteq R$ is $\mathcal{R}$-magmal.

1. Note that we may construct $\mathcal{S}^*$ while viewing S as a subset in $\mathcal{S}$. Using Remark 2.3.4 and Proposition 2.3.13, it is straightforward to verify that the resulting distance magma $\mathcal{S}^*$ does not depend on this choice of context. For example, given $r, s, t \in S$, the triple (r, s, t) is an $\mathcal{R}$-triangle if and only if it is an $\mathcal{S}$-triangle. Therefore $\mathcal{S}$-metric spaces coincide with $\mathcal{R}$-metric spaces with distances in S. Note also that an $\mathcal{S}^*$-metric space is approximately $(S, \mathcal{R})$-metric if and only if it is approximately $(S, \mathcal{S})$-metric.

2. By Theorem 2.4.3(b) and Proposition 2.5.2, we may consider $\mathcal{S}$ as an $\mathcal{L}_{\text{om}}$-substructure of $\mathcal{S}^*$. Note, however, that $\mathcal{S}$ is not likely to be an elementary substructure. For example, every element of S has an immediate successor in $\mathcal{S}^*$ (see Remark 2.3.2), but not necessarily in $\mathcal{S}$.

For the rest of this section, we fix an $\mathcal{R}$-magmal subset $S \subseteq R$. Note that this setting generalizes the situation described in Example 2.1.4(3). Our first goal is to obtain the converse of Proposition 2.4.5. We start by observing that, when $\omega_S \notin S$, the magmality of S forces ω_S to behave much like an infinite element of $\mathcal{S}^*$.

Proposition 2.5.4. *Suppose $\omega_S \notin S$. If $\alpha, \beta \in S^*$ are such that $\alpha, \beta <^* \omega_S$, then $\alpha \oplus^*_S \beta <^* \omega_S$.*

Proof. If S is $\mathcal{R}$-magmal and $\omega_S \notin S$, then $r \oplus^*_S s <^* \omega_S$ for all $r, s \in S$. So the result follows by density of S. □

Next, we define certain well-behaved S-approximations of subsets of S^*.

Definition 2.5.5. Suppose $X \subseteq S^*$.

1. X is S-**bounded** if, for all $\alpha \in X$, there is $s \in S$ with $\alpha \leq^* s$, i.e., if $\omega_S \notin S$ implies $\omega_S \notin X$.

2. An S-approximation Φ of X is **standard** if $\Phi^+(X)$ is S-bounded, i.e., if $\Phi^+(X) \subseteq S$.

3. Suppose X is S-bounded and Φ is a standard S-approximation of X. Then Φ is **metric** if

   (i) for all $\alpha, \beta \in X$, if $\alpha <^* \beta$ then $\Phi^+(\alpha) \leq \Phi^-(\beta)$;

   (ii) for all $\alpha, \beta, \gamma \in X$, if $\alpha \leq^* \beta \oplus^*_S \gamma$ then $\Phi^+(\alpha) \leq \Phi^+(\beta) \oplus_S \Phi^+(\gamma)$.

4. If Φ and Ψ are S-approximations of X then Φ **refines** Ψ if $\Phi(\alpha) \subseteq \Psi(\alpha)$ for all $\alpha \in X$.

Lemma 2.5.6. *Let $\mathcal{R}$ be a distance magma and fix an $\mathcal{R}$-magmal subset $S \subseteq R$. Suppose $X \subseteq S^*$ is finite and S-bounded. For any S-approximation Ψ of X, there is a metric S-approximation Φ of X, which refines Ψ.*

Proof. For convenience, assume $0 \in X$. Let $X = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, and assume $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n$. Fix an S-approximation Ψ of X. Since X is S-bounded, we may assume Ψ is standard. By density of S, we may also assume $\Psi^+(\alpha_k) <^* \alpha_{k+1}$ for all $1 \leq k < n$. Given $1 \leq k \leq n$, define $I_k = \{(i,j) : 1 \leq i,j < k,\ \alpha_k \leq^* \alpha_i \oplus^*_S \alpha_j\}$. We inductively define $s_0, s_1, \ldots, s_n \in S$ such that

(1) $\alpha_k \leq^* s_k \leq \Psi^+(\alpha_k)$ for all $1 \leq k \leq n$;

(2) $s_k <^* \alpha_{k+1}$ for all $0 \leq k < n$;

(3) for all $1 \leq k \leq n$, if $(i,j) \in I_k$ then $s_k \leq s_i \oplus_S s_j$.

Let $s_0 = 0$. Fix $1 \leq k \leq n$ and suppose we have defined s_i for all $1 \leq i < k$. Define

$$s_k = \min(\{\Psi^+(\alpha_k)\} \cup \{s_i \oplus_S s_j : (i,j) \in I_k\}).$$

Then properties (2) and (3) above are satisfied. For (1), we have $s_k \leq \Psi^+(\alpha_k)$, so it remains to show $\alpha_k \leq^* s_k$. Given $(i,j) \in I_k$, we have, by induction, $\alpha_k \leq^* \alpha_i \oplus^*_S \alpha_j \leq^* s_i \oplus^*_S s_j = s_i \oplus_S s_j$.

Define $\Phi : X \longrightarrow I(S)$ such that $\Phi(0) = \{0\}$ and, for $k > 0$, $\Phi(\alpha_k) = (\max\{\Psi^-(\alpha_k), s_{k-1}\}, s_k]$. Then, by (1) and (2), Φ is an S-approximation of X, which refines Ψ. So it remains to show Φ is metric. Condition (i) of Definition 2.5.5($2a$) is clear. So fix $\alpha_i, \alpha_j, \alpha_k \in X$ such that $\alpha_k \leq^* \alpha_i \oplus^*_S \alpha_j$. We want to show $s_k \leq s_i \oplus_S s_j$. By construction, $(s_i)_{i=0}^k$ is increasing, so we may assume $i, j < k$. Then $(i,j) \in I_k$, and so $s_k \leq s_i \oplus_S s_j$ by (3). □

Using this result, we can obtain the converse of Proposition 2.4.5, for $\mathcal{R}$-magmal sets S.

Theorem 2.5.7. *Let $\mathcal{R}$ be a distance magma and fix an $\mathcal{R}$-magmal subset $S \subseteq R$. Suppose $\mathcal{A}$ is an $\mathcal{S}^*$-colored space. Then $\mathcal{A}$ is an $\mathcal{S}^*$-metric space if and only if $\mathcal{A}$ is approximately $(S, \mathcal{R})$-metric.*

Proof. We have the reverse direction by Proposition 2.4.5. For the forward direction, assume $\mathcal{A}$ is an $\mathcal{S}^*$-metric space. Fix a finite subset $A_0 \subseteq A$ and an S-approximation Ψ of $\operatorname{Spec}(A_0, d_A)$. We want to find an $\mathcal{R}$-metric $d_\Psi : A_0 \times A_0 \longrightarrow S$ such that, for all $x, y \in A_0$, $d_\Psi(x,y) \in \Psi(d_A(x,y))$.

Suppose first that $\operatorname{Spec}(A_0, d_A)$ is not S-bounded. Then we may fix $t \in S$, with $\Psi^-(\omega_S) < t$ and $\alpha \leq^* t$ for all $\alpha \in \operatorname{Spec}(A_0, d_A) \backslash \{\omega_S\}$. Define $d'_A : A_0 \times A_0 \longrightarrow S^*$ such that

$$d'_A(x,y) = \begin{cases} d_A(x,y) & \text{if } d_A(x,y) <^* \omega_S \\ t & \text{otherwise.} \end{cases}$$

Note that $\mathrm{Spec}(A_0, d'_A)$ is S-bounded and Ψ is an S-approximation of $\mathrm{Spec}(A_0, d'_A)$. We claim d'_A is an $\mathcal{S}^*$-metric on A_0. Indeed, fix $a, b, c \in A_0$ with $d_A(a, b) = \omega_S$. Then $d'_A(a, b) = t$ and so, since $\mathrm{Spec}(A_0, d'_A)$ is bounded by t, we have $d'_A(b, c) \leq d'_A(a, b) \oplus d'_A(a, c)$ and $d'_A(a, c) \leq d'_A(a, b) \oplus d'_A(b, c)$. Moreover, by Proposition 2.5.4, $d_A(a, b) = \omega_S$ implies at least one of $d_A(a, c)$ or $d_A(b, c)$ is ω_S. Therefore $d'_A(a, b) \leq d'_A(a, c) \oplus d'_A(b, c)$ as well. Altogether, we have shown that it suffices to assume $\mathrm{Spec}(A_0, d_A)$ is S-bounded.

By Lemma 2.5.6, there is a metric S-approximation Φ of $\mathrm{Spec}(A_0, d_A)$, which refines Ψ. Define $d_\Psi : A_0 \times A_0 \longrightarrow S$ such that $d_\Psi(x, y) = \Phi^+(d_A(x, y))$. Since Φ is metric, it follows that d_Ψ is an $\mathcal{R}$-metric. □

2.6 Metrically Complete Distance Structures

Given a distance magma $\mathcal{R}$ and a subset $S \subseteq R$, with $0 \in S$, the distance magma $\mathcal{S}^*$ is constructed to satisfy nice analytic properties, such as being a complete order and containing S as a dense subset. In this section, we further show that the construction of $\mathcal{S}^*$ automatically creates a certain level of continuity, which will be an essential tool for later results. We start with the following general definition.

Definition 2.6.1. Let $\mathcal{R} = (R, \oplus, \leq, 0)$ be a distance magma.

1. Given $r, s \in R$, define $M_\mathcal{R}(r, s) = \{x \in R : r \leq s \oplus x \text{ and } s \leq r \oplus x\}$. Note that $M_\mathcal{R}(r, s)$ is an end segment in R.
2. $\mathcal{R}$ is **metrically complete** if, for all $r, s \in R$, $M_\mathcal{R}(r, s)$ is a principal noncut.

The purpose of this definition is that it allows for a generalized notion of absolute value of the difference between two distances.

Definition 2.6.2. Suppose $\mathcal{R}$ is a metrically complete distance magma. Given $r, s \in R$, define $|r \ominus s| := \inf M_\mathcal{R}(r, s)$.

The following properties illustrate that this difference operation is well behaved.

Proposition 2.6.3. *Suppose $\mathcal{R}$ is a metrically complete distance magma.*

(a) *For all $r, s \in R$, if $s \leq r$ then $|r \ominus s| = \inf\{x \in R : r \leq s \oplus x\}$.*

(b) *For all $r, s, t \in R$, $|r \ominus s| \leq t$ if and only if $r \leq s \oplus t$ and $s \leq r \oplus t$.*

(c) *For all $r, s \in R$, $|r \ominus s| \leq \max\{r, s\} \leq r \oplus s$.*

(d) *For all $r, s \in R$, $|r \ominus s| = |s \ominus r|$, and $|r \ominus s| = 0$ if and only if $r = s$.*

(e) *Define $d : R \times R \longrightarrow R$ such that $d(r, s) = |r \ominus s|$. Then d is an $\mathcal{R}$-metric if and only if $\oplus$ is associative.*

Proof. Parts (a) through (d) follow trivially from the definitions.

Part (e). First, suppose $\oplus$ is associative. Given $r, s, t \in R$, we want to show $|r \ominus t| \leq |r \ominus s| \oplus |s \ominus t|$. It suffices to assume $t \leq r$ and show $r \leq (|r \ominus s| \oplus |s \ominus t|) \oplus t$. By associativity, it suffices to show $r \leq |r \ominus s| \oplus (|s \ominus t| \oplus t)$. But this follows from $r \leq |r \ominus s| \oplus s$ and $s \leq |s \ominus t| \oplus t$.

Conversely, suppose $\oplus$ is not associative. By commutativity of $\oplus$, we may assume there are $a, b, c \in R$ such that $(a \oplus b) \oplus c < a \oplus (b \oplus c)$. Let $r = a \oplus (b \oplus c)$, $s = b \oplus c$, and $t = c$. Then $|r \ominus s| \leq a$ and $|s \ominus t| \leq b$, and so

$$(|r \ominus s| \oplus |s \ominus t|) \oplus t \leq (a \oplus b) \oplus c < r.$$

Therefore $|r \ominus s| \oplus |s \ominus t| < |r \ominus t|$, which implies (R, d) is not an $\mathcal{R}$-metric space. □

We will frequently see that metrically complete distance magmas satisfy nice continuity properties, which one usually takes for granted when working with structures like $(\mathbb{R}^{\geq 0}, +, \leq, 0)$. The following proposition gives one such example.

Proposition 2.6.4. *Suppose $\mathcal{R} = (R, \oplus, \leq, 0)$ is a metrically complete distance magma. Fix nonempty subsets $X, Y \subseteq R$ and suppose $r, s \in R$ are such that $r = \inf X$ and $s = \inf Y$. Then $r \oplus s = \inf\{x \oplus y : x \in X,\ y \in Y\}$.*

Proof. We clearly have $r \oplus s \leq x \oplus y$ for all $x \in X$ and $y \in Y$. Suppose, toward a contradiction, there is $t \in R$ such that $r \oplus s < t$ and $t \leq x \oplus y$ for all $x \in X$ and $y \in Y$. Then $r < |t \ominus s|$ and so, by choice of r, there is $x \in X$ such that $x < |t \ominus s|$. Since $s < t$, we must have $x \oplus s < t$, and so $s < |t \ominus x|$. Again, this means there is $y \in Y$ such that $y < |t \ominus x|$, and so $x \oplus y < t$, which contradicts the choice of t. □

We now proceed to the main result of this section.

Theorem 2.6.5. *Suppose $\mathcal{R}$ is a distance magma and $S \subseteq R$, with $0 \in S$. Then $\mathcal{S}^*$ is a metrically complete distance magma.*

Proof. Fix $\alpha, \beta \in S^*$ and let $\gamma = \inf M_{\mathcal{S}^*}(\alpha, \beta)$. We want to show $\gamma \in M_{\mathcal{S}^*}(\alpha, \beta)$ and, without loss of generality, we may assume $\beta \leq^* \alpha$. By Proposition 2.3.5(b), we may also assume $\gamma \notin S$. In particular, this implies that if $s \in \overline{\nu}_S(\gamma)$ then $\gamma <^* s$, and so $\alpha \leq^* \beta \oplus_S s$. We want to show $\alpha \leq^* \beta \oplus^*_S \gamma$. Let $\mu = \sup P_S(\beta, \gamma)$, and note that $\mu \leq^* \beta \oplus^*_S \gamma$ by Proposition 2.3.12(a), so we may assume $\mu <^* \alpha$.
Claim: $\mu \in \nu(S)$ and $\alpha = \mu^+$.
Proof: By density of S, and the construction of $(S^*, \leq^*)$, it suffices to show that, for all $t \in S$, if $t \leq^* \alpha$ then $t \leq^* \mu$. So suppose, toward a contradiction, $\mu <^* t \leq^* \alpha$ for some $t \in S$. Then $t \notin P_S(\beta, \gamma)$ so there are $r \in \overline{\nu}_S(\beta)$ and $s \in \overline{\nu}_S(\gamma)$ such that $t > r \oplus s$. By Proposition 2.3.13(b), we have $r \oplus^*_S s \leq^* t$, and so $\alpha \leq^* \beta \oplus^*_S s \leq^* r \oplus^*_S s \leq^* t$. Therefore $\alpha = t \in S$, and so $\alpha > r \oplus s$.

By density of S, there is some $t' \in S$ such that $\mu \leq^* t' < \alpha$. If $\mu <^* t'$ then, applying the same argument above with t replaced by t', we obtain $t' = \alpha$, which is a contradiction. Therefore $\mu = t' \in S$. So altogether, we have shown $\mu, \alpha \in S$ and α is the immediate successor of μ. But then $\alpha > r \oplus s$ and Proposition 2.3.13(b) imply $r \oplus^*_S s \leq^* \mu$, which contradicts $\mu < \alpha \leq^* \beta \oplus^*_S s \leq^* r \oplus^*_S s$. $\dashv_{\text{claim}}$

By the claim, we need to show $\beta \oplus^*_S \gamma = \mu^+$, which by Proposition 2.3.13(a), means showing $\mu <_S r \oplus s$ for all $r \in \overline{\nu}_S(\beta)$ and $s \in \overline{\nu}_S(\gamma)$. So fix $r \in \overline{\nu}_S(\beta)$ and $s \in \overline{\nu}_S(\gamma)$, and suppose $\mu \not<_S r \oplus s$. By Proposition 2.3.13(b), it follows that $r \oplus^*_S s \leq^* \mu$. But then $\mu^+ = \alpha \leq^* \beta \oplus^*_S s \leq^* r \oplus^*_S s \leq^* \mu$, which is a contradiction. □

From the previous result, we obtain the following continuity property in $\mathcal{S}^*$.

Corollary 2.6.6. *Suppose $\mathcal{R}$ is a distance magma and $S \subseteq R$, with $0 \in S$. Given $\alpha, \beta, \gamma \in S^*$, if $\alpha \leq^* \beta \oplus^*_S s$ for all $s \in \overline{\nu}_S(\gamma)$ then $\alpha \leq^* \beta \oplus^*_S \gamma$.*

Proof. We have $\gamma = \inf \overline{\nu}_S(\gamma)$ and $\overline{\nu}_S(\gamma) \subseteq M_{\mathcal{S}^*}(\alpha, \beta)$, which implies $\inf M_{\mathcal{S}^*}(\alpha, \beta) \leq^* \gamma$. Therefore $\alpha \leq^* \beta \oplus^*_S \gamma$ by Theorem 2.6.5. □

For clarity, we repeat the definition of the generalized difference operation on $\mathcal{S}^*$.

Definition 2.6.7. Fix a distance magma $\mathcal{R}$ and $S \subseteq R$, with $0 \in S$. Given $\alpha, \beta \in S^*$, define

$$|\alpha \ominus^*_S \beta| := \inf M_{\mathcal{S}^*}(\alpha, \beta) = \inf\{x \in S^* : \alpha \leq^* \beta \oplus^*_S x \text{ and } \beta \leq^* \alpha \oplus^*_S x\}.$$

Recall that $\alpha \oplus^*_S \beta$ was originally defined as the largest possible length for the third side of a logical S^*-triangle, in which the other two sides are length α and β. We now note that $|\alpha \ominus^*_S \beta|$ satisfies the expected property of being the shortest possible length.

Corollary 2.6.8. *Suppose $\mathcal{R}$ is a distance magma and $S \subseteq R$, with $0 \in S$. Given $\alpha, \beta \in S^*$, we have $\Delta(\alpha, \beta) = \{\gamma \in S^* : |\alpha \ominus^*_S \beta| \leq^* \gamma \leq^* \alpha \oplus^*_S \beta\}$, and so $|\alpha \ominus^*_S \beta| = \inf \Delta(\alpha, \beta)$.*

Proof. From the equivalence of $\mathcal{S}^*$-triangles and logical S^*-triangles, we have $\gamma \in \Delta(\alpha, \beta)$ if and only if $\gamma \leq^* \alpha \oplus^*_S \beta$ and $\gamma \in M_{\mathcal{S}^*}(\alpha, \beta)$. Combined with Theorem 2.6.5, it follows that $\gamma \in \Delta(\alpha, \beta)$ if and only if $|\alpha \ominus^*_S \beta| \leq^* \gamma \leq^* \alpha \oplus^*_S \beta$. □

Finally, we use the established continuity in $\mathcal{S}^*$ to show that, in order to check associativity of $\oplus^*_S$, it suffices to check only the elements of S.

Proposition 2.6.9. *Suppose $\mathcal{R}$ is a distance magma and $S \subseteq R$, with $0 \in S$. If $r \oplus^*_S (s \oplus^*_S t) = (r \oplus^*_S s) \oplus^*_S t$ for all $r, s, t \in S$, then $\oplus^*_S$ is associative on S^*.*

Proof. Suppose $\oplus^*_S$ is not associative on S^*. Since $\oplus^*_S$ is commutative, we may fix $\alpha, \beta, \gamma \in S^*$ such that $\alpha \oplus^*_S (\beta \oplus^*_S \gamma) <^* (\alpha \oplus^*_S \beta) \oplus^*_S \gamma$. By Corollary 2.6.6, we may fix $r \in \overline{\nu}_S(\alpha)$ such that $r \oplus^*_S (\beta \oplus^*_S \gamma) <^* (\alpha \oplus^*_S \beta) \oplus^*_S \gamma$. Let $\eta = (\alpha \oplus^*_S \beta) \oplus^*_S \gamma$. It suffices to find $s \in \overline{\nu}_S(\beta)$ and $t \in \overline{\nu}_S(\gamma)$ such that $r \oplus^*_S (s \oplus^*_S t) <^* \eta$. So suppose, toward a contradiction, $\eta \leq^* r \oplus^*_S (s \oplus^*_S t)$ for all $s \in \overline{\nu}_S(\beta)$ and $t \in \overline{\nu}_S(\gamma)$. Let $\mu = \sup P_S(\beta, \gamma)$.

Suppose first that $\beta \oplus^*_S \gamma \notin S$. Fix $z \in \overline{\nu}_S(\beta \oplus^*_S \gamma)$. Then $\mu \leq^* \beta \oplus^*_S \gamma <^* z$, and so there are $s \in \overline{\nu}_S(\beta)$ and $t \in \overline{\nu}_S(\gamma)$ such that $s \oplus t < z$. Then $s \oplus^*_S t \leq^* z$, and so, by assumption, $\eta \leq^* r \oplus^*_S z$. Altogether, by Corollary 2.6.6, we have $\eta \leq^* r \oplus^*_S (\beta \oplus^*_S \gamma)$, which contradicts the choice of r.

Finally, suppose $\beta \oplus^*_S \gamma \in S$. Then, by Proposition 2.3.13(a), we must have $\beta \oplus^*_S \gamma = \mu$. We claim there are $s \in \overline{\nu}_S(\beta)$ and $t \in \overline{\nu}_S(\gamma)$ such that $\beta \oplus^*_S \gamma \not<_S s \oplus t$. Indeed, if $\mu \in \nu(S)$ then this follows directly from Proposition 2.3.13(a). On the other hand, if μ has an immediate successor $v \in S$, then $\mu < v$ implies there are $s \in \overline{\nu}_S(\beta)$ and $t \in \overline{\nu}_S(\gamma)$ such that $s \oplus t < v$. Since v is the immediate successor of μ, we must have $\beta \oplus^*_S \gamma = \mu \not<_S s \oplus t$, as desired. By Proposition 2.3.13(b), it follows that $\beta \oplus^*_S \gamma = s \oplus^*_S t$. By assumption, $\eta \leq^* r \oplus^*_S (\beta \oplus^*_S \gamma)$, which contradicts the choice of r. □

Corollary 2.6.10. *Suppose $\mathcal{R}$ is a distance magma and $S \subseteq R$ is $\mathcal{R}$-magmal. Then $\mathcal{S}^*$ is a distance monoid if and only if $\mathcal{S}$ is a distance monoid.*

2.7 Associativity, Amalgamation, and the Four-Values Condition

In this section, we turn to a specific class of generalized metric spaces. The motivating example is the *rational Urysohn space*, i.e., the unique countable, universal, and homogeneous metric space with rational distances. In [27], generalizations of this space are obtained by replacing $\mathbb{Q}^{\geq 0}$ with arbitrary countable subsets $S \subseteq \mathbb{R}^{\geq 0}$. The sets S for which an analogous metric space exists are characterized in [27] by a property called the *four-values condition.*

We first generalize the four-values condition to arbitrary distance magmas. Our treatment closely follows [27]. In particular, Proposition 2.7.4, which is the main result of this section, is a direct generalization of the main result of [27, Section 1.3]. Throughout the section, we fix a distance magma $\mathcal{R} = (R, \oplus, \leq, 0)$.

Definition 2.7.1. A subset $S \subseteq R$ satisfies the **four-values condition in $\mathcal{R}$** if for all $u_1, u_2, v_1, v_2 \in S$, if there is some $s \in S$ such that (s, u_1, u_2) and (s, v_1, v_2) are $\mathcal{R}$-triangles, then there is some $t \in S$ such that (t, u_1, v_1) and (t, u_2, v_2) are $\mathcal{R}$-triangles.

The four-values condition describes the amalgamation of two 3-point metric spaces over a common 2-point subspace (Figure 3). In Proposition 2.7.4, we show that this instance of amalgamation is enough to show amalgamation for any two finite $\mathcal{R}$-metric spaces with distances in S. Toward this goal, we first show that, when checking the four-values condition, it suffices to only consider nonzero values. This is a direct generalization of [27, Lemma 1.3].

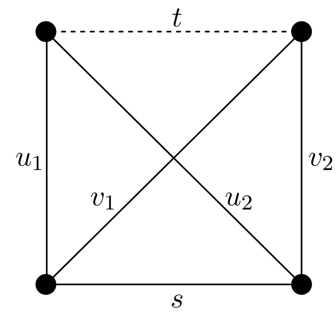

Figure 3: The four-values condition.

Lemma 2.7.2. *Fix $S \subseteq R$ and suppose $u_1, u_2, v_1, v_2, s \in S$ are such that (s, u_1, u_2) and (s, v_1, v_2) are $\mathcal{R}$-triangles.*

(a) *If any one of u_1, u_2, v_1, v_2 is 0 then there is some $t \in S$ such that (t, u_1, v_1) and (t, u_2, v_2) are $\mathcal{R}$-triangles.*

(b) *If $u_1, u_2, v_1, v_2 > 0$ then there is some $s' \in S^{>0}$ such that (s', u_1, u_2) and (s', v_1, v_2) are $\mathcal{R}$-triangles.*

Proof. Part (a). Without loss of generality, suppose $u_1 = 0$. Then (v_1, u_1, v_1) is clearly an $\mathcal{R}$-triangle. Moreover,

$$\begin{aligned}
u_2 &\le s \oplus u_1 = s \le v_1 \oplus v_2,\\
v_1 &\le s \oplus v_2 \le (u_1 \oplus u_2) \oplus v_2 = u_2 \oplus v_2, \text{ and}\\
v_2 &\le s \oplus v_1 \le (u_1 \oplus u_2) \oplus v_1 = u_2 \oplus v_1.
\end{aligned}$$

Therefore (v_1, u_2, v_2) is an $\mathcal{R}$-triangle.

Part (b). If $s > 0$ then we may let $s' = s$. Suppose $s = 0$. Let $s' = \min\{u_1, v_1\}$, and note that $s' > 0$. We have

$$\begin{aligned}
s' &\le u_1 \oplus u_2,\\
u_1 &\le s \oplus u_2 \le s' \oplus u_2,\\
u_2 &\le s \oplus u_1 \le s' \oplus u_1,\\
s' &\le v_1 \oplus v_2,\\
v_1 &\le s \oplus v_2 \le s' \oplus v_2, \text{ and}\\
v_2 &\le s \oplus v_1 \le s' \oplus v_1,
\end{aligned}$$

and so, altogether, (s', u_1, u_2) and (s', v_1, v_2) are $\mathcal{R}$-triangles. □

Definition 2.7.3. Given $S \subseteq R$, with $0 \in S$, let $\mathcal{K}^S_{\mathcal{R}}$ denote the class of finite $\mathcal{R}$-metric spaces with distances in S. Let $\mathcal{K}_{\mathcal{R}} = \mathcal{K}^R_{\mathcal{R}}$.

Given a distance magma $\mathcal{R}$ and a subset $S \subseteq R$, with $0 \in S$, we use our original interpretation of $\mathcal{R}$-metric spaces as $\mathcal{L}_S$-structures to view $\mathcal{K}^S_{\mathcal{R}}$ as a class of relational $\mathcal{L}_S$-structures, which is therefore amenable to classical Fraïssé theory (see [40, Chapter 7]). In particular, it is straightforward to see that the class $\mathcal{K}^S_{\mathcal{R}}$ always satisfies the hereditary property and the joint embedding property. Therefore, our focus is on the amalgamation property.

The next result uses the four-values condition to characterize the amalgamation property for $\mathcal{K}^S_{\mathcal{R}}$. This result is a direct generalization of [27, Proposition 1.6]. The proof is the same as what can be found in [27], modulo adjustments made to account for the possibility that $\mathcal{R}$ is not metrically complete.

Proposition 2.7.4. *Fix $S \subseteq R$, with $0 \in S$. The following are equivalent.*

(i) $\mathcal{K}^S_{\mathcal{R}}$ *has the amalgamation property.*

(ii) $\mathcal{K}^S_{\mathcal{R}}$ *has the disjoint amalgamation property.*

(iii) *For all (X_1, d_1) and (X_2, d_2) in $\mathcal{K}^S_{\mathcal{R}}$ such that $d_1|_{X_1 \cap X_2} = d_2|_{X_1 \cap X_2}$, $|X_1| = |X_2| = 3$, and $|X_1 \cap X_2| = 2$, there is an $\mathcal{R}$-pseudometric d on $X_1 \cup X_2$, with* $\operatorname{Spec}(X_1 \cup X_2, d) \subseteq S$, *such that $d|_{X_1} = d_1$ and $d|_{X_2} = d_2$.*

(iv) *S satisfies the four-values condition in $\mathcal{R}$.*

Proof. *(ii)* $\Rightarrow$ *(i)* $\Rightarrow$ *(iii)*. Trivial.

(iii) $\Rightarrow$ *(iv)*. Fix $u_1, u_2, v_1, v_2, s \in S$ such that (s, u_1, u_2) and (s, v_1, v_2) are $\mathcal{R}$-triangles. By Lemma 2.7.2(a), we may assume u_1, u_2, v_1, v_2 are all nonzero. Let $X_1 = \{x, y_1, y_2\}$ and $X_2 = \{x', y_1, y_2\}$, where x, x', y_1, y_2 are four distinct points. Define d_i on X_i such that

$$d_i(x, y_j) = u_j,\ d_i(x', y_j) = v_j, \text{ and } d_i(y_1, y_2) = s.$$

By assumption, each (X_i, d_i) is an $\mathcal{R}$-metric space. Therefore, by *(iii)*, there is an $\mathcal{R}$-pseudometric d on $X_1 \cup X_2$, with $\operatorname{Spec}(X_1 \cup X_2, d) \subseteq S$, such that $d|_{X_i} = d_i$. Let $t = d(x, x') \in S$. Then (t, u_1, v_1) and (t, u_2, v_2) are $\mathcal{R}$-triangles, and so S satisfies the four-values condition in $\mathcal{R}$.

(iv) $\Rightarrow$ *(ii)*. Assume S satisfies the four-values condition in $\mathcal{R}$. Fix (X_1, d_1) and (X_2, d_2) in $\mathcal{K}^S_{\mathcal{R}}$ such that $d_1|_{X_1 \cap X_2} = d_2|_{X_1 \cap X_2}$. We may assume $X_1 \not\subseteq X_2$ and $X_2 \not\subseteq X_1$. Let $m = |(X_1 \backslash X_2) \cup (X_2 \backslash X_1)|$ and set $X = X_1 \cup X_2$. Then $m \geq 2$ by our assumptions, and we proceed by induction on m.

Suppose $m = 2$. Let $X_1 \backslash X_2 = \{x_1\}$ and $X_2 \backslash X_1 = \{x_2\}$. Given $t \in S$, let $d_t : X \times X \longrightarrow S$ be such that $d_t|_{X_1} = d_1$, $d_t|_{X_2} = d_2$, and $d_t(x_1, x_2) = t$. Then d_t is an $\mathcal{R}$-metric if and only if

$$t > 0 \text{ and } (t, d_1(x_1, x), d_2(x_2, x)) \text{ is an } \mathcal{R}\text{-triangle for all } x \in X_1 \cap X_2. \qquad (\dagger)$$

Therefore, it suffices to find $t \in S$ satisfying $(\dagger)$.

Fix $y \in X_1 \cap X_2$ such that

$$d_1(x_1, y) \oplus d_2(x_2, y) = \min_{x \in X_1 \cap X_2} (d_1(x_1, x) \oplus d_2(x_2, x)).$$

Next, recall that $M_{\mathcal{R}}(r, s)$ is an end segment in $(R, \leq, 0)$ for any $r, s \in R$. Therefore we may fix $y' \in X_1 \cap X_2$ such that

$$M_{\mathcal{R}}(d_1(x_1, y'), d_2(x_2, y')) = \bigcap_{x \in X_1 \cap X_2} M_{\mathcal{R}}(d_1(x_1, x), d_2(x_2, x)).$$

Note that $(d_1(y, y'), d_1(x_1, y), d_1(x_1, y'))$ and $(d_2(y, y'), d_2(x_2, y), d_2(x_2, y'))$ are $\mathcal{R}$-triangles. Since $d_1(y, y') = d_2(y, y')$ and S satisfies the four-values condition in $\mathcal{R}$, there is some $t \in S$ such that $(t, d_1(x_1, y), d_2(x_2, y))$ and $(t, d_1(x_1, y'), d_2(x_2, y'))$ are $\mathcal{R}$-triangles. By Lemma 2.7.2(b), we may assume $t > 0$. Since $(t, d_1(x_1, y), d_2(x_2, y))$ is an $\mathcal{R}$-triangle, we have

$$t \leq d_1(x_1, y) \oplus d_2(x_2, y) = \min_{x \in X_1 \cap X_2} (d_1(x_1, x) \oplus d_2(x_2, x)).$$

Therefore, to show t satisfies (†), it remains to show that, for all $x \in X_1 \cap X_2$, we have the inequalities $d_1(x_1, x) \leq d_2(x_2, x) \oplus t$ and $d_2(x_2, x) \leq d_1(x_1, x) \oplus t$. Since $(t, d_1(x_1, y'), d_2(x_2, y'))$ is an $\mathcal{R}$-triangle, we have $t \in M_{\mathcal{R}}(d_1(x_1, y'), d_2(x_2, y'))$. Therefore, by choice of y', we have $t \in M_{\mathcal{R}}(d_1(x_1, x), d_2(x_2, x))$ for all $x \in X_1 \cap X_2$, which yields the desired result. This completes the base case $m = 2$.

We now proceed with the induction step. Fix $x_1 \in X_1 \backslash X_2$ and $x_2 \in X_2 \backslash X_1$. By induction, we may disjointly amalgamate (X_1, d_1) and $(X_2 \backslash \{x_2\}, d_2)$ to obtain a space (Y_1, d'_1), where $Y_1 = X \backslash \{x_2\}$. Note that the spaces $(Y_1 \backslash \{x_1\}, d'_1)$ and (X_2, d_2) coincide on their intersection $X_2 \backslash \{x_2\}$. So, by induction again, we may disjointly amalgamate $(Y_1 \backslash \{x_1\}, d'_1)$ and (X_2, d_2) to obtain a space (Y_2, d'_2), where $Y_2 = X \backslash \{x_1\}$. Then $Y_1 \cap Y_2 = X \backslash \{x_1, x_2\}$, and d'_1 and d'_2 agree on $X \backslash \{x_1, x_2\}$. By the base case, we disjointly amalgamate (Y_1, d'_1) and (Y_2, d'_2) over $X \backslash \{x_1, x_2\}$ to obtain the desired disjoint amalgamation of (X_1, d_1) and (X_2, d_2). □

Using the previous characterization, we proceed as follows. Fix a distance magma $\mathcal{R}$ and a subset $S \subseteq R$, with $0 \in S$. In order to apply classical Fraïssé theory, we also assume S is countable, which means $\mathcal{K}^S_{\mathcal{R}}$ is a countable (up to isomorphism) class of $\mathcal{L}_S$-structures. If we assume, moreover, S satisfies the four-values condition in $\mathcal{R}$ then, altogether, $\mathcal{K}^S_{\mathcal{R}}$ is a Fraïssé class and so we may define the Fraïssé limit (see [40, Theorem 7.1.2]).

Definition 2.7.5. Given a distance magma $\mathcal{R}$ and a countable subset $S \subseteq R$, such that $0 \in S$ and S satisfies the four-values condition in $\mathcal{R}$, let $\mathcal{U}^S_{\mathcal{R}}$ denote the Fraïssé limit of $\mathcal{K}^S_{\mathcal{R}}$. Let $\mathcal{U}_{\mathcal{R}} = \mathcal{U}^R_{\mathcal{R}}$.

We now obtain a countable $\mathcal{L}_S$-structure $\mathcal{U}^S_{\mathcal{R}}$, and it is clear that $\mathcal{U}^S_{\mathcal{R}} \models T^{\text{ms}}_{S,\mathcal{R}}$. By Proposition 2.4.2($b$), we may consider $\mathcal{U}^S_{\mathcal{R}}$ as an $\mathcal{S}^*$-metric space. However, since the age of $\mathcal{U}_{\mathcal{R}}^S$ is precisely $\mathcal{K}_{\mathcal{R}}^S$, it follows that $\mathrm{Spec}(\mathcal{U}_{\mathcal{R}}^S) = S$. In particular, we may view $\mathcal{U}_{\mathcal{R}}^S$ as an $\mathcal{R}$-metric space with spectrum S, which justifies the next definition.

Definition 2.7.6. Given a distance magma $\mathcal{R}$ and a countable subset $S \subseteq R$, such that $0 \in S$ and S satisfies the four-values condition in $\mathcal{R}$, we call $\mathcal{U}_{\mathcal{R}}^S$ the **$\mathcal{R}$-Urysohn space with spectrum** S.

We summarize our results with the following combinatorial description of $\mathcal{U}_{\mathcal{R}}^S$.

Theorem 2.7.7. *Suppose $\mathcal{R}$ is a distance magma and $S \subseteq R$ is countable, with $0 \in S$.*

(a) *If S satisfies the four-values condition in $\mathcal{R}$ then $\mathcal{U}_{\mathcal{R}}^S$ is the unique $\mathcal{R}$-metric space satisfying the following properties:*

  (i) *$\mathcal{U}_{\mathcal{R}}^S$ is countable and $\mathrm{Spec}(\mathcal{U}_{\mathcal{R}}^S) = S$;*

  (ii) *(ultrahomogeneity) any partial isometry between two finite subspaces of $\mathcal{U}_{\mathcal{R}}^S$ extends to a total isometry of $\mathcal{U}_{\mathcal{R}}^S$;*

  (iii) *(universality) any element of $\mathcal{K}_{\mathcal{R}}^S$ is isometric to a subspace of $\mathcal{U}_{\mathcal{R}}^S$.*

(b) *If there is a countable, universal, and ultrahomogeneous $\mathcal{R}$-metric space $\mathcal{A}$, with $\mathrm{Spec}(\mathcal{A}) = S$, then S satisfies the four-values condition in $\mathcal{R}$ and $\mathcal{A}$ is isometric to $\mathcal{U}_{\mathcal{R}}^S$.*

Remark 2.7.8.

1. Consider the distance monoid $\mathcal{Q} = (\mathbb{Q}^{\geq 0}, +, \leq, 0)$. Then $\mathcal{U}_{\mathcal{Q}}$ is precisely the classical *rational Urysohn space*, which is an important example in model theory, descriptive set theory, Ramsey theory, and topological dynamics of isometry groups. The completion of the rational Urysohn space is called the *Urysohn space*, and is the universal separable metric space. Both the rational Urysohn space and the complete Urysohn space were first constructed by Urysohn (see [89], [90]). Further details and results can be found in [64].

2. In Proposition 2.7.4, there is no restriction on the cardinality of S. However, in order to apply classical Fraïssé theory and construct a countable space $\mathcal{U}_{\mathcal{R}}^S$, we must assume S is countable. In [77], Sauer considers arbitrary subsets $S \subseteq \mathbb{R}^{\geq 0}$ and, combining the four-values condition with certain topological properties, characterizes the existence of a universal separable complete metric space with distances in S (e.g. if $S = \mathbb{R}^{\geq 0}$ then this produces the Urysohn space).

Note that if $S \subseteq R$ is countable and $\mathcal{R}$-magmal then $\mathcal{K}_{\mathcal{S}} = \mathcal{K}_{\mathcal{R}}^S$ and $\mathcal{U}_{\mathcal{S}} = \mathcal{U}_{\mathcal{R}}^S$. In this case, we have the following nice characterization of when $\mathcal{U}_{\mathcal{S}}$ exists. This result was first shown for (topologically) closed subsets of $(\mathbb{R}^{\geq 0}, +, \leq, 0)$ by Sauer in [78, Theorem 5], and the following is, once again, a direct generalization.

Proposition 2.7.9. *Suppose $S \subseteq R$ is $\mathcal{R}$-magmal. Then S satisfies the four-values condition in $\mathcal{R}$ if and only if $\oplus_S$ is associative on S.*

Proof. Suppose S satisfies the four-values condition in $\mathcal{R}$, and fix $r, s, t \in S$. Since $\oplus_S$ is commutative, it suffices to show $(r \oplus_S s) \oplus_S t \leq r \oplus_S (s \oplus_S t)$. Let $u = (r \oplus_S s) \oplus_S t$. Then $(r \oplus_S s, r, s)$ and $(r \oplus_S s, u, t)$ are both $\mathcal{R}$-triangles. By the four-values condition, there is $v \in S$ such that (v, r, u) and (v, s, t) are $\mathcal{R}$-triangles. Therefore $u \leq r \oplus_S v \leq r \oplus_S (s \oplus_S t)$, as desired.

Conversely, assume $\oplus_S$ is associative on S. Fix $u_1, u_2, v_1, v_2, s \in S$ such that (s, u_1, u_2) and (s, v_1, v_2) are $\mathcal{R}$-triangles. Without loss of generality, assume $u_1 \oplus v_1 \leq u_2 \oplus v_2$. Let $t = u_1 \oplus_S v_1$. Then (t, u_1, v_1) is clearly an $\mathcal{R}$-triangle, so it suffices to show (t, u_2, v_2) is an $\mathcal{R}$-triangle. We have $t \leq u_1 \oplus v_1 \leq u_2 \oplus v_2$ by assumption, so it remains to show $v_2 \leq u_2 \oplus t$ and $u_2 \leq v_2 \oplus t$. Note that $s \leq u_2 \oplus_S u_1$ and $v_2 \leq s \oplus_S v_1$ since (s, u_1, u_2) and (s, v_1, v_2) are $\mathcal{R}$-triangles. Therefore

$$v_2 \leq s \oplus_S v_1 \leq (u_2 \oplus_S u_1) \oplus_S v_1 = u_2 \oplus_S (u_1 \oplus_S v_1) \leq u_2 \oplus t.$$

Similarly, note that $s \leq v_2 \oplus_S v_1$ and $u_2 \leq s \oplus_S u_1$ since (s, v_1, v_2) and (s, u_1, u_2) are $\mathcal{R}$-triangles. Therefore

$$u_2 \leq (v_2 \oplus_S v_1) \oplus_S u_1 = v_2 \oplus_S (v_1 \oplus_S u_1) = v_2 \oplus_S (u_1 \oplus_S v_1),$$

as desired. □

By Corollary 2.6.10 and Proposition 2.7.9, we obtain the following corollary.

Corollary 2.7.10. *If $S \subseteq R$ is $\mathcal{R}$-magmal, and satisfies the four-values condition in $\mathcal{R}$, then $\mathcal{S}^*$ is a distance monoid.*

Example 2.7.11. We show that, in the previous corollary, the magmality assumption is necessary. Let $\mathcal{R} = (\mathbb{R}^{\geq 0}, +, \leq, 0)$ and $S = [0, 2) \cup (4, \infty)$. Note that S is not $\mathcal{R}$-magmal since $P_S(1, 1)$ does not contain a maximal element. To verify the four-values condition for S, fix $u_1, u_2, v_1, v_2, s \in S$ such that $\max\{|u_1 - u_2|, |v_1 - v_2|\} \leq s \leq \min\{u_1 + u_2, v_1 + v_2\}$. Then we have $\max\{|u_1 - v_1|, |u_2 - v_2|\} \leq \min\{u_1 + v_1, u_2 + v_2\}$, so it suffices to show that, if $\max\{|u_1 - v_1|, |u_2 - v_2|\} \geq 2$, then we must have $\min\{u_1 + v_1, u_2 + v_2\} > 4$. This immediate from the choice of S. On the other hand, $+^*_S$ is not associative on S^*. Indeed, we have $X := (4, \infty) \in \kappa(S)$, and, using Proposition 2.3.13(a), it is straightforward to show $(1 +^*_S 1) +^*_S g_X = 8^+$ and $1 +^*_S (1 +^*_S g_X) = 6^+$.

The next result will be useful when checking the four-values condition. It is a generalization of [27, Example 1.6.3].[1]

[1] The reader is cautioned of an error in [27, Example 1.6.3]. Specifically, the set $\{2^{-n} : n > 0\}$ does *not* satisfy the four-values condition in $(\mathbb{R}^{\geq 0}, +, \leq, 0)$ (e.g. let $u_1 = u_2 = \frac{1}{4}$, $v_1 = \frac{1}{2}$, and $v_2 = 1$).

Definition 2.7.12. A subset $S \subseteq R$ is a **good value set in** $\mathcal{R}$ if $0 \in S$ and, for all $r, s \in S$, if there some $t \in S$ such that $r \oplus s \leq t$, then $r \oplus s \in S$.

Proposition 2.7.13. *Assume $\mathcal{R}$ is metrically complete. Fix $S \subseteq R$, with $0 \in S$, and consider the following property of S.*

$(*)$ *For all $u_1, u_2, v_1, v_2 \in S$, if $|u_1 \ominus u_2| \leq v_1 \oplus v_2$ there is some $t \in S$ such that $|u_1 \ominus u_2| \leq t \leq v_1 \oplus v_2$.*

Then:

(a) *If $S \subseteq R$ is a good value set in $\mathcal{R}$ then $(*)$ holds.*

(b) *If $(*)$ holds then, assuming $\oplus$ is associative, S satisfies the four-values condition in $\mathcal{R}$.*

Proof. Part (a). Suppose S is a good value set in $\mathcal{R}$ and suppose $u_1, u_2, v_1, v_2 \in S$ are such that $|u_1 \ominus u_2| \leq v_1 \oplus v_2$. If $v_1 \oplus v_2 \in S$ then we may let $t = v_1 \oplus v_2$. Otherwise, we must have $\max\{u_1, u_2\} < v_1 \oplus v_2$, and we may let $t = \max\{u_1, u_2\}$.

Part (b). Assume $\oplus$ is associative and suppose $(*)$ holds. Fix $u_1, u_2, v_1, v_2, s \in S$ such that (s, u_1, u_2) and (s, v_1, v_2) are $\mathcal{R}$-triangles. We want to find $t \in S$ such that (t, u_1, v_1) and (t, u_2, v_2) are $\mathcal{R}$-triangles. By assumption, we have

$$\max\{|u_1 \ominus u_2|, |v_1 \ominus v_2|\} \leq s \leq \min\{u_1 \oplus u_2, v_1 \oplus v_2\}.$$

Fix $i, j \in \{1, 2\}$ such that

$$|u_i \ominus v_i| = \max\{|u_1 \ominus v_1|, |u_2 \ominus v_2|\} \text{ and } u_j \oplus v_j = \min\{u_1 \oplus v_1, u_2 \oplus v_2\}.$$

Claim: $|u_i \ominus v_i| \leq u_j \oplus v_j$.
Proof: By Proposition 2.6.3(c), we may assume $i \neq j$. We need to show $u_j \leq (u_i \oplus v_i) \oplus v_j$ and $v_j \leq (u_i \oplus v_i) \oplus v_j$. By associativity, we may show $u_j \leq u_i \oplus (v_i \oplus v_j)$ and $v_j \leq u_i \oplus (v_i \oplus v_j)$. These inequalities follow from $|u_1 \ominus u_2| \leq v_1 \oplus v_2$ and $|v_1 \ominus v_2| \leq u_1 \oplus u_2$, respectively. $\dashv_{\text{claim}}$

By the claim and $(*)$, there is $t \in S$ such that $|u_i \ominus v_i| \leq t \leq u_j \oplus v_j$. Therefore t is as desired. □

Remark 2.7.14. In Proposition 2.7.13, the assumption that $\mathcal{R}$ is metrically complete is made only for the purposes of smoother presentation. For general $\mathcal{R}$, we only need to replace all instances of "$|u \ominus v| \leq s$" with "$s \in M_{\mathcal{R}}(u, v)$", and all instances of "$|u \ominus v| \leq |r \ominus s|$" with "$M_{\mathcal{R}}(r, s) \subseteq M_{\mathcal{R}}(u, v)$".

On the other hand, if $\mathcal{R}$ is metrically complete then one may define a **good difference set in** $\mathcal{R}$ to be a subset $S \subseteq R$ such that $0 \in S$ and, for all $r, s \in S$, if there is $t \in S^{>0}$ such that $t \leq |r \ominus s|$ then $|r \ominus s| \in S$. Then, using a similar argument, one may show that good difference sets satisfy $(*)$ as stated in Proposition 2.7.13. The analog of *good difference* set for general distance magmas $\mathcal{R}$ could be formulated as: for all $r, s \in S$, if there is some $t \in S^{>0}$ such that $t \notin M_{\mathcal{R}}(r, s)$ then, for all $w \in M_{\mathcal{R}}(r, s)$ there is $u \in S \cap M_{\mathcal{R}}(r, s)$, with $u \leq w$.

Finally, it is worth noting that if $\mathcal{R}$ is a countable distance *monoid* then there is a more direct way to demonstrate that $\mathcal{K}_{\mathcal{R}}$ is a Fraïssé class. In particular, to prove $\mathcal{K}_{\mathcal{R}}$ has the amalgamation property, one may use the natural generalization of the notion of "free amalgamation of metric spaces." We first define this notion for generalized metric spaces over an arbitrary distance magma.

Definition 2.7.15. Let $\mathcal{R}$ be a distance magma.

1. Suppose $\mathcal{A} = (A, d_A)$ and $\mathcal{B} = (B, d_B)$ are finite $\mathcal{R}$-metric spaces such that $A \cap B \neq \emptyset$ and $d_A|_{A\cap B} = d_B|_{A\cap B}$. Define the $\mathcal{R}$-colored space $\mathcal{A} \otimes \mathcal{B} = (C, d_C)$ where $C = A \cup B$ and

$$d_C(x,y) = \begin{cases} d_A(x,y) & \text{if } x,y \in A \\ d_B(x,y) & \text{if } x,y \in B \\ \min\limits_{z\in A\cap B}(d_A(x,z) \oplus d_B(z,y)) & \text{if } x \in A\backslash B \text{ and } y \in B\backslash A. \end{cases}$$

2. $\mathcal{R}$ **admits free amalgamation of metric spaces** if $\mathcal{A} \otimes \mathcal{B}$ is an $\mathcal{R}$-metric space for all finite $\mathcal{R}$-metric spaces $\mathcal{A}$ and $\mathcal{B}$.

Proposition 2.7.16. *Let $\mathcal{R}$ be a distance magma. Then $\mathcal{R}$ admits free amalgamation of metric spaces if and only if $\oplus$ is associative.*

Proof. The forward direction follows from Proposition 2.7.4 and Proposition 2.7.9. Conversely, assume $\oplus$ is associative. We check the nontrivial triangle inequalities. Let $d_* = d_A|_{A\cap B} = d_B|_{A\cap B}$.
Case 1: $x_1, x_2 \in A\backslash B$ and $y \in B\backslash A$.

First,

$$\begin{aligned} d_C(x_1,y) \oplus d_C(x_2,y) &= \min_{z\in A\cap B}(d_A(x_1,z) \oplus d_B(z,y)) \oplus \min_{z\in A\cap B}(d_A(x_2,z) \oplus d_B(z,y)) \\ &= \min_{z,z'\in A\cap B}(d_A(x_1,z) \oplus d_B(z,y) \oplus d_A(x_2,z') \oplus d_B(z',y)) \\ &\geq \min_{z,z'\in A\cap B}(d_A(x_1,z) \oplus d_A(x_2,z') \oplus d_*(z,z')) \\ &\geq \min_{z\in A\cap B}(d_A(x_1,z) \oplus d_A(x_2,z)) \\ &\geq d_A(x_1,x_2) \\ &= d_C(x_1,x_2). \end{aligned}$$

Next, for any $z \in A \cap B$,

$$\begin{aligned} d_C(x_1,y) &= \min_{z'\in A\cap B}(d_A(x_1,z') \oplus d_B(z',y)) \\ &\leq d_A(x_1,z) \oplus d_B(z,y) \\ &\leq d_A(x_1,x_2) \oplus d_A(x_2,z) \oplus d_B(z,y). \end{aligned}$$

Therefore

$$d_C(x_1, y) \leq d_A(x_1, x_2) \oplus \min_{z \in A \cap B}(d_A(x_2, z) \oplus d_B(z, y)) = d_C(x_1, x_2) \oplus d_C(x_2, y).$$

Case 2: $x \in A$, $y \in B$, $z \in A \cap B$.

First,

$$d_C(x, y) = \min_{z' \in A \cap B}(d_A(x, z') \oplus d_B(z', y)) \leq d_A(x, z) \oplus d_B(z, y) = d_C(x, z) \oplus d_C(z, y).$$

Next, for any $z' \in A \cap B$,

$$\begin{aligned} d_C(x, z) &= d_A(x, z) \\ &\leq d_A(x, z') \oplus d_*(z', z) \\ &\leq d_A(x, z') \oplus d_B(z', y) \oplus d_B(y, z). \end{aligned}$$

Therefore,

$$d_C(x, z) \leq \min_{z' \in A \cap B}(d_A(x, z') \oplus d_B(z', y) \oplus d_B(y, z)) = d_C(x, y) \oplus d_C(y, z). \qquad \square$$

2.8 Quantifier Elimination in Theories of Generalized Urysohn Spaces

In this section, we consider quantifier elimination in the theory of a generalized Urysohn space of the kind constructed in Section 2.7. The setup is as follows. We have a distance magma $\mathcal{R} = (R, \oplus, \leq, 0)$ and a countable subset $S \subseteq R$, such that $0 \in S$ and S satisfies the four-values condition in $\mathcal{R}$. We will also assume S is $\mathcal{R}$-magmal. The reason for this is that Lemma 2.8.10, which is a key tool in this section, crucially relies on the existence of an associative binary operation on S. In light of Remark 2.5.3(1), in order to cover this setup it suffices to just fix a countable distance *monoid* $\mathcal{R} = (R, \oplus, \leq, 0)$ and let $S = R$. By previous results we have:

1. $\mathcal{U}_\mathcal{R}$ exists. Let $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ denote the complete $\mathcal{L}_R$-theory of $\mathcal{U}_\mathcal{R}$.

2. $\mathcal{R}^* = (R^*, \oplus^*, \leq^*, 0)$ is a metrically complete distance monoid (where $\oplus^* := \oplus^*_R$). Let $\ominus^* := \ominus^*_R$ be the generalized difference operation defined on $\mathcal{R}^*$ (Definition 2.6.7).

3. Given $\alpha, \beta \in R^*$, if $\mu = \sup P_R(\alpha, \beta)$, then

$$\alpha \oplus^* \beta = \begin{cases} \mu^+ & \text{if } \mu \in \nu(R) \text{ and } \mu < r \oplus s \text{ for all } r \in \overline{\nu}_R(\alpha) \text{ and } s \in \overline{\nu}_R(\beta) \\ \mu & \text{otherwise.} \end{cases}$$

We continue to consider $\mathcal{R}$ as an $\mathcal{L}_{\text{om}}$-substructure (in particular, submonoid) of $\mathcal{R}^*$. Therefore, to ease notation, we omit the asterisks on the symbols in $\mathcal{L}_{\text{om}}$, and let $\mathcal{R}^* = (R^*, \oplus, \leq, 0)$. We will also omit the asterisk on $\ominus^*$ (note, however, that R is not necessarily closed under $\ominus^*$).

By universality of $\mathcal{U}_{\mathcal{R}}$ and Theorem 2.5.7, we obtain the following fact.

Proposition 2.8.1. *Any $\mathcal{R}^*$-metric space is isometric to a subspace of some model of* $\text{Th}(\mathcal{U}_{\mathcal{R}})$.

The goal of this section is Theorem B, a characterization of quantifier elimination for $\text{Th}(\mathcal{U}_{\mathcal{R}})$. The proof will rely on *extension axioms*, i.e., $\mathcal{L}_R$-sentences approximating one-point extensions of finite $\mathcal{R}^*$-metric spaces. We begin with several definitions in this direction.

Definition 2.8.2. Fix an $\mathcal{R}^*$-metric space $\mathcal{A} = (A, d_A)$.

1. A function $f : A \longrightarrow R^*$ is an **$\mathcal{R}^*$-Katětov map on** $\mathcal{A}$ if, for all $x, y \in A$, the triple $(d_A(x, y), f(x), f(y))$ is an $\mathcal{R}^*$-triangle.

2. Let $E_{\mathcal{R}^*}(\mathcal{A})$ be the set of $\mathcal{R}^*$-Katětov maps on $\mathcal{A}$.

Remark 2.8.3. Note that the definition of Katětov map makes sense in the context of an arbitrary distance magma. These maps take their name from [45], in which Katětov uses them to construct the Urysohn space, as well as similar metric spaces in larger cardinalities. See [64] for more on Katětov maps in the classical distance structure $(\mathbb{R}^{\geq 0}, +, \leq, 0)$, including an analysis of $E_{\mathcal{R}}(\mathcal{A})$ as a topological space.

It is also worth mentioning that Katětov maps have a natural model theoretic characterization as quantifier-free 1-types. In particular, if $\mathcal{A}$ is an $\mathcal{R}^*$-metric space then, by Proposition 2.8.1, we may fix $M \models \text{Th}(\mathcal{U}_{\mathcal{R}})$ such that $\mathcal{A}$ is a subspace of (M, d_M). Let $S_1^{\text{qf}}(A)$ be the space of quantifier-free 1-types over the parameter set A. Given $f \in E_{\mathcal{R}^*}(\mathcal{A})$, define $q_f(x) = \bigcup_{a \in A} p_{f(a)}(x, a)$. Conversely, given $q(x) \in S_1^{\text{qf}}(A)$, let $f_q : A \longrightarrow R^*$ such that $p_{f_q(a)}(x, a) \subseteq q(x)$. Then one may verify $f \mapsto q_f$ is a bijection from $E_{\mathcal{R}^*}(\mathcal{A})$ to $S_1^{\text{qf}}(A)$, with inverse $q \mapsto f_q$.

Going forward, we will only consider *non-principal* Katětov maps, i.e., those not containing 0 in their image.

Definition 2.8.4. Fix an $\mathcal{R}^*$-metric space $\mathcal{A} = (A, d_A)$.

1. Let $E^+_{\mathcal{R}^*}(\mathcal{A}) = \{f \in E_{\mathcal{R}^*}(\mathcal{A}) : f(a) > 0 \text{ for all } a \in A\}$.

2. Given $f \in E^+_{\mathcal{R}^*}(\mathcal{A})$, define an $\mathcal{R}^*$-metric space $\mathcal{A}^f = (A^f, d_A)$ where $A^f = A \cup \{z_f\}$, with $z_f \notin A$, and, for all $x \in A$, $d_A(x, z_f) = f(x) = d_A(z_f, x)$.

Next, we give a variation of the notion of R-approximation, which will simplify some steps of the arguments in this section. In particular, we currently think of R-approximations as functions defined on sets of distances in $\mathcal{R}^*$ and, given an $\mathcal{R}^*$-metric space $\mathcal{A}$, we consider R-approximations of the distance set $\operatorname{Spec}(\mathcal{A})$. In the following arguments, it will be more convenient to think of R-approximations as functions on the space $\mathcal{A}$ itself. In other words, if $a, b, a', b' \in A$ and $d_A(a,b) = d_A(a',b') = \alpha \in R^*$, then we allow for the possibility that the approximation of α differs when considering the pair of points (a,b) versus the pair (a',b').

Definition 2.8.5. Fix an $\mathcal{R}^*$-metric space $\mathcal{A} = (A, d_A)$.

1. A symmetric function $\Phi : A \times A \longrightarrow I(R)$ is an R-**approximation** of $\mathcal{A}$ if $d_A(a,b) \in \Phi(a,b)$ for all $a, b \in A$.
2. Given $f \in E^+_{\mathcal{R}^*}(\mathcal{A})$, if Φ is an R-approximation of $\mathcal{A}^f$ and $x \in A$, then we let $\Phi(x) = \Phi(x, z_f)$.

Given an $\mathcal{R}^*$-metric space $\mathcal{A}$, and an R-approximation Φ of $\operatorname{Spec}(\mathcal{A})$, we can naturally consider Φ as an R-approximation of $\mathcal{A}$ in the obvious way. Conversely, given an R-approximation Φ of $\mathcal{A}$, if A is finite then we can construct a refinement $\hat{\Phi}$, which is an R-approximation of $\operatorname{Spec}(A)$, in the following way.

Definition 2.8.6. Suppose $\mathcal{A} = (A, d_A)$ is an $\mathcal{R}^*$-metric space, with A finite. Given an R-approximation Φ of $\mathcal{A}$ and $\alpha \in \operatorname{Spec}(\mathcal{A})$, define

$$\begin{aligned} \hat{\Phi}^-(\alpha) &= \max\{\Phi^-(a,b) : d_A(a,b) = \alpha\} \text{ and} \\ \hat{\Phi}^+(\alpha) &= \min\{\Phi^+(a,b) : d_A(a,b) = \alpha\}. \end{aligned}$$

Let $\hat{\Phi}(\alpha) = (\hat{\Phi}^-(\alpha), \hat{\Phi}^+(\alpha)]$, and note that $\hat{\Phi}$ is an R-approximation of $\operatorname{Spec}(\mathcal{A})$ in the sense of Definition 2.2.2 and Notation 2.3.6.

We now define some specific $\mathcal{L}_R$-formulas. Recall that ω_R denotes the maximal element of R^*.

Definition 2.8.7.

1. Given $I \in I(R)$, define the $\mathcal{L}_R$-formula

$$d(x,y) \in I := \begin{cases} r < d(x,y) \leq s & \text{if } I = (r,s], \text{ and } s < \omega_R \text{ or } s = \omega_R \in R \\ d(x,y) > r & \text{if } I = (r, \omega_R] \text{ and } \omega_R \notin R \\ x = y & \text{if } I = \{0\}. \end{cases}$$

2. Fix a finite $\mathcal{R}^*$-metric space $\mathcal{A}$ and $f \in E^+_{\mathcal{R}^*}(\mathcal{A})$. Suppose Φ is an R-approximation of $\mathcal{A}^f$. Let $A = \{a_1, \ldots, a_n\}$, and fix a tuple $\bar{x} = (x_1, \ldots, x_n)$ of variables.

(*a*) Define the quantifier-free $\mathcal{L}_R$-formulas

$$C^{\Phi}_{\mathcal{A}}(\bar{x}) := \bigwedge_{1\leq i,j\leq n} d(x_i,x_j)\in\Phi(a_i,a_j) \text{ and } K^{\Phi}_{\mathcal{A}}(\bar{x},y) := \bigwedge_{1\leq i\leq n} d(x_i,y)\in\Phi(a_i).$$

(*b*) Define the $\mathcal{L}_R$-sentence

$$\epsilon^{\Phi}_{\mathcal{A}} := \forall x_1\dots x_n\Big(C^{\Phi}_{\mathcal{A}}(\bar{x})\to\exists y K^{\Phi}_{\mathcal{A}}(\bar{x},y)\Big).$$

Sentences of the form $\epsilon^{\Phi}_{\mathcal{A}}$ should be viewed as extension axioms approximating Katětov maps. Note that if Φ is a poor approximation of $\mathcal{A}^f$ then there is no reason to expect $\mathcal{U}_{\mathcal{R}}\models\epsilon^{\Phi}_{\mathcal{A}}$. This observation motivates our final definition.

Definition 2.8.8.

1. An **extension scheme** is a triple $(\mathcal{A},f,\Psi)$, where $\mathcal{A}$ is a finite $\mathcal{R}^*$-metric space, $f\in E^+_{\mathcal{R}^*}(\mathcal{A})$, and Ψ is an R-approximation of $\mathcal{A}^f$.
2. $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ **admits extension axioms** if, for all extension schemes $(\mathcal{A},f,\Psi)$, there is an R-approximation Φ of $\mathcal{A}^f$ such that Φ refines Ψ and $\mathcal{U}_{\mathcal{R}}\models\epsilon^{\Phi}_{\mathcal{A}}$.

To avoid inconsequential complications when $\omega_R\notin R$, we make the following reduction. Call an extension scheme $(\mathcal{A},f,\Psi)$ **standard** if $\Psi^+(A^f\times A^f)\subseteq S$.

Proposition 2.8.9. $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ *admits extension axioms if and only if, for all standard extension schemes* $(\mathcal{A},f,\Psi)$*, there is an* R*-approximation* Φ *of* $\mathcal{A}^f$ *such that* Φ *refines* Ψ *and* $\mathcal{U}_{\mathcal{R}}\models\epsilon^{\Phi}_{\mathcal{A}}$.

Proof. The forward direction is trivial. If $\omega_R\in R$ then the reverse direction is also trivial. So we assume $\omega_R\notin R$. Fix an extension scheme $(\mathcal{A},f,\Psi)$, with $\mathcal{A}=(A,d_A)$. Define the set $A_0=\{a\in A : f(a)<\omega_R\}$. If $A_0=\emptyset$ then we claim $\mathcal{U}_{\mathcal{R}}\models\epsilon^{\Psi}_{\mathcal{A}}$. Indeed, if $\mathcal{U}_{\mathcal{R}}\models C^{\Psi}_{\mathcal{A}}(\bar{b})$ and $s\in R$ is such that $d(b_i,b_j)\leq s$ for all $b_i,b_j\in\bar{b}$, then, by universality and homogeneity, there is some $c\in\mathcal{U}_{\mathcal{R}}$ such that $d(b_i,c)=s$ for all $b_i\in\bar{b}$. If, moreover, $\max\{\Psi^-(a):a\in A\}<s$, then $\mathcal{U}_{\mathcal{R}}\models K^{\Psi}_{\mathcal{A}}(\bar{b},c)$.

So we may assume $A_0\neq\emptyset$. Set $f_0=f|_{A_0}$, $\mathcal{A}_0=(A_0,d_A)$, and $\Psi_0=\Psi|_{A_0\times A_0}$. From Proposition 2.5.4, it follows that $d_A(a,b)<\omega_R$ for all $a,b\in A_0$, and so we may assume $(\mathcal{A}_0,f_0,\Psi_0)$ is a standard extension scheme. By assumption, there is an R-approximation Φ_0 of $\mathcal{A}_0^{f_0}$ such that Φ_0 refines Ψ_0 and $\mathcal{U}_{\mathcal{R}}\models\epsilon^{\Phi_0}_{\mathcal{A}_0}$. We define an R-approximation Φ of $\mathcal{A}^f$ such that, given $a,b\in A^f$,

$$\Phi(a,b)=\begin{cases}\Phi_0(a,b) & \text{if } a,b\in A_0\cup\{z_f\}\\ \hat{\Psi}(d(a,b)) & \text{otherwise.}\end{cases}$$

Then Φ refines Ψ, and we show $\mathcal{U}_{\mathcal{R}}\models\epsilon^{\Phi}_{\mathcal{A}}$. Note, in particular, that if $a,b\in A^f$ and $d_A(a,b)=\omega_R$ then $\Phi(a,b)=\hat{\Psi}(\omega_R)$.

Let $A = \{a_1, \ldots, a_n\}$, with $A_0 = \{a_1, \ldots, a_k\}$ for some $1 \leq k \leq n$. Suppose $\bar{b} \in \mathcal{U}_\mathcal{R}$ is such that $\mathcal{U}_\mathcal{R} \models C^\Phi_\mathcal{A}(\bar{b})$. If $\bar{b}_0 = (b_1, \ldots, b_k)$ then $\mathcal{U}_\mathcal{R} \models C^{\Phi_0}_{\mathcal{A}_0}(\bar{b}_0)$ so there is some $c \in \mathcal{U}_\mathcal{R}$ such that $\mathcal{U}_\mathcal{R} \models K^{\Phi_0}_{\mathcal{A}_0}(\bar{b}_0, c)$. By homogeneity of $\mathcal{U}_\mathcal{R}$ and Proposition 2.7.16, we may assume $c\bar{b}$ is isometric to $c\bar{b}_0 \otimes \bar{b}$. We claim $\mathcal{U}_\mathcal{R} \models K^\Phi_\mathcal{A}(\bar{b}, c)$, and it suffices to show $d(b_i, c) > \Phi^-(a_i)$ for all $k < i \leq n$. For this, given $k < i \leq n$, there is some $1 \leq j \leq k$ such that $d(b_i, c) = d(b_i, b_j) \oplus d(b_j, c)$. Since $a_j \in A_0$ and $a_i \in A \backslash A_0$, we have $d_A(a_i, a_j) = \omega_R$ by Proposition 2.5.4. Since $\mathcal{U}_\mathcal{R} \models C^\Phi_\mathcal{A}(\bar{b})$, we have $d(b_i, c) \geq d(b_i, b_j) > \Phi^-(a_i, a_j) = \hat{\Psi}^-(\omega_R) = \Phi^-(a_i)$. □

Next, we give sufficient conditions for when, in a standard extension scheme $(\mathcal{A}, f, \Phi)$, Φ is a good enough approximation of $\mathcal{A}^f$ to ensure $\mathcal{U}_\mathcal{R} \models \epsilon^\Phi_\mathcal{A}$.

Lemma 2.8.10. *Suppose $(\mathcal{A}, f, \Phi)$ is a standard extension scheme such that:*

(i) for all $a, b \in A$, $\Phi^+(a, b) \leq \Phi^+(a) \oplus \Phi^+(b)$;

(ii) for all $a, b \in A$ and $s \in R$, if $\Phi^-(a, b) < s$ then $\Phi^-(a) < s \oplus \Phi^+(b)$.

Then $\mathcal{U}_\mathcal{R} \models \epsilon^\Phi_\mathcal{A}$.

Proof. Let $\{a_1, \ldots, a_n\}$ be an enumeration of A such that $\Phi^+(a_1) \leq \ldots \leq \Phi^+(a_n)$. Suppose there are $b_1, \ldots, b_n \in \mathcal{U}_\mathcal{R}$ such that $\mathcal{U}_\mathcal{R} \models C^\Phi_\mathcal{A}(\bar{b})$. We inductively construct $s_1, \ldots, s_n \in R$ such that:

1. $\Phi^-(a_k) < s_k \leq \Phi^+(a_k)$ for all $1 \leq k \leq n$,
2. for all $1 \leq k \leq n$, if $s_k < \Phi^+(a_k)$ then $s_k = s_i \oplus d(b_i, b_k)$ for some $i < k$,
3. for all $1 \leq k \leq n$, if $i < k$ then $(d(b_i, b_k), s_i, s_k)$ is an $\mathcal{R}$-triangle.

Let $s_1 = \Phi^+(a_1)$. Fix $1 < k \leq n$ and suppose we have defined s_i, for $i < k$, satisfying the desired properties. Define

$$s_k = \min(\{\Phi^+(a_k)\} \cup \{s_i \oplus d(b_i, b_k) : i < k\}).$$

Note that (2) is satisfied. We need to verify (1) and (3).
Case 1: $s_k = \Phi^+(a_k)$.

Then (1) is satisfied. For (3), note that for any $i < k$, we have

$$s_k = \Phi^+(a_k) \leq s_i \oplus d(b_i, b_k) \text{ and } s_i \leq \Phi^+(a_i) \leq \Phi^+(a_k) \leq s_k \oplus d(b_i, b_k).$$

So we have left to fix $i < k$ and show $d(b_i, b_k) \leq s_i \oplus s_k$. Toward this end, we construct a sequence $i = i_0 > i_1 > \ldots > i_t$, for some $t \geq 0$, such that $s_{i_t} = \Phi^+(a_{i_t})$ and, for all $0 \leq l < t$, $s_{i_l} = s_{i_{l+1}} \oplus d(b_{i_l}, b_{i_{l+1}})$. Note that such a sequence exists by (2), and since $s_1 = \Phi^+(a_1)$. By construction, we have

$$s_i = d(b_{i_0}, b_{i_1}) \oplus \ldots \oplus d(b_{i_{t-1}}, b_{i_t}) \oplus \Phi^+(a_{i_t}).$$

Therefore, using (i), we have

$$\begin{aligned} d(b_i, b_k) &\leq d(b_{i_0}, b_{i_1}) \oplus \ldots \oplus d(b_{i_{t-1}}, b_{i_t}) \oplus d(b_{i_t}, b_k) \\ &\leq d(b_{i_0}, b_{i_1}) \oplus \ldots \oplus d(b_{i_{t-1}}, b_{i_t}) \oplus \Phi^+(a_{i_t}, a_k) \\ &\leq d(b_{i_0}, b_{i_1}) \oplus \ldots \oplus d(b_{i_{t-1}}, b_{i_t}) \oplus \Phi^+(a_{i_t}) \oplus \Phi^+(a_k) \\ &= s_i \oplus s_k. \end{aligned}$$

Case 2: $s_k = s_i \oplus d(b_i, b_k)$ for some $i < k$.

Then, for any $j < k$, using (3) and induction we have

- $d(b_j, b_k) \leq d(b_i, b_j) \oplus d(b_i, b_k) \leq s_i \oplus s_j \oplus d(b_i, b_k) = s_j \oplus s_k$,
- $s_j \leq s_i \oplus d(b_i, b_j) \leq s_i \oplus d(b_i, b_k) \oplus d(b_j, b_k) = s_k \oplus d(b_j, b_k)$, and
- $s_k = s_i \oplus d(b_i, b_k) \leq s_j \oplus d(b_j, b_k)$,

and so (3) is satisfied. For (1), we must show $\Phi^-(a_k) < s_i \oplus d(b_i, b_k)$. As in Case 1, we construct a sequence $i = i_0 > i_1 > \ldots > i_t$ such that

$$s_i = d(b_{i_0}, b_{i_1}) \oplus \ldots \oplus d(b_{i_{t-1}}, b_{i_t}) \oplus \Phi^+(a_{i_t}).$$

We want to show

$$\Phi^-(a_k) < d(b_{i_0}, b_{i_1}) \oplus \ldots \oplus d(b_{i_{t-1}}, b_{i_t}) \oplus \Phi^+(a_{i_t}) \oplus d(b_i, b_k).$$

By the triangle inequality, it suffices to show

$$\Phi^-(a_k) < d(b_k, b_{i_t}) \oplus \Phi^+(a_{i_t}).$$

Since $\Phi^-(a_k, a_{i_t}) < d(b_k, b_{i_t})$, this follows from (ii).

This finishes the construction of the sequence $s_1, \ldots, s_n$. Let $g : \bar{b} \longrightarrow R$ such that $g(b_i) = s_i$. Then $g \in E^+_{\mathcal{R}^*}(\bar{b}, d)$ by (3), with $\mathrm{Spec}(\bar{b}^g, d) \subseteq R$. Therefore, by universality and homogeneity of $\mathcal{U}_\mathcal{R}$, there is some $c \in \mathcal{U}_\mathcal{R}$ such that $d(b_i, c) = s_i$ for all $1 \leq i \leq n$. By (1), $\mathcal{U}_\mathcal{R} \models K^\Phi_\mathcal{A}(\bar{b}, c)$. □

We can now restate and prove Theorem B.

Theorem 2.8.11. *Suppose $\mathcal{R}$ is a countable distance monoid. The following are equivalent.*

(i) $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ *has quantifier elimination (in the language $\mathcal{L}_R$).*

(ii) $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ *admits extension axioms.*

(iii) *For all $\alpha \in R^*$, if α is nonzero with no immediate predecessor in R^*, then, for all $s \in R$,*

$$\alpha \oplus s = \sup\{x \oplus s : x < \alpha\}.$$

Remark 2.8.12. Condition (*iii*) of Theorem 2.8.11 is clearly a statement about continuity of $\oplus$ from below. On the other hand, the analogous statement concerning continuity from above is always true. In particular, it follows from Corollary 2.6.6 that if $\mathcal{R}$ is a distance magma then, for all $\alpha \in R^*$, if α has no immediate successor then $\alpha \oplus s = \inf\{x \oplus s : \alpha < x\}$ for all $s \in R$ (in fact, for all $s \in R^*$). In other words, we always have some level of continuity in $\mathcal{R}^*$, and quantifier elimination for $\text{Th}(\mathcal{U}_{\mathcal{R}})$ is characterized by further continuity.

Proof of Theorem 2.8.11. (*iii*) $\Rightarrow$ (*ii*): Fix an extension scheme $(\mathcal{A}, f, \Psi)$. By Proposition 2.8.9, we may assume $(\mathcal{A}, f, \Psi)$ is standard. By Proposition 2.5.6, there is a metric R-approximation Ψ_0 of $\text{Spec}(\mathcal{A}^f)$ such that Ψ_0 refines $\hat{\Psi}$. We may consider Ψ_0 as an R-approximation of $\mathcal{A}^f$, which refines Ψ. We define an R-approximation Φ of $\mathcal{A}^f$ such that Φ refines Ψ_0 and $\mathcal{U}_{\mathcal{R}} \models \epsilon^{\Phi}_{\mathcal{A}}$. By Lemma 2.8.10, it suffices to define Φ, refining Ψ_0, so that:

(1) for all $a, b \in A$, $\Phi^+(a, b) \leq \Phi^+(a) \oplus \Phi^+(b)$,

(2) for all $a, b \in A$ and $s \in R$, if $\Phi^-(a, b) < s$ then $\Phi^-(a) < s \oplus \Phi^+(b)$.

Let $\Phi(a) = \Psi_0(a)$ for all $a \in A$. Given distinct $a, b \in A$, let $\Phi^+(a, b) = \Psi_0^+(a, b)$. Since Ψ_0 is metric, we have that for any $a, b \in A$,

$$\Phi^+(a, b) = \Psi_0^+(a, b) \leq \Psi_0^+(a) \oplus \Psi_0^+(b) = \Phi^+(a) \oplus \Phi^+(b),$$

and so (1) is satisfied.

We have left to define $\Phi^-(a, b)$ so that (2) is satisfied. By construction, (2) is equivalent to

$(2)^*$ for all $a, b \in A$ and $s \in R$, if $\Phi^-(a, b) < s$ then $\Psi_0^-(a) < s \oplus \Psi_0^+(b)$.

Note that $(2)^*$ is trivially satisfied when $f(a) \leq f(b)$. Therefore, we fix $a, b \in A$ with $f(b) < f(a)$, and define $\Phi^-(a, b)$ so that $(2)^*$ is satisfied. We will then set $\Phi^-(b, a) = \Phi^-(a, b)$.

Case 1: $d_A(a, b)$ has an immediate predecessor $u \in R^*$.

Then $u \in R$, and we have $\Psi_0^-(a, b) \leq u < d_A(a, b)$. So we may set $\Phi^-(a, b) = u$. For any $s \in R$, if $\Phi^-(a, b) < s$ then $d_A(a, b) \leq s$, and so

$$\Psi_0^-(a) < f(a) \leq d_A(a, b) \oplus f(b) \leq s \oplus \Psi_0^+(b),$$

which verifies $(2)^*$.

Case 2: $d_A(a, b)$ has no immediate predecessor in R^*.

Subcase 2.1: There is $u \in R$ such that $u < d_A(a, b)$ and $u \oplus \Psi_0^+(b) = d_A(a, b) \oplus \Psi_0^+(b)$.

Let $\Phi^-(a, b) = \max\{u, \Psi_0^-(a, b)\}$. For any $s \in R$, if $\Phi^-(a, b) < s$ then

$$\Psi_0^-(a) < f(a) \leq d_A(a, b) \oplus \Psi_0^+(b) = \Phi^-(a, b) \oplus \Psi_0^+(b) \leq s \oplus \Psi_0^+(b),$$

which verifies $(2)^*$.

Subcase 2.2: For all $u \in R$, if $u < d_A(a,b)$ then $u \oplus \Psi_0^+(b) < d_A(a,b) \oplus \Psi_0^+(b)$.

Note that $|\Psi_0^-(a) \ominus \Psi_0^+(b)| \leq d_A(a,b)$. Indeed, we have $\Psi_0^-(a) < f(a) \leq d_A(a,b) \oplus \Psi_0^+(b)$, and, since Ψ_0 is metric and $f(b) < f(a)$, we also have $\Psi_0^+(b) \leq \Psi_0^-(a) \leq d_A(a,b) \oplus \Psi_0^-(a)$.

Claim: $|\Psi_0^-(a) \ominus \Psi_0^+(b)| < d_A(a,b)$.

Proof: Suppose not. Let $\alpha = |\Psi_0^-(a) \ominus \Psi_0^+(b)| = d_A(a,b)$. Then α has no immediate predecessor, and so, by (*iii*), we have

$$\Psi_0^-(a) < f(a) \leq \alpha \oplus \Psi_0^+(b) = \sup\{x \oplus \Psi_0^+(b) : x < \alpha\}.$$

In particular, there is $x \in R^*$ such that $x < |\Psi_0^-(a) \ominus \Psi_0^+(b)|$ and $\Psi_0^-(a) < x \oplus \Psi_0^+(b)$. It follows that $x \oplus \Psi_0^-(a) < \Psi_0^+(b)$, which contradicts $\Psi_0^+(b) \leq \Psi_0^-(a)$. $\dashv_{\text{claim}}$

By the claim, and density of R, there is some $t \in R$ such that $|\Psi_0^-(a) \ominus \Psi_0^+(b)| \leq t < d_A(a,b)$. We may assume $\Psi_0^-(a,b) \leq t$. Note that $t \oplus \Psi_0^+(b) < d_A(a,b) \oplus \Psi_0^+(b)$ by the assumption of this case. Therefore, by (*iii*) and density of R, there is $u \in R$ such that $u < d_A(a,b)$ and $t \oplus \Psi_0^+(b) < u \oplus \Psi_0^+(b)$. Let $\Phi^-(a,b) = u$ and note that $\Psi_0^-(a,b) < \Phi^-(a,b) < d_A(a,b)$. For any $s \in R$, if $\Phi^-(a,b) < s$ then

$$\Psi_0^-(a) \leq t \oplus \Psi_0^+(b) < \Phi^-(a,b) \oplus \Psi_0^+(b) \leq s \oplus \Psi_0^+(b),$$

which verifies $(2)^*$.

(*ii*) $\Rightarrow$ (*i*): Fix $M, N \models \mathrm{Th}(\mathcal{U}_\mathcal{R})$ and suppose $C \subseteq M \cap N$ is a substructure. Fix a quantifier-free formula $\varphi(\bar{x}, y)$. Suppose there is $\bar{a} \in C$ and some $b \in M$ such that $M \models \varphi(\bar{a}, b)$. We want to show there is some $c \in N$ such that $N \models \varphi(\bar{a}, c)$. Without loss of generality, we may assume $\varphi(\bar{x}, y)$ is a conjunction of atomic and negated atomic formulas. If $b \in \bar{a}$ then we may set $c = b$. Otherwise, we may assume $x_i \neq y$ is a conjunct of $\varphi(\bar{x}, y)$ for all $1 \leq i \leq \ell(\bar{x})$.

By Theorem 2.4.3, we have $\mathcal{R}^*$-metrics d_M and d_N on M and N, respectively. Let $\mathcal{A} = (\bar{a}, d_M)$ and define $f : \bar{a} \longrightarrow R^*$ such that $f(a_i) = d_M(a_i, b)$. Then $f \in E^+_{\mathcal{R}^*}(\mathcal{A})$. Moreover, there is some R-approximation Ψ of $\mathcal{A}^f = (\bar{a}b, d_M)$ such that $\varphi(\bar{x}, y)$ is equivalent to $C^\Psi_{\bar{a}}(\bar{x}) \wedge K^\Psi_{\bar{a}}(\bar{x}, y)$. Since R admits extension axioms, there is an R-approximation Φ of $\mathcal{A}^f$ such that Φ refines Ψ and $\mathcal{U}_\mathcal{R} \models \epsilon^\Phi_\mathcal{A}$. Then $N \models C^\Phi_\mathcal{A}(\bar{a})$, so there is some $c \in N$ such that $N \models K^\Phi_\mathcal{A}(\bar{a}, c)$. Since Φ refines Ψ, it follows that $N \models \varphi(\bar{a}, c)$, as desired.

(*i*) $\Rightarrow$ (*iii*): Suppose (*iii*) fails. Fix $s \in R$ and $\alpha \in R^*$ such that $\alpha > 0$ has no immediate predecessor in R^* and $\sup\{x \oplus s : x < \alpha\} < \alpha \oplus s$. By density of R, we may fix $t \in R$ such that $\sup\{x \oplus s : x < \alpha\} \leq t < \alpha \oplus s$.

By Proposition 2.8.1, there is $M \models \mathrm{Th}(\mathcal{U}_\mathcal{R})$, with $a_1, a_2, b \in M$, such that $d_M(a_1, a_2) = \alpha$, $d_M(a_1, b) = s$, and $d_M(a_2, b) = \alpha \oplus s$. Define the $\mathcal{L}_R$-formula

$$\varphi(x_1, x_2, y) := d(x_1, y) \leq s \wedge d(x_2, y) > t,$$

and note that $M \models \varphi(a_1, a_2, b)$.
Claim: There is $N \models \mathrm{Th}(\mathcal{U}_\mathcal{R})$, with $a_1', a_2' \in N$, such that $d_N(a_1', a_2') = \alpha$ and $N \models \neg\exists y\varphi(a_1', a_2', y)$.
Proof: By compactness it suffices to fix $u, v \in R$, with $u < \alpha \leq v$, and show

$$\mathcal{U}_\mathcal{R} \models \exists x_1 x_2 (u < d(x_1, x_2) \leq v \wedge \neg\exists y\varphi(x_1, x_2, y)).$$

Since α has no immediate predecessor, we may use density of R to fix $w \in R$ such that $u < w < \alpha$. Then $w \oplus s \leq t$ by choice of t. Pick $a_1', a_2' \in \mathcal{U}_\mathcal{R}$ with $d(a_1', a_2') = w$. Then $\mathcal{U}_\mathcal{R} \models u < d(a_1', a_2') \leq v$. Moreover, if $\mathcal{U}_\mathcal{R} \models \varphi(a_1', a_2', b')$ then

$$t < d(a_2', b') \leq d(a_1', a_2') \oplus d(a_1', b') = w \oplus d(a_1', b') \leq w \oplus s \leq t,$$

which is a contradiction. So $\mathcal{U}_\mathcal{R} \models \neg\exists y\varphi(a_1', a_2', y)$. $\dashv_{\text{claim}}$

Let N be as in the claim. Then $M \models \exists y\varphi(a_1, a_2, y)$ and $N \models \neg\exists y\varphi(a_1', a_2', y)$. Moreover, (a_1, a_2) and (a_1', a_2') both realize $p_\alpha(x_1, x_2)$, and thus have the same quantifier-free type. Therefore $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ does not have quantifier elimination. □

It is worth observing that the characterization of quantifier elimination can be given in terms of properties of $\mathcal{R}$, although the formulation for $\mathcal{R}^*$ is much cleaner.

Corollary 2.8.13. *The following are equivalent.*

(*i*) $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ *has quantifier elimination.*

(*ii*) *The following continuity properties hold in* $\mathcal{R}$*:*

(*a*) *For all* $r, s \in R$*, if* $s < r$ *and, for all* $x \in R$ *such that* $s \oplus x < r$*, there is some* $y \in R$ *such that* $x < y$ *and* $s \oplus y < r$*, then there is some* $z \in R$ *such that*
$$z \oplus s = \sup\{x \oplus s : x \in R,\ x \oplus s < r\}.$$

(*b*) *For all* $r, s \in R$*, if* r *is nonzero and has no immediate predecessor in* R *then*
$$r \oplus s = \sup\{x \oplus s : x \in R,\ x < r\}.$$

Proof. We use (a) and (b) to refer to the properties stated in (ii) above. We use $(iii)^*$ to refer to property (iii) of Theorem 2.8.11. We need to show $\mathcal{R}$ satisfies (a) and (b) if and only if $\mathcal{R}^*$ satisfies $(iii)^*$.

Suppose $\mathcal{R}$ satisfies (a) and (b). To show $(iii)^*$, fix $\alpha \in R^*$ and $s \in R$, such that α is nonzero and has no immediate predecessor in R^*. We want to show

$$\alpha \oplus s = \sup\{x \oplus s : x < \alpha\}. \qquad (\dagger)$$

If $\alpha \in R$ then ($\dagger$) follows from (b) and density of R. So we may assume $\alpha \notin R$. Since α has no immediate predecessor in R^*, it follows that $\alpha = g_X$ for some $X \in \kappa(S)$.

Suppose, toward a contradiction, (†) fails. Using Proposition 2.3.13, it follows that there is some $r \in R$ such that $r > y \oplus s$ for all $y \notin X$ and $r \leq x \oplus s$ for all $x \in X$. In particular, it follows that $g_X \leq |r \ominus s|$. Note also that $s < r$. We claim $g_X = |r \ominus s|$. If not, then there is some $x \in X$ such that $x < |r \ominus s|$, and so $s \oplus x < r$, which contradicts the choice of r. Altogether, we have that $s < r$ and that $|r \ominus s| = g_X$ has no immediate predecessor in R^*. By (a), there is some $z \in R$ such that

$$z \oplus s = \sup\{x \oplus s : x \in R,\ x \oplus s < r\}.$$

Then $|r \ominus s| \leq z$ and so, since (†) fails, we have

$$\sup\{x \oplus s : x < |r \ominus s|\} < |r \ominus s| \oplus s \leq z \oplus s = \sup\{x \oplus s : x \in R,\ x \oplus s < r\},$$

which, by density of R, is a contradiction.

Conversely, suppose $\mathcal{R}^*$ satisfies $(iii)^*$. From density of R, we immediately obtain that $\mathcal{R}$ satisfies (b). To show (a), fix $r, s \in R$ such that $s < r$ and, for all $x \in R$ such that $s \oplus x < r$, there is some $y \in R$ such that $x < y$ and $s \oplus y < r$. It follows immediately that $|r \ominus s|$ has no immediate predecessor in R^*, and so, by $(iii)^*$, we have

$$|r \ominus s| \oplus s = \sup\{x \oplus s : x < |r \ominus s|\} = r.$$

We want to find $z \in R$ such that $\sup\{x \oplus s : x \in R,\ x \oplus s < r\} = z \oplus s$, and it suffices to find $z \in R$ such that $r = z \oplus s$. If $|r \ominus s| \in R$ then we may set $z = |r \ominus s|$. Otherwise, we have $|r \ominus s| = g_X$ for some $X \in \kappa(R)$, and $g_X \oplus s = r$. By Proposition 2.3.13 it follows that $r = \sup P_R(g_X, s)$ and, if $r \in \nu(R)$, then there is some $z \in X$ such that $r = z \oplus s$. Therefore, we may assume $r \notin \nu(R)$. Let t be the immediate successor of r in R. Suppose, toward a contradiction, $r < s \oplus z$ for all $z \in X$. Then $t \leq s \oplus z$ for all $z \in X$, and so $t \leq s \oplus g_X$ by Corollary 2.6.6, which is a contradiction. Therefore, there is $z \in X$ such that $s \oplus z \leq r$. Since $r = s \oplus g_X \leq s \oplus z$, we have $r = s \oplus z$, as desired. □

The primary reason that we explicate a characterization of quantifier elimination for $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$, which uses only properties of $\mathcal{R}$, is to obtain the following interesting corollary.

Corollary 2.8.14. *There is a first-order $\mathcal{L}_{\mathrm{om}}$-sentence φ_{QE} such that, for any countable distance monoid $\mathcal{R}$, $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ has quantifier elimination if and only if $\mathcal{R} \models \varphi_{\mathrm{QE}}$.*

Proof. It is easily seen that properties (a) and (b) in Corollary 2.8.13(ii) are expressible as a single first-order sentence in $\mathcal{L}_{\mathrm{om}}$. □

In Section 2.9, we will give a number of natural examples, which illustrate that quantifier elimination for $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ holds in many sufficiently nice situations. For now, we give examples where quantifier elimination fails.

Example 2.8.15.

1. Let $\mathcal{R} = (R, +, \leq, 0)$, where $R^{>0} = (\mathbb{Q} \cap [2, \infty)) \backslash \{3\}$. Let $X = (3, \infty) \cap \mathbb{Q} \in \kappa(R)$. Then g_X has no immediate predecessor and

$$\sup\{x + 2 : x < g_X\} = 5 < 5^+ = g_X + 2,$$

and so $\mathcal{R}^*$ fails Theorem 2.8.11(*iii*).

2. Let $\mathcal{R} = (R, +_R, \leq, 0)$, where $R^{>0} = (\mathbb{Q} \cap [1, 2]) \cup \{3\}$ and, by definition, we set $r +_R s = \max\{x \in R : x \leq r + s\}$. For any nonzero $r, s, t \in R$, we have $r +_R (s +_R t) = 3$, and so $+_R$ is associative. Moreover, 2 has no immediate predecessor and

$$\sup\{x +_R 1 : x < 2\} = 2 < 3 = 2 +_R 1,$$

and so $\mathcal{R}^*$ fails Theorem 2.8.11(*iii*).

We invite the reader to observe basic model theoretic facts about $\mathrm{Th}(\mathcal{U}_\mathcal{R})$, which follow from quantifier elimination and classical results in model theory (see e.g. [62]). For instance, assuming quantifier elimination, one may show $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is $\aleph_0$-categorical if and only if R is finite; and $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is *small* (i.e. has a countable saturated model) if and only if R^* is countable. We end this section with an $\forall\exists$-axiomatization of $\mathrm{Th}(\mathcal{U}_\mathcal{R})$, in the case that quantifier elimination holds.

Definition 2.8.16. Suppose $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ has quantifier elimination.

1. Given an extension scheme $(\mathcal{A}, f, \Psi)$, let Φ be an R-approximation of $\mathcal{A}^f$ such that Φ refines Ψ and $\mathcal{U}_\mathcal{R} \models \epsilon^\Phi_\mathcal{A}$. Define $\epsilon(\mathcal{A}, f, \Psi) := \epsilon^\Phi_\mathcal{A}$.

2. Define $T^{\mathrm{ax}}_\mathcal{R} = T^{\mathrm{ms}}_{R,\mathcal{R}} \cup \{\epsilon(\mathcal{A}, f, \Psi) : (\mathcal{A}, f, \Psi) \text{ is an extension scheme}\}$.

Theorem 2.8.17. *Assuming quantifier elimination,* $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ *is axiomatized by* $T^{\mathrm{ax}}_\mathcal{R}$.

Proof. We clearly have $T^{\mathrm{ax}}_\mathcal{R} \subseteq \mathrm{Th}(\mathcal{U}_\mathcal{R})$, so it suffices to show $T^{\mathrm{ax}}_\mathcal{R}$ is a complete $\mathcal{L}_R$-theory. To accomplish this, we fix saturated models M and N of $T^{\mathrm{ax}}_\mathcal{R}$ of the same cardinality κ, and show M and N are isomorphic. Let $(a_i)_{i<\kappa}$ and $(b_i)_{i<\kappa}$ be enumerations of M and N, respectively. We build a sequence of partial $\mathcal{L}_R$-embeddings $\varphi_0 \subseteq \varphi_1 \subseteq \ldots \subseteq \varphi_i \subseteq \ldots$, for $i < \kappa$, such that $\varphi_0 = \emptyset$ and, for all $i < \kappa$,

(*i*) $a_i \in \mathrm{dom}(\varphi_{i+1}) \subseteq M$ and $b_i \in \mathrm{Im}(\varphi_{i+1}) \subseteq N$, and

(*ii*) $|\mathrm{dom}(\varphi_i)| < \kappa$ and $|\mathrm{Im}(\varphi_i)| < \kappa$.

Suppose we have constructed φ_i as above, for $i < j$. If j is a limit ordinal, let $\varphi_j = \bigcup_{i<j} \varphi_i$. Otherwise, let $j = i + 1$ for some $i < \kappa$. We first construct $\psi_j \supseteq \varphi_i$ as follows.

If $a_i \in \mathrm{dom}(\varphi_i)$ then let $\psi_j = \varphi_i$. Otherwise, suppose $a_i \notin \mathrm{dom}(\varphi_i)$. Let $X_i = \mathrm{dom}(\varphi_i)$. Consider the type

$$q(x) = \bigcup_{a \in X_i} p_{d_M(a_i,a)}(x, \varphi_i(a)),$$

and note that $p(x)$ is a partial type over $\mathrm{Im}(\varphi_i)$. We want to show $p(x)$ is realized in N. By saturation, it suffices to realize a formula of the form

$$\theta(x) = \bigwedge_{a \in A} \Phi^-(a_i, a) < d(x, \varphi_i(a)) \leq \Phi^+(a_i, a),$$

where $A \subseteq X_i$ is finite and Φ is an R-approximation of $(A \cup \{a_i\}, d_M)$. Let $\mathcal{A} = (A, d_M)$ and let $f : A \longrightarrow R^*$ such that $f(a) = d(a_i, a)$. Then $(\mathcal{A}, f, \Phi)$ is an extension scheme, and so we have $\epsilon(\mathcal{A}, f, \Phi) \in T^{\mathrm{ax}}_{\mathcal{R}}$. Enumerate $A = \{c_1, \ldots, c_n\}$ and suppose

$$\epsilon(\mathcal{A}, f, \Phi) = \forall x_1 \ldots x_n \Big(C^{\Psi}_{\mathcal{A}}(\bar{x}) \rightarrow \exists y K^{\Psi}_{\mathcal{A}}(\bar{x}, y) \Big),$$

where Ψ is an R-approximation of (A^f, d_M) refining Φ.

Then $M \models C^{\Psi}_{\mathcal{A}}(\bar{c})$. Since $\bar{c} \in \mathrm{dom}(\varphi_i)$ and φ_i is a partial $\mathcal{L}_R$-embedding, it follows that $N \models C^{\Psi}_{\mathcal{A}}(\varphi_i(\bar{c}))$. Since $N \models \epsilon(\mathcal{A}, f, \Phi)$, it follows that there is some $e \in N$ such that $N \models K^{\Psi}_{\mathcal{A}}(\varphi_i(\bar{c}), e)$, which clearly implies $N \models \theta(e)$.

Therefore, we may find $b \in N$ such that $N \models p(b)$. Define $\psi_j : \mathrm{dom}(\varphi_i) \cup \{a_i\} \longrightarrow N$ such that $\psi_j \supset \varphi_i$ and $\psi_j(a_i) = b$. By construction ψ_j is a partial $\mathcal{L}_R$-embedding.

By a similar argument, we may find a partial $\mathcal{L}_R$-embedding $\varphi_j \supseteq \psi_j$ such that $b_i \in \mathrm{Im}(\varphi_j)$. Finally, let $\varphi = \bigcup_{i<\kappa} \varphi_i$, and altogether we have that φ is an $\mathcal{L}_R$-isomorphism from M onto N. $\square$

2.9 Examples

In this section, we consider examples of Urysohn spaces, which arise naturally in the literature, and we verify they all have quantifier elimination.

Definition 2.9.1. Let $\mathcal{R} = (R, \oplus, \leq, 0)$ be a countable distance monoid.

1. $\mathcal{R}$ is **right-closed** if every nonempty subset of R, with an upper bound in R, contains a maximal element.

2. $\mathcal{R}$ is **ultrametric** if $r \oplus s = \max\{r, s\}$ for all $r, s \in R$.

3. Suppose $\mathcal{G} = (G, +, \leq, 0)$ is an ordered abelian group. Let $\mathcal{G}_{\geq 0}$ denote the distance monoid $(G_{\geq 0}, + \leq, 0)$, where $G_{\geq 0} = \{x \in G : x \geq 0\}$.

   $\mathcal{R}$ is **convex** if there is a countable ordered abelian group $\mathcal{G}$ such that $R = I \cup \{0\}$, for some convex subset $I \subseteq G_{\geq 0}$, and, given $r, s \in R$, $r \oplus s = \min\{r + s, \omega_R\}$.

Remark 2.9.2.

1. Note, in particular, that any finite distance monoid is right-closed. Urysohn spaces over finite distance sets in $\mathbb{R}^{\geq 0}$ are studied in [69] and [76] from the perspectives of infinitary Ramsey properties and topological dynamics of isometry groups.

2. Suppose $\mathcal{R}$ is ultrametric. Then $\mathcal{U}_{\mathcal{R}}$ is an ultrametric space with spectrum R. It is important to mention that, in this case, $\text{Th}(\mathcal{U}_{\mathcal{R}})$ is essentially the theory of infinitely refining equivalence relations, indexed by $(R, \leq, 0)$. These are standard examples, often used in a first course in model theory to exhibit a variety of behavior in the stability spectrum (see e.g. [7, Section III.4]). Ultrametric Urysohn spaces are also studied in descriptive set theory and topological dynamics of isometry groups (e.g. [33], [68]).

3. Urysohn spaces over convex monoids have appeared frequently in the literature in the case when $\mathcal{G}$ is a countable subgroup of $(\mathbb{R}, +, \leq, 0)$. These examples are often included in the general study of Urysohn spaces, since closure of a countable distance set under (truncated) addition easily yields a Urysohn space over that set. See, for example, [6], [9], and [85].

We will show that right-closed, ultrametric, and convex distance monoids all yield Urysohn spaces with quantifier elimination. First, however, we record the following fact concerning ultrametric monoids.

Lemma 2.9.3. *Suppose $\mathcal{R}$ is an ultrametric monoid. Then for all $\alpha, \beta \in R^*$ we have $\alpha \oplus \beta = \max\{\alpha, \beta\}$.*

Proof. Fix $\alpha, \beta \in R^*$ and suppose, toward a contradiction, $\alpha \leq \beta < \alpha \oplus \beta$. By density of R, there is $u \in R$ such that $\beta \leq u < \alpha \oplus \beta$. Let Φ be an R-approximation of $(\alpha, \beta, \alpha \oplus \beta)$ such that $\Phi^+(\beta) \leq u \leq \Phi^-(\alpha \oplus \beta)$. Let (r, s, t) be an $\mathcal{R}$-triangle realizing Φ. Then $t \leq r \oplus s = \max\{r, s\} \leq \max\{\Phi^+(\alpha), \Phi^+(\beta)\} < t$, which is a contradiction. □

Proposition 2.9.4. *Suppose $\mathcal{R}$ is a countable distance monoid. If $\mathcal{R}$ is right-closed, ultrametric, or convex, then $\text{Th}(\mathcal{U}_{\mathcal{R}})$ has quantifier elimination.*

Proof. Suppose $\mathcal{R}$ is right-closed. Then it is easy to verify that, if $\alpha \in R^*$ has no immediate predecessor in R^*, then we must have $\alpha = \omega_R \notin R$. From this it follows that any right-closed monoid satisfies Theorem 2.8.11(*iii*).

Next, suppose $\mathcal{R}$ is ultrametric. We want to verify Theorem 2.8.11(iii) holds for $\mathcal{R}^*$. So fix $\alpha \in R^*$ and $s \in R$, such that α is nonzero and has no immediate predecessor in R^*. By Lemma 2.9.3, we must show

$$\max\{\alpha, s\} = \sup\{\max\{x, s\} : x < \alpha\}.$$

If $\alpha \leq s$ then this is trivial, and if $s < \alpha$ then this follows since α has no immediate predecessor in R^*.

Finally, suppose $\mathcal{R}$ is convex. Fix an ordered abelian group $\mathcal{G} = (G, +, \leq, 0)$ such that $R = I \cup \{0\}$ for some convex subset $I \subseteq G_{\geq 0}$. Toward verifying Theorem 2.8.11(iii), we first make the following observations.

(i) If $\alpha \in R^*$ cannot be identified with an element of $G^*_{\geq 0}$, then either $\alpha = \omega_R \notin R$ or $\alpha = 0^+$.

(ii) Given $r, s \in G_{\geq 0}$, if $|r -^* s|$ is the generalized difference operation on $\mathcal{G}^*_{\geq 0}$ (see Definition 2.6.7), then $|r -^* s| = |r - s| := \max\{r, s\} - \min\{r, s\}$. In particular, $|r -^* s| \in G_{\geq 0}$.

Fix a nonzero $\alpha \in R^*$, with no immediate predecessor in R^*, and some $s \in R$. We want to show

$$\alpha \oplus s = \sup\{x \oplus s : 0 < x < \alpha,\ x \in R\}.$$

We may assume $\alpha < \omega_R$, and therefore, by remark (i) above, identify α with an element of $G^*_{\geq 0}$. If $x \oplus s = \omega_R$ for some $x \in R$, with $x < \alpha$, then the result follows. So it suffices to show

$$\alpha \oplus s = \sup\{x + s : x < \alpha,\ x \in G_{\geq 0}\},$$

where this supremum is calculated in $\mathcal{R}^*$. Suppose this fails. Note that $\alpha \oplus s \leq \alpha + s$, and so

$$\sup\{x + s : x < \alpha,\ x \in G_{\geq 0}\} < \alpha + s,$$

where this supremum is calculated in $\mathcal{G}^*_{\geq 0}$. By density of $G_{\geq 0}$ in $G^*_{\geq 0}$, there is $v \in G_{\geq 0}$ such that $\sup\{x + s : x < \alpha,\ x \in G_{\geq 0}\} \leq v < \alpha + s$. By remark ($ii$) above, it follows that $v - s < \alpha$. Since α has no immediate predecessor, we may fix $x \in G_{\geq 0}$ such that $v - s < x < \alpha$. But $x + s \leq v$ by choice of v, which is a contradiction. $\square$

We end this section with a discussion of a particular family of generalized Urysohn spaces, which have been used in previous work to obtain exotic behavior in model theory. First, however, we give a more explicit axiomatization of $\mathrm{Th}(\mathcal{U}_\mathcal{R})$, in the case that $\mathcal{R}$ is finite. Recall that, if $\mathcal{R}$ is a finite distance monoid, then $\mathcal{R}$ is right-closed and so $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ has quantifier elimination. This conclusion also follows from classical results in general Fraïssé theory in finite relational languages (see [40, Theorem 7.4.1]).

Note that, if $\mathcal{R}$ is a finite distance monoid, then we have $\mathcal{R}^* = \mathcal{R}$. In this case, given $r \in R$ with $r > 0$, we let r^- denote the immediate predecessor of r.

Definition 2.9.5. Suppose $\mathcal{R}$ is a finite distance monoid. Given a finite $\mathcal{R}$-metric space $\mathcal{A}$, the **canonical R-approximation of $\mathcal{A}$** is the function $\Phi_{\mathcal{A}} : A \times A \longrightarrow I(R)$ such that, given distinct $a, b \in A$, $\Phi^+_{\mathcal{A}}(a,b) = d_A(a,b)$ and $\Phi^-_{\mathcal{A}}(a,b) = d_A(a,b)^-$. If $f \in E^+_{\mathcal{R}}(\mathcal{A})$, we let $\epsilon(\mathcal{A}, f)$ denote $\epsilon^{\Phi_{\mathcal{A}} f}_{\mathcal{A}}$.

If $\mathcal{R}$ is a finite distance monoid, and $\mathcal{A}$ is a finite $\mathcal{R}$-metric space, then $\Phi_{\mathcal{A}}$ refines any R-approximation of $\mathcal{A}$. Moreover, if $f \in E^+_{\mathcal{R}}(\mathcal{A})$ then $\mathcal{U}_{\mathcal{R}} \models \epsilon(\mathcal{A}, f)$ (this can be shown directly or as an easy consequence of Lemma 2.8.10). Altogether, given an extension scheme $(\mathcal{A}, f, \Psi)$, we may define the axiomatization $T^{\text{ax}}_{\mathcal{R}}$ so that $\epsilon(A, f, \Psi) = \epsilon(\mathcal{A}, f)$. In particular, $\epsilon(\mathcal{A}, f, \Psi)$ does not depend on Ψ.

We now turn to a specific family of examples. Given $n > 0$, set

$$R_n = \{0, 1, 2, \dots, n\} \text{ and } S_n = \{0, \tfrac{1}{n}, \tfrac{2}{n}, \dots, 1\},$$

and let $+_n$ denote addition truncated at n. Let $S = \mathbb{Q} \cap [0,1]$. Define $\mathcal{R}_n = (R_n, +_n \leq, 0)$, $\mathcal{S}_n = (S_n, +_1, \leq, 0)$, and $\mathcal{S} = (S, +_1, \leq, 0)$. Note that $\mathcal{S}_n$ is a submonoid of $\mathcal{S}$.

In [15], Casanovas and Wagner construct T_n, the theory of the *free n^{th} root of the complete graph*, for $n > 0$. In particular, T_1 is the theory of an infinite complete graph; and T_2 is the theory of the random graph. The reader familiar their work will recognize that, for general $n > 0$, T_n is precisely $\text{Th}(\mathcal{U}_{\mathcal{R}_n})$. Moreover, the axiomatization of $\text{Th}(\mathcal{U}_{\mathcal{R}_n})$ given in [15] uses the same canonical extension axioms described above. In order to form a directed system of first-order theories, Casanovas and Wagner then replace $\mathcal{R}_n$ with $\mathcal{S}_n$ and define $T_\infty = \bigcup_{n>0} \text{Th}(\mathcal{U}_{\mathcal{S}_n})$. We now verify T_∞ is precisely $\text{Th}(\mathcal{U}_{\mathcal{S}})$, the theory of the rational Urysohn sphere.

Proposition 2.9.6. *$T_\infty = \text{Th}(\mathcal{U}_{\mathcal{S}})$.*

Proof. We first fix $n > 0$ and show $\text{Th}(\mathcal{U}_{\mathcal{S}_n}) \subseteq \text{Th}(\mathcal{U}_{\mathcal{S}})$. Recall that S_n is $\mathcal{S}$-metrically dense over $\mathcal{U}_{\mathcal{S}}$ (see Example 2.2.4(2)), and so $T^{\text{ms}}_{S_n,\mathcal{S}} \subseteq \text{Th}(\mathcal{U}_{\mathcal{S}})$ by Proposition 2.2.5. Therefore, we must fix a finite $\mathcal{S}_n$-metric space $\mathcal{A}$ and $f \in E^+_{\mathcal{S}_n}(\mathcal{A})$, and show $\mathcal{U}_{\mathcal{S}} \models \epsilon(\mathcal{A}, f)$. In particular, we use Lemma 2.8.10. Let Φ be the canonical S_n-approximation of $\mathcal{A}^f$. Given distinct $a, b \in A$, we clearly have $\Phi^+(a,b) \leq \Phi^+(a) +_1 \Phi^+(b)$. Next, fix $a, b \in A$ and $s \in S$ with $\Phi^-(a,b) < s$. Let $d_A(a,b) = \frac{k}{n}$, $f(a) = \frac{i}{n}$, and $f(b) = \frac{j}{n}$, where $0 < i, j, k \leq n$. Then we have $s > \frac{k-1}{n}$, and we want to show $\frac{i-1}{n} < s +_1 \frac{j}{n}$. We obviously have $\frac{i-1}{n} < 1$, so it suffices to show $i - 1 < ns + j$. Since $f \in E^+_{\mathcal{S}_n}(\mathcal{A})$, we have $i \leq k + j$, and so $i - 1 \leq k - 1 + j < ns + j$, as desired.

We have shown $T_\infty \subseteq \text{Th}(\mathcal{U}_{\mathcal{S}})$, and so T_∞ is consistent. Since $\text{Th}(\mathcal{U}_{\mathcal{S}_n})$ is a complete $\mathcal{L}_{S_n}$-theory for all $n > 0$, and $\mathcal{L}_S = \bigcup_{n>0} \mathcal{L}_{S_n}$, it follows that T_∞ is complete. Therefore $T_\infty = \text{Th}(\mathcal{U}_{\mathcal{S}})$. □

Casanovas and Wagner remark that a saturated model of $\text{Th}(\mathcal{U}_{\mathcal{S}})$ could be considered a metric space with nonstandard distances in $(\mathbb{Q} \cap [0,1])^*$, but it is not observed that the theory they have constructed is the theory of such a classical structure. The main result of [15] is that $\mathrm{Th}(\mathcal{U}_\mathcal{S})$ does not eliminate hyperimaginaries. In particular, let

$$E(x,y) = \{d(x,y) \leq r : r \in \mathbb{Q} \cap (0,1]\}$$

be the type-definable equivalence relation describing infinitesimal distance. Then the equivalence class of any singleton element (in some sufficiently saturated model) is a non-eliminable hyperimaginary. In Chapter 3, we will generalize their methods in the setting of an arbitrary countable distance monoid $\mathcal{R}$, such that $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ has quantifier elimination, and obtain necessary conditions for elimination of hyperimaginaries in $\mathrm{Th}(\mathcal{U}_\mathcal{R})$. Casanovas and Wagner also show $\mathrm{Th}(\mathcal{U}_\mathcal{S})$ is non-simple and without the strict order property. In [26], it is shown that the continuous theory of the complete Urysohn sphere has SOP_n for all $n > 0$, but does not have the fully finite strong order property (see Section 1.4 for definitions). In Chapter 3, we show that the same arguments work to prove that the theory of the rational Urysohn sphere in classical logic has SOP_n for all $n > 0$ (i.e. SOP_ω). Moreover, we strengthen and refine the methods in [26] to prove that, for any countable distance monoid $\mathcal{R}$, if $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ has quantifier elimination then it does not have the finitary strong order property. Furthermore, we characterize the strong order rank of $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ in terms of algebraic properties of $\mathcal{R}$.

Chapter 3

Neostability in Homogeneous Metric Spaces

3.1 Introduction

In this chapter, we consider model theoretic properties of $\mathcal{R}$-Urysohn spaces, in the case that $\text{Th}(\mathcal{U}_\mathcal{R})$ has quantifier elimination. Our results will show that this class of metric spaces exhibits a rich spectrum of complexity in the classification of first-order theories without the strict order property.

First, we repeat some motivating examples of $\mathcal{R}$-Urysohn spaces, which have frequently appeared in previous literature. Note that, in particular, each of these examples is still a classical metric space over the distance monoid $(\mathbb{R}^{\geq 0}, +, \leq, 0)$.

Example 3.1.1.

1. Let $\mathcal{Q} = (\mathbb{Q}^{\geq 0}, +, \leq, 0)$ and $\mathcal{Q}_1 = (\mathbb{Q} \cap [0,1], +_1, \leq, 0)$, where $+_1$ is addition truncated at 1. Then $\mathcal{U}_\mathcal{Q}$ and $\mathcal{U}_{\mathcal{Q}_1}$ are, respectively, the *rational Urysohn space* and *rational Urysohn sphere*. The completion of the rational Urysohn space is called the *Urysohn space*, and is the unique complete, separable metric space, which is homogeneous and universal for separable metric spaces. These spaces were originally constructed by Urysohn in 1925 (see [89], [90]).

2. Let $\mathcal{R}_2 = (\{0,1,2\}, +_2, \leq, 0)$, where $+_2$ is addition truncated at 2. Then $\mathcal{U}_{\mathcal{R}_2}$ is isometric to the *countable random graph* or *Rado graph* (when equipped with the minimal path metric). A directed version of this graph was first constructed by Ackermann in 1937 [1]. The standard graph construction is usually attributed to Erdős and Rényi (1963) [30] or Rado (1964) [72].

3. Generalize the previous example as follows. Fix $n > 0$ and let

$$\mathcal{R}_n = (\{0, 1, \ldots, n\}, +_n, \leq, 0),$$

where $+_n$ is addition truncated at n. Let $\mathcal{N} = (\mathbb{N}, +, \leq, 0)$. We refer to $\mathcal{U}_{\mathcal{R}_n}$ as the *integral Urysohn space of diameter* n, and to $\mathcal{U}_{\mathcal{N}}$ as the *integral Urysohn space*. Alternatively, in [15], Casanovas and Wagner construct the *free* n^{th} *root of the complete graph*. As with the case $n = 2$, equipping this graph with the path metric yields $\mathcal{U}_{\mathcal{R}_n}$.

4. Generalize all of the previous examples as follows. Let $S \subseteq \mathbb{R}^{\geq 0}$ be a countable good value set. Let $\mathcal{S} = (S, +_S, \leq, 0)$, where $+_S$ is addition truncated at $\sup S$. Urysohn spaces of the form $\mathcal{U}_{\mathcal{S}}$ are often used as interesting examples in the study of automorphism groups of countable structures (e.g. [6], [9], [85], [87], [88]).

5. Generalize all of the previous examples as follows. Fix a countable subset $S \subseteq \mathbb{R}^{\geq 0}$, with $0 \in S$. Assume, moreover, that S is closed under the induced binary operation $r +_S s := \sup\{x \in S : x \leq r + s\}$ and that $+_S$ is associative. Let $\mathcal{S} = (S, +_S, \leq, 0)$. For sets S closed under $+_S$, associativity of $+_S$ characterizes the existence of $\mathcal{U}_{\mathcal{S}}$ (see [78, Theorem 5] or Proposition 2.7.9).

6. For an example of a different flavor, fix a countable linear order $(R, \leq, 0)$, with least element 0, and let $\mathcal{R} = (R, \max, \leq, 0)$. We refer to $\mathcal{U}_{\mathcal{R}}$ as the *ultrametric Urysohn space over* $(R, \leq, 0)$. Explicit constructions of these spaces are given in [33]. Alternatively, $\mathcal{U}_{\mathcal{R}}$ can be viewed as a countable model of the theory of infinitely refining equivalence relations indexed by $(R, \leq, 0)$. These are standard model theoretic examples, often used to illustrate various behavior in the stability spectrum (see [7, Section III.4]).

We will consider model theoretic properties of $\mathcal{R}$-Urysohn spaces. Recall that $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ denotes the complete $\mathcal{L}_R$-theory of $\mathcal{U}_{\mathcal{R}}$, where $\mathcal{L}_R = \{d(x, y) \leq r : r \in R\}$. In Chapter 2, we constructed a "nonstandard" distance monoid extension $\mathcal{R}^*$ of $\mathcal{R}$, with the property that any model of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is canonically an $\mathcal{R}^*$-metric space. We let $\mathbb{U}_{\mathcal{R}}$ denote a sufficiently saturated monster model of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$. Then $\mathbb{U}_{\mathcal{R}}$ is, of course, a κ^+-universal and κ-homogeneous $\mathcal{L}_R$-structure, where κ is the saturation cardinal of $\mathbb{U}_{\mathcal{R}}$. Moreover, $\mathbb{U}_{\mathcal{R}}$ is κ^+-universal as an $\mathcal{R}^*$-metric space by Proposition 2.8.1. We focus on the case when, in addition, $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ has quantifier elimination, in which case $\mathbb{U}_{\mathcal{R}}$ is also κ-homogeneous as an $\mathcal{R}^*$-metric space.

Definition 3.1.2. A countable distance monoid $\mathcal{R}$ is **Urysohn** if $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ has quantifier elimination.

Recall that, in Theorem 2.8.11, we characterized quantifier elimination for $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ as a natural continuity property of $\mathcal{R}^*$. This motivates a general schematic for analyzing the model theoretic behavior of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$.

Definition 3.1.3. Let **RUS** denote the class of $\mathcal{R}$-Urysohn spaces $\mathcal{U}_{\mathcal{R}}$, where $\mathcal{R}$ is a Urysohn monoid. We say a property P of **RUS** is **axiomatizable** (resp.

finitely axiomatizable) if there is an $\mathcal{L}_{\omega_1,\omega}$-sentence (resp. $\mathcal{L}_{\omega,\omega}$-sentence) φ_P, in the language of ordered monoids, such that, if $\mathcal{R}$ is a Urysohn monoid, then $\mathcal{U}_{\mathcal{R}}$ satisfies P if and only if $\mathcal{R} \models \varphi_P$.

Although we have relativized this notion of axiomatizability to the class **RUS**, recall that, by Corollary 2.8.13, there is a first-order $\mathcal{L}_{\text{om}}$-sentence φ_{QE} such that a countable distance monoid $\mathcal{R}$ is Urysohn if and only if $\mathcal{R} \models \varphi_{\text{QE}}$. Therefore, if some property P is axiomatizable with respect to the class of *all* $\mathcal{R}$-Urysohn spaces, then P is also axiomatizable relative to **RUS**. This remark will be especially pertinent when we show certain properties are not axiomatizable (relative to **RUS**).

Concerning axiomatizable properties of **RUS**, we begin with notions around stability and simplicity. In particular, the ultrametric spaces in Example 3.1.1(6) are well-known to be stable when considered as theories of refining equivalence relations. We also have the random graph as a canonical example of a simple unstable theory. Toward a general understanding of the role of stability and simplicity in Urysohn spaces, we consider, in Section 3.3, several ternary relations defined on subsets of the monster model $\mathbb{U}_{\mathcal{R}}$, where $\mathcal{R}$ is Urysohn. First to be considered are the notions of independence given by nonforking and nondividing. We state a combinatorial characterization of forking and dividing for complete types in $\text{Th}(\mathcal{U}_{\mathcal{R}})$, when $\mathcal{R}$ is Urysohn. This characterization is identical to the same result for the complete Urysohn sphere in continuous logic, which was proved in joint work with Caroline Terry [26]. The proof of this result in our present setting closely follows the strategy of [26]. Finally in Section 3.3, we define three more ternary relations on $\mathbb{U}_{\mathcal{R}}$, including the *stationary independence relation* of free amalgamation of metric spaces, which was used by Tent and Ziegler [87], [88] to analyze the algebraic structure of the isometry groups of $\mathcal{U}_{\mathcal{Q}}$ and $\mathcal{U}_{\mathcal{Q}_1}$.

In Section 3.5, we use this network of ternary relations to prove the following result.

Theorem C.

(*a*) *Stability and simplicity are finitely axiomatizable properties of* **RUS**. *In particular, given a Urysohn monoid* $\mathcal{R}$*,*

  (*i*) $\text{Th}(\mathcal{U}_{\mathcal{R}})$ *is stable if and only if* $\mathcal{U}_{\mathcal{R}}$ *is ultrametric, i.e., for all* $r, s \in R$, $r \oplus s = \max\{r, s\}$*;*

  (*ii*) $\text{Th}(\mathcal{U}_{\mathcal{R}})$ *is simple if and only if, for all* $r, s \in R$*, if* $r \leq s$ *then* $r \oplus r \oplus s = r \oplus s$*.*

(*b*) *Superstability and supersimplicity are not axiomatizable properties of* **RUS**.

Concerning part (*b*) of the previous result, we show that superstability and supersimplicity are detected via relatively straightforward properties of $\mathcal{R}$, but not in a first-order way.

Having established the presence of generalized Urysohn spaces in the most well-behaved regions of classification theory, we then turn to the question of how complicated $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ can be. For example, Theorem C immediately implies that the rational Urysohn space is not simple. This is a well-known fact, which was observed for the complete Urysohn sphere in continuous logic by Pillay (see [29]). Casanovas and Wagner give a similar argument in [15] to show $\mathrm{Th}(\mathcal{U}_{\mathcal{R}_n})$ is not simple when $n \geq 3$. In [26], it is shown that the complete Urysohn sphere in fact has SOP_n for all $n \geq 3$, and these methods can be easily adjusted to show that, if $n \geq 3$, then $\mathrm{Th}(\mathcal{U}_{\mathcal{R}_n})$ is SOP_n and NSOP_{n+1}. Regarding an upper bound in complexity, it is shown in [26] that the complete Urysohn sphere does not have the *fully finite strong order property*. Altogether, this work sets the stage for the main result of Section 3.6, which gives the following upper bound for the complexity of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$.

Theorem D. *If $\mathcal{R}$ is Urysohn then $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ does not have the finitary strong order property.*

This result is obtained by generalizing work in [26], which analyzes when an indiscernible sequence is cyclic (Definition 1.4.7). In particular, given a subset $C \subset \mathbb{U}_{\mathcal{R}}$ and a C-indiscernible sequence $(\bar{a}^l)_{l<\omega}$ in $\mathbb{U}_{\mathcal{R}}$, if $\bar{a}^0$ has finite length $n < \omega$, then $(\bar{a}^l)_{l<\omega}$ is $(n+1)$-cyclic.

In Section 3.7, we address the region of complexity between simplicity and the finitary strong order property, which, in general, is stratified by Shelah's SOP_n-hierarchy. Concerning $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$, we first use the characterizations of stability and simplicity to formulate a purely algebraic notion of the *archimedean complexity*, $\mathrm{arch}(\mathcal{R})$, of a general distance monoid $\mathcal{R}$ (see Definition 3.7.1). In particular, $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is stable (resp. simple) if and only if $\mathrm{arch}(\mathcal{R}) \leq 1$ (resp. $\mathrm{arch}(\mathcal{R}) \leq 2$). We then use this rank to pinpoint the exact complexity of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$.

Theorem E. *If $\mathcal{R}$ is Urysohn and $n \geq 3$, then $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is SOP_n if and only if $\mathrm{arch}(\mathcal{R}) \geq n$.*

As an immediate consequence, we obtain that any non-simple $\mathcal{U}_{\mathcal{R}}$ is SOP_3; and we further show that the failure of simplicity also implies TP_2. Altogether, this provides the first class of examples in which the entirety of the SOP_n-hierarchy has a meaningful interpretation independent of combinatorial dividing lines.

In Section 3.8, we consider the question of *elimination of hyperimaginaries*. This builds on work of Casanovas and Wagner [15], which was motivated by the search for a theory without the strict order property that does not eliminate hyperimaginaries. In particular, they showed $\mathrm{Th}(\mathcal{U}_{\mathcal{Q}_1})$ is such a theory (although they did not identify their theory as such, see Proposition 2.9.6). We adapt their methods to give necessary conditions for elimination of hyperimaginaries and weak elimination of imaginaries for $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$, where $\mathcal{R}$ is any Urysohn monoid. Finally, we conjecture these conditions are sufficient, and discuss consequences of this conjecture.

Remark 3.1.4. This chapter has been rewritten in preparation for submission for publication. A preprint is available on the arXiv [24].

3.2 Preliminaries

In this section we repeat our conventions concerning $\mathcal{R}$-Urysohn spaces. For clarity, we also give a succinct summary of the technical results from Chapter 2 that will be used in this chapter.

Suppose $\mathcal{R}$ is a distance monoid. We let $\mathcal{R}^* = (R^*, \oplus, \leq, 0)$ denote the distance monoid extension of $\mathcal{R}$ given to us by Theorem 2.4.3. Note that, as in Section 2.8, we continue to omit the asterisks on $\oplus$ and $\leq$. We will do the same with the generalized difference operation $|\alpha \ominus \beta|$ defined on $\mathcal{R}^*$. Recall that, when considering $\emptyset$ as a subset of R^*, we let $\sup \emptyset = 0$ and $\inf \emptyset = \omega_R$.

The following reformulation of Corollary 2.6.6 will be used frequently.

Proposition 3.2.1. *Suppose $\mathcal{R}$ is a distance monoid and $\alpha, \beta, \gamma \in R^*$. If $\gamma \leq r \oplus s$ for all $r, s \in R$, with $\alpha \leq r$ and $\beta \leq s$, then $\gamma \leq \alpha \oplus \beta$.*

Recall that we define a *Urysohn monoid* to be a countable distance monoid $\mathcal{R}$ such that $\text{Th}(\mathcal{U}_{\mathcal{R}})$ has quantifier elimination. For example, it follows from the results in Section 2.9 that each monoid in Example 3.1.1 is Urysohn, except for the full generality of (5).

Next, we recall the important properties of the monster model $\mathbb{U}_{\mathcal{R}}$, when $\mathcal{R}$ is Urysohn. We let d denote the $\mathcal{R}^*$-metric on $\mathbb{U}_{\mathcal{R}}$ given by Theorem 2.4.3.

Proposition 3.2.2. *Suppose $\mathcal{R}$ is a Urysohn monoid and $\mathbb{U}_{\mathcal{R}}$ has cardinality κ. Then $\mathbb{U}_{\mathcal{R}}$ is a κ-homogeneous and κ^+-universal $\mathcal{R}^*$-metric space, i.e., any isometry between subspaces of $\mathbb{U}_{\mathcal{R}}$, of cardinality less than κ, extends to an isometry of $\mathbb{U}_{\mathcal{R}}$, and any $\mathcal{R}^*$-metric space of cardinality at most κ is isometric to a subspace of $\mathbb{U}_{\mathcal{R}}$.*

Finally, we define natural multiplicative operations on elements of $\mathcal{R}^*$.

Definition 3.2.3. Given $\alpha \in R^*$ and $n > 0$, we define

$$n\alpha := \underbrace{\alpha \oplus \ldots \oplus \alpha}_{n \text{ times}} \quad \text{and} \quad \tfrac{1}{n}\alpha := \inf\{\beta \in R^* : \alpha \leq n\beta\}.$$

These notions allow us to treat $\mathcal{R}^*$ as a module over the semiring $(\mathbb{N}, +, \cdot)$, but not necessarily over $(\mathbb{Q}^{\geq 0}, +, \cdot)$. For example, if $\mathcal{S} = (\{0, 1, 3, 4\}, +_S, \leq, 0)$, then $\frac{1}{2}(1 \oplus 3) = 3$ and $\frac{1}{2}1 \oplus \frac{1}{2}3 = 4$. However, the following observation will be sufficient for our results.

Proposition 3.2.4. *If $\alpha, \beta \in R^*$ then, for any $n > 0$, $\frac{1}{n}(\alpha \oplus \beta) \leq \frac{1}{n}\alpha \oplus \frac{1}{n}\beta$.*

Proof. We want to show that, for all $\alpha,\beta\in R^*$, $\alpha\oplus\beta\leq n(\frac{1}{n}\alpha\oplus\frac{1}{n}\beta)$. Since multiplication by n is clearly distributive, it suffices to show that, for all $\alpha\in R^*$, $\alpha\leq n(\frac{1}{n}\alpha)$. By Proposition 2.6.4, we have $n(\frac{1}{n}\alpha)=\inf\{n\beta:\beta\in R^*,\ \alpha\leq n\beta\}$, and so the desired result follows. □

3.3 Notions of Independence

In this section, we consider various ternary relations on subsets of $\mathbb{U}_{\mathcal{R}}$, where $\mathcal{R}$ is a Urysohn monoid. The first examples are nonforking and nondividing independence. Toward a characterization of these notions, we define the following distance calculations.

Definition 3.3.1. Fix a Urysohn monoid $\mathcal{R}$. Given $C\subset\mathbb{U}_{\mathcal{R}}$ and $b_1,b_2\in\mathbb{U}_{\mathcal{R}}$, we define

$$d_{\max}(b_1,b_2/C)=\inf_{c\in C}(d(b_1,c)\oplus d(c,b_2))$$
$$d_{\min}(b_1,b_2/C)=\max\left\{\sup_{c\in C}|d(b_1,c)\ominus d(c,b_2)|,\tfrac{1}{3}d(b_1,b_2)\right\}.$$

Note that $d_{\max}(b_1,b_2/C)$ is reminiscent of the notion of free amalgamation of $\mathcal{R}$-metric spaces (see Definition 2.7.15). Model theoretically, $d_{\max}(b_1,b_2/C)$ can be interpreted as the largest possible distance between realizations of $\mathrm{tp}(b_1/C)$ and $\mathrm{tp}(b_2/C)$. On the other hand, $d_{\min}$ does not have as straightforward an interpretation, and has to do with the behavior of indiscernible sequences in $\mathbb{U}_{\mathcal{R}}$. We use these values to give a completely combinatorial description of $\mathop{\smile\!\!\!\!\!\mid}^{d}$ and $\mathop{\smile\!\!\!\!\!\mid}^{f}$, which, in particular, shows forking and dividing are the same for complete types in $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$.

Theorem 3.3.2. *Suppose $\mathcal{R}$ is a Urysohn monoid. Given $A,B,C\subset\mathbb{U}_{\mathcal{R}}$, $A\mathop{\smile\!\!\!\!\!\mid}^{d}_{C}B$ if and only if $A\mathop{\smile\!\!\!\!\!\mid}^{f}_{C}B$ if and only if, for all $b_1,b_2\in B$,*

$$d_{\max}(b_1,b_2/AC)=d_{\max}(b_1,b_2/C)\ \text{ and }\ d_{\min}(b_1,b_2/AC)=d_{\min}(b_1,b_2/C).$$

The proof of this result is long and technical. Therefore, for smoother exposition, we give the full proof in Section 3.4. However, we note the following inequalities, which, while most useful in Section 3.4, will also be used in this section.

Lemma 3.3.3. *Fix $b_1,b_2,b_3\in\mathbb{U}$ and $C\subset\mathbb{U}$.*

(a) $d_{\max}(b_1,b_3/C)\leq d_{\max}(b_1,b_2/C)\oplus d_{\min}(b_2,b_3/C)$.

(b) $d_{\min}(b_1,b_3/C)\leq d_{\min}(b_1,b_2/C)\oplus d_{\min}(b_2,b_3/C)$.

Proof. Part (a): For any $c' \in C$, we have

$$\begin{aligned} d_{\max}(b_1, b_3/C) &\leq d(b_1, c') \oplus d(b_3, c') \\ &\leq d(b_1, c') \oplus d(b_2, c') \oplus |d(b_2, c') \ominus d(b_3, c')| \\ &\leq d(b_1, c') \oplus d(b_2, c') \oplus d_{\min}(b_2, b_3/C). \end{aligned}$$

Therefore $d_{\max}(b_1, b_3/C) \leq d_{\max}(b_1, b_2/C) \oplus d_{\min}(b_2, b_3/C)$.

Part (b): If $d_{\min}(b_1, b_3/C) = \frac{1}{3}d(b_1, b_3)$ then the result follows from Proposition 3.2.4. So we may assume $d_{\min}(b_1, b_3/C) = \sup_{c \in C} |d(b_1, c) \ominus d(b_3, c)|$. If $c \in C$ then $|d(b_1, c) \ominus d(b_3, c)| \leq |d(b_1, c) \ominus d(b_2, c)| \oplus |d(b_2, c) \ominus d(b_3, c)|$ by Proposition 2.6.3(e). Therefore

$$\begin{aligned} d_{\min}(b_1, b_3/C) &\leq \sup_{c \in C} |d(b_1, c) \ominus d(b_2, c)| \oplus \sup_{c \in C} |d(b_2, c) \ominus d(b_3, c)| \\ &\leq d_{\min}(b_1, b_2/C) \oplus d_{\min}(b_2, b_3/C). \end{aligned}$$ □

The rest of this section is devoted to several more natural ternary relations on $\mathbb{U}_{\mathcal{R}}$, which will be useful in understanding stability and simplicity.

Definition 3.3.4. Suppose $\mathcal{R}$ is a countable distance monoid.

1. Given $a \in \mathbb{U}_{\mathcal{R}}$ and $C \subset \mathbb{U}_{\mathcal{R}}$, define $d(a, C) = \inf\{d(a, c) : c \in C\}$.
2. Given $A, B, C \subset \mathbb{U}_{\mathcal{R}}$, define

$$\begin{aligned} A \mathop{\smile\!\!\!\!\!\mid}^{\text{dist}}_C B &\Leftrightarrow d(a, BC) = d(a, C) \text{ for all } a \in A; \\ A \mathop{\smile\!\!\!\!\!\mid}^{\otimes}_C B &\Leftrightarrow d(a, b) = d_{\max}(a, b/C) \text{ for all } a \in A,\ b \in B; \\ A \mathop{\smile\!\!\!\!\!\mid}^{d_{\max}}_C B &\Leftrightarrow d_{\max}(b_1, b_2/AC) = d_{\max}(b_1, b_2/C) \text{ for all } b_1, b_2 \in B. \end{aligned}$$

The relation $\mathop{\smile\!\!\!\!\!\mid}^{\text{dist}}$ has obvious significance as a notion of independence in metric spaces. The relation $A \mathop{\smile\!\!\!\!\!\mid}^{\otimes}_C B$ should be viewed as asserting that, as $\mathcal{R}^*$-metric spaces, ABC is isometric to the free amalgamation of AC and BC over C (since $\mathcal{R}^*$ has a maximal element, this still makes sense for $C = \emptyset$). The final relation $\mathop{\smile\!\!\!\!\!\mid}^{d_{\max}}$ is a simplification of the characterization of $\mathop{\smile\!\!\!\!\!\mid}^{f}$ in Theorem 3.3.2.

Finally, we note the following implications between these ternary relations.

Proposition 3.3.5. *Suppose $\mathcal{R}$ is a Urysohn monoid.*

(a) $\mathop{\smile\!\!\!\!\!\mid}^{f}$ *implies* $\mathop{\smile\!\!\!\!\!\mid}^{d_{\max}}$.

(b) $\mathop{\smile\!\!\!\!\!\mid}^{\otimes}$ *is a stationary independence relation on* $\mathbb{U}_{\mathcal{R}}$*, and so* $\mathop{\smile\!\!\!\!\!\mid}^{\otimes}$ *implies* $\mathop{\smile\!\!\!\!\!\mid}^{f}$.

(c) $\mathop{\smile\!\!\!\!\!\mid}^{\otimes}$ *implies* $\mathop{\smile\!\!\!\!\!\mid}^{\text{dist}}$.

(d) $\mathop{\smile\!\!\!\!\!\mid}^{\text{dist}}$ *satisfies local character.*

Proof. Part (a) is immediate from Theorem 3.3.2.

Part (b). First, recall that if we show $\mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}$ is a stationary independence relation, then we will have $\mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}$ implies $\mathop{\smile\!\!\!\!\!\!\mid}^{f}$ by Proposition 1.3.11. Therefore, we only need to verify $\mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}$ satisfies the axioms of a stationary independence relation.

Invariance, symmetry, and finite character are trivial. Stationarity follows from quantifier elimination. We verify full transitivity and full existence.

First, fix $A, B, C, D \subset \mathbb{U}_{\mathcal{R}}$, and suppose $A \mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}_{C} BD$. We clearly have $A \mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}_{C} B$. To show $A \mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}_{BC} D$, fix $a \in A$ and $e \in D$. Then $d(a,e) \leq d_{\max}(a,e/BC) \leq d_{\max}(a,e/C) = d(a,e)$, as desired. Conversely, suppose $A \mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}_{C} B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}_{BC} D$. To show $A \mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}_{C} BD$, fix $e \in BD$. If $e \in B$ then we have $d(a,e) = d_{\max}(a,e/C)$ by assumption. Assume $e \in D$. Then, for any $b \in BC$, we have, by Lemma 3.3.3(a),

$$d_{\max}(a,e/C) \leq d_{\max}(a,b/C) \oplus d(b,e) = d(a,b) \oplus d(b,e),$$

and so $d(a,e) \leq d_{\max}(a,e/C) \leq d_{\max}(a,e/BC) = d(a,e)$. This completes the verification of full transitivity.

Finally, we verify full existence. Fix $A, B, C \subset \mathbb{U}_{\mathcal{R}}$. We want to find $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}_{C} B$. In particular, we fix $\bar{x} = (x_a)_{a \in A}$ and define the $\mathcal{R}^*$-colored space $\mathcal{A} = (\bar{x}BC, d)$ such that

- $d(x_{a_1}, x_{a_2}) = d(a_1, a_2)$ for all $a_1, a_2 \in A$,
- $d(x_a, b) = d_{\max}(a, b/C)$, and
- $d(x_a, c) = d(a, c)$,

and we show $\mathcal{A}$ is an $\mathcal{R}^*$-metric space. Note that, for any $c \in B \cap C$, we have $d(a,c) = d_{\max}(a,b/C)$, and so $\mathcal{A}$ is well-defined. For the triangle inequality, the nontrivial cases are triangles of the form (x_{a_1}, x_{a_2}, b), (x_a, b_1, b_2), or (x_a, b, c). Each inequality in the first two triangles follows from Lemma 3.3.3(a). For the triangle (x_a, b, c), the only inequality not following directly from Lemma 3.3.3(a) is $d_{\max}(a,b/C) \leq d(a,c) \oplus d(b,c)$, which is trivial.

Part (c). Fix $A, B, C \subset \mathbb{U}_{\mathcal{R}}$, with $A \mathop{\smile\!\!\!\!\!\!\mid}^{\otimes}_{C} B$. Given $a \in A$ and $b \in B$, we have

$$d(a,C) = \inf_{c \in C} d(a,c) \leq d_{\max}(a,b/C) = d(a,b).$$

Therefore $d(a,BC) = d(a,C)$, and so $A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{dist}}_{C} B$.

Part (d). Fix $A, B \subset \mathbb{U}_{\mathcal{R}}$. We may assume $A, B \neq \emptyset$. We show there is $C \subseteq B$ such that $|C| \leq |A| + \aleph_0$ and $A \mathop{\smile\!\!\!\!\!\!\mid}^{\text{dist}}_{C} B$. It suffices to show that, for all $a \in A$, there is $C_a \subseteq B$ such that $|C_a| \leq \aleph_0$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^{\text{dist}}_{C_a} B$. We will then set $C = \bigcup_{a \in A} C_a$.

Fix $a \in A$. If there is some $b \in B$ such that $d(a,b) = d(a,B)$ then set $C_a = \{b\}$. Otherwise, define $X = \{r \in R : d(a,B) \leq r\}$. Given $r \in X$, we claim there is some $b_r \in B$ such that $d(a,b_r) \leq r$. Indeed, this follows simply from the observation that any non-maximal $r \in R$ has an immediate successor in R^*. Set $C_a = \{b_r : r \in X\}$. By assumption and density of R in R^*, for any $b \in B$ there is some $r \in X$, with $r < d(a, b)$. We have $b_r \in C$ and $d(a, b_r) \leq r < d(a, b)$, as desired. □

3.4 Forking and Dividing in Generalized Urysohn Spaces

The goal of this section is to prove Theorem 3.3.2. The statement of this theorem is identical to the characterization of forking and dividing for the complete Urysohn sphere as a metric structure in continuous logic, which was proved in joint work with Caroline Terry [26]. The proof of this result for $\mathcal{R}$-Urysohn spaces in discrete logic closely follows the strategy of [26], with only minor modifications.

Throughout this section, we fix a countable Urysohn monoid $\mathcal{R}$. Our first task is a combinatorial characterization of dividing for complete types. We begin by applying the recurring theme that, by quantifier elimination, the consistency of complete types in $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ reduces to a verification of the triangle inequality. As a result, we can strengthen the usual "finite character of dividing", and show dividing is always detected by three points.

Lemma 3.4.1. *Given $A, B, C \subset \mathbb{U}_\mathcal{R}$, $A \mathop{\smile\!\!\!\!|}^d_C B$ if and only if $a \mathop{\smile\!\!\!\!|}^d_C b_1 b_2$ for all $a \in A$ and $b_1, b_2 \in B$.*

Proof. The forward direction follows from finite character of $\mathop{\smile\!\!\!\!|}^d$ (Fact 1.2.3). For the reverse direction, fix $A, B, C \subset \mathbb{M}$ such that $\mathrm{tp}(A/BC)$ divides over C. Let $\bar{a}$ and $\bar{b}$ enumerate A and B, respectively, and set $p(\bar{x}, \bar{y}) = \mathrm{tp}(\bar{a}, \bar{b}/C)$. Then there is a C-indiscernible sequence $(\bar{b}^l)_{l<\omega}$, with $\bar{b}^0 = \bar{b}$, such that $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$ is inconsistent. By Proposition 3.2.2, there is some failure of the triangle inequality in $\bigcup_{l<\omega} p(\bar{x}, \bar{b}^l)$. Since $C \cup \bigcup_{l<\omega} \bar{b}^l$ is a subspace of $\mathbb{U}_\mathcal{R}$ and $(\bar{b}^l)_{l<\omega}$ is C-indiscernible, this failure must come from three points of the form x_i, b^m_j, b^n_k for some $m, n < \omega$, $i \in \ell(\bar{a})$, and $j, k \in \ell(\bar{b})$. Setting $q(x, y_j, y_k) = \mathrm{tp}(a_i, b_j, b_k/C)$, we then have that $\bigcup_{l<\omega} q(x, b^l_j, b^l_k)$ is inconsistent. Therefore $a_i \not\mathop{\smile\!\!\!\!|}^d_C b_j, b_k$, as desired. □

From this result, we see that in order to understand dividing, it is enough to consider indiscernible sequences of 2-tuples.

Definition 3.4.2. Fix $C \subset \mathbb{U}_\mathcal{R}$ and $b_1, b_2 \in \mathbb{U}_\mathcal{R}$. Define $\Gamma(b_1, b_2/C) \subseteq R^*$ such that $\gamma \in \Gamma(b_1, b_2/C)$ if and only if there is a C-indiscernible sequence $(b^l_1, b^l_2)_{l<\omega}$, with $(b^0_1, b^0_2) = (b_1, b_2)$, such that $d(b^0_1, b^1_2) = \gamma$.

Proposition 3.4.3. *If $b_1, b_2 \in \mathbb{U}_\mathcal{R}$ and $C \subset \mathbb{U}_\mathcal{R}$ then $\Gamma(b_1, b_2/C) = \Gamma(b_2, b_1/C)$.*

Proof. It suffices to show $\Gamma(b_2, b_1/C) \subseteq \Gamma(b_1, b_2/C)$. Suppose $\mathcal{I} = (\bar{b}^l)_{l<\omega}$ is a C-indiscernible sequence, with $\bar{b} = (b_2, b_1)$. Let $\omega^* = \{l^* : l < \omega\}$, ordered so that $l^* > (l+1)^*$. By compactness, we may stretch $\mathcal{I}$ so that $\mathcal{I} = (\bar{b}^l)_{l<\omega+\omega^*}$. Define

$(\bar{a}^l)_{l<\omega}$ such that $\bar{a}^l = (b_1^{l^*}, b_2^{l^*})$. Then $(\bar{a}^l)_{l<\omega}$ is C-indiscernible and $\bar{a}^0 \equiv_C (b_1, b_2)$. In particular, we may assume $\bar{a}^0 = (b_1, b_2)$. Moreover, we have

$$d(a_1^0, a_2^1) = d(b_1^{0^*}, b_2^{1^*}) = d(b_1^1, b_2^0) = d(b_2^0, b_1^1),$$

as desired. □

Lemma 3.4.4. *Given $C \subset \mathbb{U}_\mathcal{R}$ and $a, b_1, b_2 \in \mathbb{U}_\mathcal{R}$, $a \mathop{\smile\!\!\!\!\!\mid}^d_C b_1 b_2$ if and only if, for all $i, j \in \{1, 2\}$,*

$$d(b_i, a) \oplus d(a, b_j) \geq \sup \Gamma(b_i, b_j/C) \ \text{ and } \ |d(b_i, a) \ominus d(a, b_j)| \leq \inf \Gamma(b_i, b_j/C).$$

Proof. Let $p(x, y_1, y_2) = \mathrm{tp}(a, b_1, b_2/C)$.

For the forward direction, suppose first that $d(a, b_i) \oplus d(a, b_j) < \sup \Gamma(b_i, b_j/C)$ for some $i, j \in \{1, 2\}$. By definition, there is a C-indiscernible sequence $(\bar{b}^l)_{l<\omega}$, with $\bar{b}^0 = (b_i, b_j)$, and some $\alpha \in R^*$ such that $d(a, b_i) \oplus d(a, b_j) < \alpha$ and $d(b_i^0, b_j^1) = \alpha$. In particular, we have $d(b_i^l, b_j^m) = \alpha$ for all $l < m < \omega$. It follows that if a' realizes $\bigcup_{l<\omega} p(x, b_i^l, b_j^l)$ then we have

$$\alpha = d(b_i^0, b_j^1) \leq d(b_i^0, a') \oplus d(a', b_j^1) = d(a, b_i) \oplus d(a, b_j),$$

which is a contradiction. Therefore $\bigcup_{l<\omega} p(x, b_i^l, b_j^l)$ is inconsistent, and so $a \not\mathop{\smile\!\!\!\!\!\mid}^d_C b_i, b_j$. By finite character of $\mathop{\smile\!\!\!\!\!\mid}^d$, we have $a \not\mathop{\smile\!\!\!\!\!\mid}^d_C b_1 b_2$.

Next, suppose $|d(a, b_i) \ominus d(a, b_j)| > \inf \Gamma(b_i, b_j/C)$ for some $i, j \in \{1, 2\}$. By definition, there is a C-indiscernible sequence $(\bar{b}^l)_{l<\omega}$, with $\bar{b}^0 = (b_i, b_j)$, and some $\alpha \in R^*$ such that $|d(a, b_i) \ominus d(a, b_j)| > \alpha$ and $d(b_i^0, b_j^1) = \alpha$. In particular, we have $d(b_i^l, b_j^m) = \alpha$ for all $l < m < \omega$. It follows that if a' realizes $\bigcup_{l<\omega} p(x, b_i^l, b_j^l)$ then we have

$$\alpha = d(b_i^0, b_j^1) \geq |d(b_i^0, a') \ominus d(a', b_j^1)| = |d(a, b_i) \ominus d(a, b_j)|,$$

which is a contradiction. Therefore $\bigcup_{l<\omega} p(x, b_i^l, b_j^l)$ is inconsistent, and so $a \not\mathop{\smile\!\!\!\!\!\mid}^d_C b_i, b_j$. By finite character of $\mathop{\smile\!\!\!\!\!\mid}^d$, we have $a \not\mathop{\smile\!\!\!\!\!\mid}^d_C b_1 b_2$.

For the reverse direction, suppose $a \not\mathop{\smile\!\!\!\!\!\mid}^d_C b_1 b_2$. Then there is a C-indiscernible sequence $(\bar{b}^l)_{l<\omega}$, with $\bar{b}^0 = (b_1, b_2)$, such that $\bigcup_{l<\omega} p(x, b_1^l, b_2^l)$ is inconsistent, and therefore contains some violation of the triangle inequality. By indiscernibility, the possible failures are

(*i*) $d(b_i^l, b_j^m) > d(a, b_i) \oplus d(a, b_j)$ for some $i, j \in \{1, 2\}$ and $l, m < \omega$;

(*ii*) $d(b_i^l, b_j^m) < |d(a, b_i) \ominus d(a, b_j)|$ for some $i, j \in \{1, 2\}$ and $l, m < \omega$.

In either case, since $\bar{b}^l \equiv_C \bar{b} \equiv_C \bar{b}^m$, it follows that $l \neq m$. By Proposition 3.4.3, we may assume $l < m$, and so

$$\inf \Gamma(b_i, b_j/C) \leq d(b_i^l, b_j^m) \leq \sup \Gamma(b_i, b_j/C),$$

which gives the desired result. □

From this result, we see that an explicit characterization of dividing rests on an explicit calculation of $\Gamma(b_1, b_2/C)$, which can be given via the values $d_{\max}$ and $d_{\min}$ (see Definition 3.3.1).

Lemma 3.4.5. *Given $C \subset \mathbb{U}_{\mathcal{R}}$ and $b_1, b_2 \in \mathbb{U}_{\mathcal{R}}$,*

$$\Gamma(b_1, b_2/C) = \{\gamma \in R^* : d_{\min}(b_1, b_2/C) \leq \gamma \leq d_{\max}(b_1, b_2/C)\}.$$

Proof. For the left-to-right inclusion suppose $(\bar{b}^l)_{l<\omega}$ is a C-indiscernible sequence, with $\bar{b}^0 = (b_1, b_2)$ and $d(b_1^0, b_2^1) = \gamma$. We have $d_{\min}(b_1, b_2/C) \leq \gamma \leq d_{\max}(b_1, b_2/C)$ by the following observations (see also Figure 4):

(*i*) For any $c \in C$, $|d(b_1, c) \ominus d(b_2, c)| = |d(b_1^0, c) \ominus d(b_2^1, c)| \leq d(b_1^0, b_2^1) = \gamma$.

(*ii*) For any $c \in C$, $\gamma = d(b_1^0, b_2^1) \leq d(b_1^0, c) \oplus d(b_2^1, c) = d(b_1, c) \oplus d(b_2, c)$.

(*iii*) $d(b_1, b_2) = d(b_1^1, b_2^1) \leq d(b_2^1, b_1^0) \oplus d(b_1^0, b_2^2) \oplus d(b_2^2, b_1^1) = 3\gamma$.

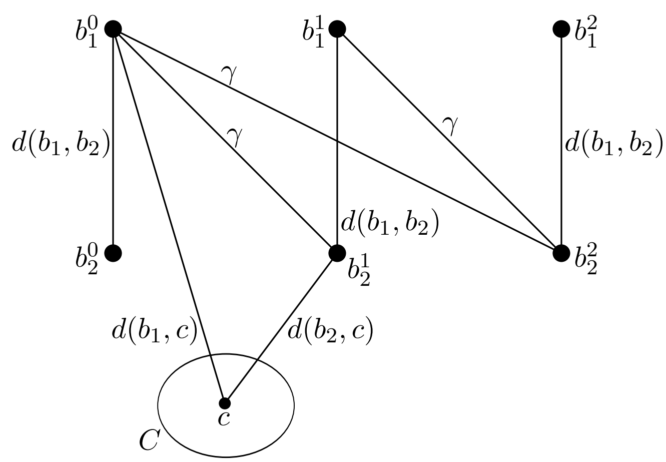

Figure 4: $d_{\max}$ and $d_{\min}$ from indiscernible sequences.

For the right-to-left inclusion fix $\gamma \in R^*$ such that

$$d_{\min}(b_1, b_2/C) \leq \gamma \leq d_{\max}(b_1, b_2/C).$$

We define a sequence $(\bar{b}^l)_{l<\omega}$ such that, for $i, j \in \{1, 2\}$ and $l \leq m < \omega$,

$$d(b_i^l, b_j^m) = \begin{cases} d(b_i, b_j) & \text{if } l = m \\ \min\{d_{\max}(b_i, b_i/C), d(b_1, b_2) \oplus \gamma, 2\gamma\} & \text{if } l < m,\ i = j \\ \gamma & \text{if } l < m,\ i \neq j. \end{cases}$$

If this sequence satisfies the triangle inequality, then it will witness $\gamma \in \Gamma(b_1, b_2/C)$. Therefore, we have left to verify the triangle inequalities, which we do via direct case analysis.

By indiscernibility in the definition of $(\bar{b}^l)_{l<\omega}$, the nontrivial triangles to check are those with the following vertex sets:

1. $\{b_i^l, b_j^m, c\}$ for some $i, j \in \{1, 2\}$, $l < m < \omega$, and $c \in C$,
2. $\{b_i^l, b_j^m, b_k^n\}$ for some $i, j, k \in \{1, 2\}$ and $l \leq m \leq n < \omega$.

Case 1: $\{b_i^l, b_j^m, c\}$ for some $i, j \in \{1, 2\}$, $l < m < \omega$, and $c \in C$.

We need to show

$$|d(b_i, c) \ominus d(b_j, c)| \leq d(b_i^l, b_j^m) \leq d(b_i, c) \oplus d(b_j, c).$$

(*i*) $d(b_i^l, b_j^m) \leq d(b_i, c) \oplus d(b_j, c)$.

In all cases, we have $d(b_i^l, b_j^m) \leq d_{\max}(b_i, b_j/C) \leq d(b_i, c) \oplus d(b_j, c)$.

(*ii*) $d(b_i^l, b_j^m) \geq |d(b_i, c) \ominus d(b_j, c)|$.

We may clearly assume $i \neq j$. Then $d(b_i^l, b_j^m) = \gamma \geq |d(b_i, c) \ominus d(b_j, c)|$.

Case 2: $\{b_i^l, b_j^m, b_k^n\}$ for some $i, j, k \in \{1, 2\}$ and $l \leq m \leq n < \omega$.

Note that i, j, k cannot all be distinct. By indiscernibility, and the symmetry in the definition of $(\bar{b}^l)_{l<\omega}$, we may assume $l < n = m$ or $l < m < n$.

Subcase 2.1: $l < m = n$. Then we may assume $j \neq k$, and it suffices to check the following two inequalities.

(*i*) $d(b_1, b_2) \leq d(b_i^l, b_j^m) \oplus d(b_i^l, b_k^m)$.

Without loss of generality, we may assume $i = j$. We want to show

$$d(b_1, b_2) \leq d(b_i^l, b_i^m) \oplus \gamma.$$

If $d(b_i^l, b_i^m) = d(b_1, b_2) \oplus \gamma$ then this is trivial. If $d(b_i^l, b_i^m) = 2\gamma$ then this is true since $d(b_1, b_2) \leq 3\gamma$. Suppose $d(b_i^l, b_i^m) = d_{\max}(b_i, b_i/C)$. Then, using Lemma 3.3.3(a),

$$d(b_1, b_2) \leq d_{\max}(b_1, b_2/C) \leq d_{\max}(b_i, b_i/C) \oplus d_{\min}(b_1, b_2/C) \leq d(b_i^l, b_i^m) \oplus \gamma.$$

(*ii*) $d(b_i^l, b_j^m) \leq d(b_1, b_2) \oplus d(b_i^l, b_k^m)$.

Suppose $i = j$. Then $i \neq k$ so $d(b_i^l, b_j^m) \leq d(b_1, b_2) \oplus \gamma = d(b_1, b_2) \oplus d(b_i^l, b_k^m)$.

Suppose $i = k$. Then $i \neq j$ so $d(b_i^l, b_j^m) = \gamma$. If $d(b_i^l, b_k^m) = d(b_1, b_2) \oplus \gamma$ or $d(b_i^l, b_k^m) = 2\gamma$ then the inequality is obvious. So we may assume $d(b_i^l, b_k^m) = d_{\max}(b_i, b_i/C)$. Then, using Lemma 3.3.3(a),

$$d(b_i^l, b_j^m) = \gamma \leq d_{\max}(b_1, b_2/C) \leq d(b_1, b_2) \oplus d_{\max}(b_i, b_i/C) = d(b_1, b_2) \oplus d(b_i^l, b_k^m).$$

Subcase 2.2: $l < m < n$. By indiscernibility, it suffices to check

$$d(b_i^l, b_j^m) \leq d(b_i^l, b_k^n) \oplus d(b_j^m, b_k^n).$$

If $i \neq j$ or $i = j = k$ then the inequality is trivial. So assume $i = j \neq k$. Then we have $d(b_i^l, b_j^m) \leq 2\gamma = d(b_i^l, b_k^n) \oplus d(b_j^m, b_k^n)$. □

Using this result, we can formulate Lemma 3.4.4 as follows.

Corollary 3.4.6. *Suppose $a, b_1, b_2 \in \mathbb{U}_{\mathcal{R}}$ and $C \subset \mathbb{U}_{\mathcal{R}}$. Then $a \mathop{\smile\!\!\!\!\!\!\mid}^d_C b_1 b_2$ if and only if, for all $i, j \in \{1, 2\}$,*

$$d(a, b_i) \oplus d(a, b_j) \geq d_{\max}(b_i, b_j/C) \text{ and } |d(a, b_i) \ominus d(a, b_j)| \leq d_{\min}(b_i, b_j/C).$$

Altogether, this gives the complete characterization of $\mathop{\smile\!\!\!\!\!\!\mid}^d$.

Theorem 3.4.7. *Given $A, B, C \subset \mathbb{U}_{\mathcal{R}}$, $A \mathop{\smile\!\!\!\!\!\!\mid}^d_C B$ if and only if, for all $b_1, b_2 \in B$,*

$$d_{\max}(b_1, b_2/AC) = d_{\max}(b_1, b_2/C) \text{ and } d_{\min}(b_1, b_2/AC) = d_{\min}(b_1, b_2/C).$$

Proof. This follows directly from Lemma 3.4.1 and Corollary 3.4.6, along with the observation that, for any $b_1, b_2 \in \mathbb{U}_{\mathcal{R}}$ and $A, C \subset \mathbb{U}_{\mathcal{R}}$, $d_{\min}(b_1, b_2/AC) \neq d_{\min}(b_1, b_2/C)$ if and only if there is some $a \in A$ such that $|d(a, b_1) \ominus d(a, b_2)| > d_{\min}(b_1, b_2/C)$. □

Having completed the characterization of $\mathop{\smile\!\!\!\!\!\!\mid}^d$, we pause to recall our main goal, which is to prove Theorem 3.3.2. With Theorem 3.4.7 in hand, it suffices to show $\mathop{\smile\!\!\!\!\!\!\mid}^d$ and $\mathop{\smile\!\!\!\!\!\!\mid}^f$ coincide on $\mathbb{U}_{\mathcal{R}}$. Using Fact 1.2.4, it therefore suffices to prove the following theorem.

Theorem 3.4.8. *Fix subsets $B, C \subset \mathbb{U}_{\mathcal{R}}$ and a singleton $b_* \in \mathbb{U}_{\mathcal{R}}$. For any $A \subset \mathbb{U}_{\mathcal{R}}$, if $A \mathop{\smile\!\!\!\!\!\!\mid}^d_C B$ then there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!\mid}^d_C B b_*$.*

The proof this result requires several steps. Therefore, for the rest of the section, we fix $B, C \subset \mathbb{U}_{\mathcal{R}}$ and $b_* \in \mathbb{U}_{\mathcal{R}}$. Given $b \in BC$, let $\delta_b = d_{\min}(b_*, b/C)$ and $\epsilon_b = d_{\max}(b_*, b/C)$.

Definition 3.4.9.

1. Given $\alpha, \beta \in R^*$, define

$$\alpha \mathbin{\dot{\ominus}} \beta = \begin{cases} |\alpha \ominus \beta| & \text{if } \beta \leq \alpha \\ 0 & \text{if } \alpha < \beta. \end{cases}$$

2. Given $a \in \mathbb{U}_{\mathcal{R}}$, define

$$U(a) = \inf_{b \in BC} (d(a, b) \oplus \delta_b),$$
$$L(a) = \sup_{b \in BC} \max\{\epsilon_b \mathbin{\dot{\ominus}} d(a, b), d(a, b) \mathbin{\dot{\ominus}} \delta_b\}.$$

Toward the proof of Theorem 3.4.8, we begin with two technical lemmas concerning constraints on nondividing extensions.

Lemma 3.4.10. *Fix* $a \in \mathbb{U}_{\mathcal{R}}$.

(a) *Assume* $\gamma \in R^*$ *is such that* $L(a) \leq \gamma \leq U(a)$ *and* $d_{\max}(b_*, b_*/C) \leq 2\gamma$. *If* $a' \in \mathbb{U}_{\mathcal{R}}$ *is such that* $a' \equiv_{BC} a$ *and* $d(a', b_*) = \gamma$, *then* $a' \mathop{\smile\!\!\!\!\!\!|}^d_C Bb_*$.

(b) *If* $a \mathop{\smile\!\!\!\!\!\!|}^d_C B$ *then* $L(a) \leq U(a)$ *and* $d_{\max}(b_*, b_*/C) \leq 2U(a)$.

(c) *If* $a \mathop{\smile\!\!\!\!\!\!|}^d_C B$ *and* $a' \equiv_C a$, *with* $a' \equiv_{BC} a$ *and* $d(a', b_*) = U(a)$, *then* $a' \mathop{\smile\!\!\!\!\!\!|}^d_C Bb_*$.

Proof. Part (a). Fix $a' \in \mathbb{U}_{\mathcal{R}}$, with $a' \equiv_{BC} a$ and $d(a, b_*) = \gamma$. We use Theorem 3.4.7 to prove $a' \mathop{\smile\!\!\!\!\!\!|}^d_C Bb_*$. First, note that $a' \equiv_{BC} a$ and $L(a) \leq \gamma \leq U(a)$ together imply that, for all $b \in B$,

$$\begin{aligned} d_{\max}(b, b_*/C) &= \epsilon_b \leq \gamma \oplus d(a, b) = d(a', b_*) \oplus d(a', b), \text{ and} \\ d_{\min}(b, b_*/C) &= \delta_b \geq |\gamma \ominus d(a, b)| = |d(a', b_*) \ominus d(a', b)|. \end{aligned}$$

Finally, we trivially have $|d(a', b_*) \ominus d(a', b_*)| \leq d_{\min}(b_*, b_*/C)$ and, by assumption,

$$d_{\max}(b_*, b_*/C) \leq 2\gamma = d(a', b_*) \oplus d(a', b_*).$$

By Theorem 3.4.7, this verifies $a' \mathop{\smile\!\!\!\!\!\!|}^d_C Bb_*$.

Part (b). Assume $a \mathop{\smile\!\!\!\!\!\!|}^d_C B$. By definition of $\mathop{\smile\!\!\!\!\!\!|}^d$, we have $A \mathop{\smile\!\!\!\!\!\!|}^d_C BC$. Then, for any $b_1, b_2 \in BC$, we have, by Theorem 3.4.7,

$$d_{\max}(b_1, b_2/C) \leq d(a, b_1) \oplus d(a, b_2), \qquad (\dagger)_1$$

$$d_{\min}(b_1, b_2/C) \geq |d(a, b_1) \ominus d(a, b_2)|. \qquad (\dagger)_2$$

Moreover, for any $b_1, b_2 \in BC$, we have, by Lemma 3.3.3,

$$\epsilon_{b_1} \leq d_{\max}(b_1, b_2/C) \oplus \delta_{b_2}, \qquad (*)_1$$

$$d_{\min}(b_1, b_2/C) \leq \delta_{b_1} \oplus \delta_{b_2}, \qquad (*)_2$$

$$d_{\max}(b_*, b_*/C) \leq \epsilon_{b_1} \oplus \delta_{b_2}. \qquad (*)_3$$

To show $L(a) \leq U(a)$, we fix $\alpha \in \{\epsilon \ominus d(a, b) : b \in BC\} \cup \{d(a, b) \ominus \delta_b : b \in BC\}$ and $\beta \in \{d(a, b) \oplus \delta_b : b \in BC\}$, and show $\alpha \leq \beta$. Let $b_2 \in BC$ be such that $\beta = d(a, b_2) \oplus \delta_{b_2}$.
Case 1: $\alpha = \epsilon_{b_1} \ominus d(a, b_1)$ for some $b_1 \in BC$.

Then it suffices to show $\epsilon_{b_1} \leq d(a, b_1) \oplus \beta$. By $(\dagger)_1$ and $(*)_1$, we have

$$\epsilon_{b_1} \leq d_{\max}(b_1, b_2/C) \oplus \delta_{b_2} \leq d(a, b_1) \oplus d(a, b_2) \oplus \delta_{b_2} = d(a, b_1) \oplus \beta.$$

Case 2: $\alpha = d(a, b_1) \ominus \delta_{b_1}$ for some $b_1 \in BC$.

Then it suffices to show $d(a,b_1) \leq \delta_{b_1} \oplus \beta$. By $(\dagger)_2$ and $(*)_2$, we have

$$d(a,b_1) \leq d_{\min}(b_1,b_2/C) \oplus d(a,b_2) \leq \delta_{b_1} \oplus \delta_{b_2} \oplus d(a,b_2) = \delta_{b_1} \oplus \beta.$$

Finally, we show $d_{\max}(b_*,b_*) \leq 2U(a)$. By Proposition 2.6.4, it suffices to fix $b \in BC$ and show $d_{\max}(b_*,b_*/C) \leq 2d(a,b) \oplus 2\delta_b$. By $(\dagger)_1$, $(*)_1$, and $(*)_3$, we have

$$d_{\max}(b_*,b_*/C) \leq \epsilon_b \oplus \delta_b \leq d_{\max}(b,b/C) \oplus \delta_b \oplus \delta_b \leq 2d(a,b) \oplus 2\delta_b.$$

Part (c). Immediate from (a) and (b). $\square$

Lemma 3.4.11.

(a) *If* $a \in \mathbb{U}_{\mathcal{R}}$ *then* $U(a) \leq d_{\max}(a,b_*/BC)$ *and* $\sup_{b\in BC} |d(a,b) \ominus d(b_*,b)| \leq L(a)$.

(b) *If* $a \in \mathbb{U}_{\mathcal{R}}$ *and* $a \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{C} B$ *then*

$$\sup_{b\in BC} |d(a,b) \ominus d(b_*,b)| \leq U(a) \leq d_{\max}(a,b_*/BC).$$

(c) *If* $a_1,a_2 \in \mathbb{U}_{\mathcal{R}}$ *are such that* $a_1a_2 \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{C} B$ *then*

$$|U(a_1) \ominus U(a_2)| \leq d(a_2,a_2) \leq U(a_1) \oplus U(a_2).$$

Proof. Part (a). The first inequality is immediate, since $\delta_b \leq d(b_*,b)$ for any $b \in \mathbb{U}_{\mathcal{R}}$. For the second inequality, we fix $b \in \mathbb{U}_{\mathcal{R}}$ and show

$$|d(a,b) \ominus d(b_*,b)| \leq \max\{\epsilon_b \dot{-} d(a,b), d(a,b) \dot{-} \delta_b\}.$$

If $d(a,b) \leq d(b_*,b)$ then, since $d(b_*,b) \leq \epsilon_b$, we have $\epsilon_b \dot{-} d(a,b) = |\epsilon_b \ominus d(a,b)|$ and

$$d(b_*,b) \leq \epsilon_b \leq |\epsilon_b \ominus d(a,b)| \oplus d(a,b).$$

This gives $|d(a,b) \ominus d(b_*,b)| \leq \epsilon_b \dot{-} d(a,b)$, as desired.

Otherwise, if $d(b_*,b) \leq d(a,b)$ then, since $\delta_b \leq d(b_*,b)$, we have $d(a,b) \dot{-} \delta_b = |d(a,b) \ominus \delta_b|$ and

$$d(a,b) \leq |d(a,b) \ominus \delta_b| \oplus \delta_b \leq |d(a,b) \ominus \delta_b| \oplus d(b_*,b).$$

This gives $|d(a,b) \ominus d(b_*,b)| \leq d(a,b) \dot{-} \delta_b$, as desired.

Part (b). Combine part (a) with Lemma 3.4.10(b).

Part (c). To show $d(a_1,a_2) \leq U(a_1) \oplus U(a_2)$, we fix $b,b' \in BC$ and show $d(a_1,a_2) \leq d(a_1,b) \oplus \delta_b \oplus d(a_2,b') \oplus \delta_{b'}$. Since $a_1 \mathop{\smile\!\!\!\!\!\!\mid}^{d}_{C} B$ we have, by Theorem 3.4.7 and Lemma 3.3.3(a),

$$|d(a_1,b) \ominus d(a_1,b')| \leq d_{\min}(b,b'/C) \leq \delta_b \oplus \delta_{b'}.$$

Therefore, $d(a_1, b') \leq d(a_1, b_1) \oplus \delta_b \oplus \delta_{b'}$, and so

$$d(a_1, a_2) \leq d(a_1, b') \oplus d(a_2, b') \leq d(a_1, b) \oplus \delta_b \oplus d(a_2, b') \oplus \delta_{b'},$$

as desired.

Finally, we show $|U(a_1) \ominus U(a_2)| \leq d(a_1, a_2)$. Without loss of generality, we assume $U(a_1) \leq U(a_2)$ and show $U(a_2) \leq d(a_1, a_2) \oplus U(a_1)$. For this, fix $b \in B$, and note $U(a_2) \leq d(a_2, b) \oplus \delta_b \leq d(a_1, a_2) \oplus d(a_1, b) \oplus \delta_b$. □

We can now prove Theorem 3.4.8, which completes the proof of Theorem 3.3.2.

Proof of Theorem 3.4.8. Fix variables $\bar{x} = (x_a)_{a \in A}$ and define the type

$$p(\bar{x}) := \mathrm{tp}_{\bar{x}}(A/BC) \cup \{d(x_a, b_*) = U(a) : a \in A\}.$$

If A' realizes $p(\bar{x})$ then $A' \equiv_{BC} A$ and, by Lemma 3.4.10(*c*), we have $a' \mathop{\smile\!\!\!\!\!\!\;|}^{d}_{C} B$ for all $a' \in A$, which gives $A' \mathop{\smile\!\!\!\!\!\!\;|}^{d}_{C} B$ by Lemma 3.4.1. Therefore, it suffices to show $p(\bar{x})$ is consistent, which means verifying the triangle inequalities in the definition. The nontrivial triangles to check either have distances $(d(a, b), d(b, b_*), U(a))$ for some $a \in A$ and $b \in BC$, or $(d(a_1, a_2), U(a_1), U(a_2))$ for some $a_1, a_2 \in A$. Therefore, the result follows from parts (*b*) and (*c*) of Lemma 3.4.11. □

3.5 Urysohn Spaces of Low Complexity

3.5.1 Stability

In this section, we characterize the Urysohn monoids $\mathcal{R}$ for which $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is stable. In particular, we show that, given a Urysohn monoid $\mathcal{R}$, $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is stable if and only if $\mathcal{R}$ is ultrametric. The heart of this fact lies in the observation that ultrametric spaces correspond to refining equivalence relations since, if (A, d) is an ultrametric space, then for any distance r, $d(x, y) \leq r$ is an equivalence relation on A. Altogether, the result that ultrametric monoids yield stable Urysohn spaces recovers classical results on theories of equivalence relations (see [7, Section III.4]). Therefore, our work focuses on the converse, which says that stable Urysohn spaces must be ultrametric. We will also emphasize the relationship to nonforking, and so it will be useful to have the following simplification of $d_{\max}$ for ultrametric Urysohn spaces.

Proposition 3.5.1. *Suppose $\mathcal{R}$ is a countable ultrametric monoid. Fix $C \subset \mathbb{U}_{\mathcal{R}}$ and $b_1, b_2 \in \mathbb{U}_{\mathcal{R}}$.*

(*a*) *If $d(b_1, c) \neq d(b_2, c)$ for some $c \in C$ then $d_{\max}(b_1, b_2/C) = d(b_1, b_2)$.*

(*b*) *If $d(b_1, c) = d(b_2, c)$ for all $c \in C$ then $d_{\max}(b_1, b_2/C) = d(b_1, C)$.*

Proof. Recall that, since $\mathcal{R}$ is ultrametric, we have $\alpha \oplus \beta = \max\{\alpha, \beta\}$ for all $\alpha, \beta \in R^*$ by Lemma 2.9.3. Therefore, $d_{\max}(b_1, b_2/C) = \inf_{c \in C} \max\{d(b_1, c), d(b_2, c)\}$, which immediately implies part (b). For part (a), fix $c \in C$ such that $d(b_1, c) \neq d(b_2, c)$. Then

$$d(b_1, b_2) \leq d_{\max}(b_1, b_2/C) \leq \max\{d(b_1, c), d(b_2, c)\} = d(b_1, b_2),$$

as desired. □

The characterization of stability combines Proposition 3.3.5 with the following observations.

Lemma 3.5.2. *Suppose $\mathcal{R}$ is a Urysohn monoid.*

(a) *If $\mathcal{R}$ is ultrametric then $\perp^{\otimes}$ coincides with $\perp^{\text{dist}}$.*

(b) *If $\perp^{\text{dist}}$ is symmetric then $\mathcal{R}$ is ultrametric.*

Proof. Part (a). Suppose $\mathcal{R}$ is ultrametric. By Proposition 3.3.5(c), it suffices to show $\perp^{\text{dist}}$ implies $\perp^{\otimes}$. Fix $A, B, C \subset \mathbb{U}_{\mathcal{R}}$ such that $A \perp^{\text{dist}}_C B$. We want to show that, for all $a \in A$ and $b \in B$, $d(a, b) = d_{\max}(a, b/C)$. By Proposition 3.5.1, it suffices to assume $d(a, c) = d(b, c)$ for all $c \in C$, and prove $d(a, b) = d(a, C)$. Note that $A \perp^{\text{dist}}_C B$ implies $d(a, BC) = d(a, C)$, and so we have $d(a, b) \geq d(a, C)$. On the other hand, if $d(a, b) > d(a, C)$ then there is $c \in C$ such that $d(a, b) > d(a, c)$. But then $d(b, c) = \max\{d(a, b), d(a, c)\} = d(a, b) > d(a, c)$, which contradicts our assumptions.

Part (b). Suppose $\perp^{\text{dist}}$ is symmetric. Fix $r, s \in R$. There are $a, b, c \in \mathbb{U}_{\mathcal{R}}$ such that $d(a, b) = \max\{r, s\}$, $d(a, c) = \min\{r, s\}$, and $d(b, c) = r \oplus s$. Then $d(a, b) \geq d(a, c)$, so $a \perp^{\text{dist}}_c b$. By symmetry, we have $b \perp^{\text{dist}}_c a$, which means $\max\{r, s\} = d(a, b) \geq d(b, c) = r \oplus s$. Therefore $r \oplus s = \max\{r, s\}$, and we have shown $\mathcal{R}$ is ultrametric. □

Theorem 3.5.3. *Given a Urysohn monoid $\mathcal{R}$, the following are equivalent.*

(i) $\text{Th}(\mathcal{U}_{\mathcal{R}})$ *is stable.*

(ii) $\perp^f$ *coincides with* $\perp^{\text{dist}}$.

(iii) $\perp^f$ *coincides with* $\perp^{\otimes}$.

(iv) *$\mathcal{R}$ is ultrametric, i.e., for all $r, s \in S$, if $r \leq s$ then $r \oplus s = s$.*

Proof. $(iv) \Rightarrow (iii)$: Suppose $\mathcal{R}$ is ultrametric. By Proposition 3.3.5(b), it suffices to show $\perp^f$ implies $\perp^{\otimes}$. So suppose $A \perp^f_C B$ and fix $a \in A$, $b \in B$. We want to show $d(a, b) = d_{\max}(a, b/C)$. By Theorem 3.3.2, we have $d(b, C) = d_{\max}(b, b/C) \leq \max\{d(b, a), d(a, b)\} = d(a, b)$. Suppose, toward a contradiction,

$d(a,b) < d_{\max}(a,b/C)$. By Proposition 3.5.1, it follows that $d(a,c) = d(b,c)$ for all $c \in C$, and so $d_{\max}(a,b/C) = d(b,C) \leq d(a,b)$, which is a contradiction.

$(iii) \Rightarrow (i)$: If $\mathop{\smile\!\!\!\!\!\!|}^f$ coincides with $\mathop{\smile\!\!\!\!\!\!|}^{\otimes}$ then $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies symmetry and stationarity by Proposition 3.3.5(b). Therefore $\text{Th}(\mathcal{U}_{\mathcal{R}})$ is stable by Fact 1.3.3.

$(i) \Rightarrow (iv)$: Suppose $\mathcal{R}$ is not ultrametric. Then we may fix $r \in R$ such that $r < r \oplus r$. We show that the formula $\varphi(x_1,x_2,y_1,y_2) := d(x_1,y_2) \leq r$ has the order property. Define a sequence $(a_1^l, a_2^l)_{l<\omega}$ such that, given $l < m$, $d(a_1^m, a_2^l) = r \oplus r$, and all other distances are r. This clearly satisfies the triangle inequality. We have $\varphi(a_1^l, a_2^l, a_1^m, a_2^m)$ if and only if $l \leq m$.

$(iv) \Rightarrow (ii)$: Combine $(iv) \Rightarrow (iii)$ with Lemma 3.5.2(a).

$(ii) \Rightarrow (iv)$: If $\mathop{\smile\!\!\!\!\!\!|}^f$ coincides with $\mathop{\smile\!\!\!\!\!\!|}^{\text{dist}}$ then $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies local character by Proposition 3.3.5(d). Therefore $\mathop{\smile\!\!\!\!\!\!|}^{\text{dist}}$ is symmetric by Fact 1.3.3(a), which implies $\mathcal{R}$ is ultrametric by Lemma 3.5.2(b). □

Looking back at this characterization, it is worth pointing out that (i), (ii), and (iii) could all be obtained from (iv) by showing that, when $\mathcal{R}$ is ultrametric, both $\mathop{\smile\!\!\!\!\!\!|}^{\text{dist}}$ and $\mathop{\smile\!\!\!\!\!\!|}^{\otimes}$ satisfy the axioms of a stable independence relation (Definition 1.3.4). In this way, the above theorem could be entirely obtained without using the general characterization of nonforking given by Theorem 3.3.2. It is also interesting to note this same strategy is employed in [13] to prove the ternary relation $\mathop{\smile\!\!\!\!\!\!|}^{\text{dist}}$ characterizes nonforking in the (stable) continuous theory of "richly branching" $\mathbb{R}$-trees, which is the model companion of the continuous theory of $\mathbb{R}$-trees.

3.5.2 Simplicity

Our next goal is an analogous characterization of simplicity for $\text{Th}(\mathcal{U}_{\mathcal{R}})$, when $\mathcal{R}$ is a Urysohn monoid. We will obtain similar behavior in the sense that simplicity of $\text{Th}(\mathcal{U}_{\mathcal{R}})$ is detected by "nicer" characterizations of forking, as well as low complexity in the arithmetic behavior of $\mathcal{R}$.

We begin by defining archimedean equivalence in $\mathcal{R}^*$, as well as the associated preorder.

Definition 3.5.4. Suppose $\mathcal{R}$ is a distance monoid.

1. Define the relation $\preceq_{\mathcal{R}}$ on R^* such that $\alpha \preceq_{\mathcal{R}} \beta$ if and only if $\alpha \leq n\beta$ for some $n > 0$.

2. Define the relation $\sim_{\mathcal{R}}$ on R^* such that $\alpha \sim_{\mathcal{R}} \beta$ if and only if $\alpha \preceq_{\mathcal{R}} \beta$ and $\beta \preceq_{\mathcal{R}} \alpha$.

3. Given $\alpha, \beta \in R^*$, write $\alpha \prec_{\mathcal{R}} \beta$ if $\beta \not\preceq_{\mathcal{R}} \alpha$, i.e., if $n\alpha < \beta$ for all $n > 0$.

Throughout this section, we will use the fact that, given a countable distance monoid $\mathcal{R}$, if $b \in \mathbb{U}_{\mathcal{R}}$ and $C \subset \mathbb{U}_{\mathcal{R}}$ then $d_{\max}(b,b/C) = 2d(b,C)$ (see Proposition 2.6.4).

We now focus on the ternary relation $\mathop{\smile\!\!\!\!\!\!\!\perp}^{d_{\max}}$. We have noted that $\mathop{\smile\!\!\!\!\!\!\!\perp}^{f}$ implies $\mathop{\smile\!\!\!\!\!\!\!\perp}^{d_{\max}}$ (when $\mathcal{R}$ is Urysohn), and our next result characterizes when they coincide.

Proposition 3.5.5. *Suppose $\mathcal{R}$ is a Urysohn monoid. The following are equivalent.*

(*i*) *$\mathop{\smile\!\!\!\!\!\!\!\perp}^{f}$ coincides with $\mathop{\smile\!\!\!\!\!\!\!\perp}^{d_{\max}}$.*

(*ii*) *For all $r,s\in R$, if $r\leq s$ then $r\oplus r\oplus s=r\oplus s$.*

Proof. $(i)\Rightarrow(ii)$. Suppose (ii) fails, and fix $r,s\in R$, with $r\leq s$ and $r\oplus s<r\oplus r\oplus s$. Define the space (X,d) such that $X=\{a,b_1,b_2,c,c'\}$ and

$$\begin{aligned} d(a,b_1)=d(b_1,c)=d(b_1,c')&=r, \\ d(b_2,c)=d(b_2,c')&=s, \\ d(a,c)=d(a,c')=d(c,c')&=2r, \end{aligned} \qquad \begin{aligned} d(b_1,b_2)&=r\oplus s, \\ d(a,b_2)&=2r\oplus s. \end{aligned}$$

It is straightforward to verify (X,d) is an $\mathcal{R}^*$-metric space, and so we may assume (X,d) is a subspace of $\mathbb{U}_{\mathcal{R}}$. Let $C=\{c,c'\}$. First, note that

- $d_{\max}(b_1,b_2/C)=r\oplus s\leq d(a,b_1)\oplus d(a,b_2)$,
- $d_{\max}(b_1,b_1/C)=2r\leq d(a,b_1)\oplus d(a,b_1)$, and
- $d_{\max}(b_2,b_2/C)=2s\leq d(a,b_2)\oplus d(a,b_2)$.

Therefore $a\mathop{\smile\!\!\!\!\!\!\!\perp}^{d_{\max}}_{C}b_1b_2$. So to show the failure of (i), we show $a\not\mathop{\smile\!\!\!\!\!\!\!\perp}^{f}_{C}b_1b_2$. Indeed, we have

- $|d(b_1,c)\ominus d(b_2,c)|=|r\ominus s|\leq s$,
- $|d(b_1,c')\ominus d(b_2,c')|=|r\ominus s|\leq s$, and
- $d(b_1,b_2)=r\oplus s\leq 3s$.

Altogether, this implies $d_{\min}(b_1,b_2/C)\leq s$. Therefore, since $r\oplus s<2r\oplus s$, we have

$$d_{\min}(b_1,b_2/C)\leq s<|(2r\oplus s)\ominus r|=|d(a,b_1)\ominus d(a,b_2)|,$$

as desired.

$(ii)\Rightarrow(i)$. Assume $\mathcal{R}$ satisfies (ii). By Proposition 3.2.1, it follows that the same algebraic property holds for $\mathcal{R}^*$. In particular, we have $2\alpha=3\alpha$ for all $\alpha\in R^*$, which then implies $2\alpha=n\alpha$ for all $\alpha\in R^*$ and $n>1$.

In order to prove (i), it suffices by Theorem 3.3.2 to show $\mathop{\smile\!\!\!\!\!\!\!\perp}^{d_{\max}}$ implies $\mathop{\smile\!\!\!\!\!\!\!\perp}^{f}$. So suppose $A\not\mathop{\smile\!\!\!\!\!\!\!\perp}^{f}_{C}B$. Suppose, toward a contradiction, $A\mathop{\smile\!\!\!\!\!\!\!\perp}^{d_{\max}}_{C}B$. By Theorem 3.3.2, there are $a\in A$ and $b_1,b_2\in B$ such that $d_{\min}(b_1,b_2/C)<|d(a,b_1)\ominus d(a,b_2)|$. Without loss of generality, we assume $d(a,b_1)\leq d(a,b_2)$, and so we have

$$d(a,b_1)\oplus d_{\min}(b_1,b_2/C)<d(a,b_2). \qquad (\dagger)$$

Case 1: $\alpha := \frac{1}{3}d(b_1, b_2) \leq d(a, b_1)$.

By (†),

$$d(a, b_1) \oplus \alpha < d(a, b_2) \leq d(b_1, b_2) \oplus d(a, b_1) \leq 3\alpha \oplus d(a, b_1) = 2\alpha \oplus d(a, b_1),$$

which contradicts (ii).

Case 2: $d(a, b_1) < \frac{1}{3}d(b_1, b_2)$.

Suppose, toward a contradiction, $d_{\max}(a, b_1/C) \sim_{\mathcal{R}} d(a, b_1)$. Since $2d(a, b_1) = nd(a, b_1)$ for all $n > 1$, it follows that $d_{\max}(a, b_1/C) \leq 2d(a, b_1)$. Combining this observation with (†) and Lemma 3.3.3(a), we have

$$\begin{aligned} d(a, b_1) \oplus d_{\min}(b_1, b_2/C) &< d(a, b_2) \\ &\leq d_{\max}(a, b_1/C) \oplus d_{\min}(b_1, b_2/C) \\ &\leq 2d(a, b_1) \oplus d_{\min}(b_1, b_2/C), \end{aligned}$$

which, since $d(a, b_1) < \frac{1}{3}d(b_1, b_2) \leq d_{\min}(b_1, b_2/C)$, contradicts (ii).

So we have $d(a, b_1) \prec_R d_{\max}(a, b_1/C)$. Moreover, by Lemma 3.3.3(a),

$$d_{\max}(a, b_1/C) \leq d_{\max}(b_1, b_1/C) \oplus d(a, b_1).$$

It follows that $d_{\max}(a, b_1/C) \preceq_{\mathcal{R}} d_{\max}(b_1, b_1/C)$, and so $d(a, b_1) \prec_R d_{\max}(b_1, b_1/C)$. But then $d(a, b_1) \oplus d(a, b_1) < d_{\max}(b_1, b_1/C)$, which contradicts $A \mathop{\smile\!\!\!\!\!\!|}^{d_{\max}}_C B$. □

The previous result uses an algebraic condition on $\mathcal{R}$ to isolate when $\mathop{\smile\!\!\!\!\!\!|}^{f}$ "reduces" to $\mathop{\smile\!\!\!\!\!\!|}^{d_{\max}}$, in the sense that $d_{\min}$ can be omitted from the characterization of $\mathop{\smile\!\!\!\!\!\!|}^{f}$. It is worth observing that this already indicates good model theoretic behavior, since $d_{\max}$ is a much more natural operation than $d_{\min}$. Our next result shows that this same algebraic condition on $\mathcal{R}$ yields a relationship between $\mathop{\smile\!\!\!\!\!\!|}^{f}$ and $\mathop{\smile\!\!\!\!\!\!|}^{\text{dist}}$.

Lemma 3.5.6. *Suppose $\mathcal{R}$ is a Urysohn monoid and, for all $r, s \in R$, if $r \leq s$ then $r \oplus r \oplus s = r \oplus s$. Then $\mathop{\smile\!\!\!\!\!\!|}^{\text{dist}}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{f}$.*

Proof. Suppose $A \mathop{\smile\!\!\!\!\!\!|}^{\text{dist}}_C B$. By Proposition 3.5.5, it suffices to show $\mathop{\smile\!\!\!\!\!\!|}^{\text{dist}}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{d_{\max}}$. So we fix $a \in A$ and $b_1, b_2 \in B$ and show $d_{\max}(b_1, b_2/C) \leq d(a, b_1) \oplus d(a, b_2)$.

Without loss of generality, assume $d(a, b_1) \leq d(a, b_2)$. Since $A \mathop{\smile\!\!\!\!\!\!|}^{\text{dist}}_C B$, we have $d(a, C) \leq d(a, b_1)$, which means $d_{\max}(a, a/C) \leq 2d(a, b_1)$. As in the proof of Proposition 3.5.5, if $\alpha, \beta \in R^*$ then $\alpha \leq \beta$ implies $\alpha \oplus \alpha \oplus \beta = \alpha \oplus \beta$. Altogether, with Lemma 3.3.3(a), we have

$$\begin{aligned} d_{\max}(b_1, b_2/C) &\leq d_{\max}(a, b_1/C) \oplus d(a, b_2) \\ &\leq d_{\max}(a, a/C) \oplus d(a, b_1) \oplus d(a, b_2) \\ &\leq 3d(a, b_1) \oplus d(a, b_2) \\ &= d(a, b_1) \oplus d(a, b_2). \end{aligned}$$

□

We can now give the characterization of simplicity for $\mathbb{U}_{\mathcal{R}}$. The reader should compare the statement of this result to Theorem 3.5.3.

Theorem 3.5.7. *Given a Urysohn monoid $\mathcal{R}$, the following are equivalent.*

(*i*) $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ *is simple.*

(*ii*) $\mathop{\smile\!\!\!\!\!|}^{\mathrm{dist}}$ *implies* $\mathop{\smile\!\!\!\!\!|}^{f}$.

(*iii*) $\mathop{\smile\!\!\!\!\!|}^{f}$ *coincides with* $\mathop{\smile\!\!\!\!\!|}^{d_{\max}}$.

(*iv*) *For all $r, s \in R$, if $r \leq s$ then $r \oplus r \oplus s = r \oplus s$.*

Proof. (*i*) $\Rightarrow$ (*iv*): Suppose (*iv*) fails, and fix $r, s \in R$ such that $r \leq s$ and $r \oplus s < r \oplus r \oplus s$. Define the space (X, d) such that $X = \{a, b_1, b_2, c\}$ and

$$\begin{aligned} d(a, b_1) &= d(a, c) = r, & d(b_1, c) &= 2r, \\ d(a, b_2) &= d(b_2, c) = s, & d(b_1, b_2) &= r \oplus s. \end{aligned}$$

Let $B = \{b_1, b_2\}$. Then $d(a, b_1) \oplus d(a, b_2) = r \oplus s < 2r \oplus s = d(b_1, c) \oplus d(b_2, c)$, and so $a \not\mathop{\smile\!\!\!\!\!|}^{f}_{c} B$ by Theorem 3.3.2. On the other hand, $d_{\max}(a, a/Bc) = 2r = d(a, c) \oplus d(a, c)$, and so $B \mathop{\smile\!\!\!\!\!|}^{f}_{c} a$. Therefore $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is not simple by Fact 1.3.3(*a*).

(*iv*) $\Rightarrow$ (*ii*): By Lemma 3.5.6.

(*ii*) $\Rightarrow$ (*i*): If (*ii*) holds then, by Proposition 3.3.5(*d*), $\mathop{\smile\!\!\!\!\!|}^{f}$ satisfies local character. Therefore $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is simple by Fact 1.3.3(*a*).

(*iii*) $\Leftrightarrow$ (*iv*): By Proposition 3.5.5. □

As a corollary, we obtain another characterization of simplicity via the behavior of nonforking, which has a strong connection to the equivalence of $\mathop{\smile\!\!\!\!\!|}^{f}$ and $\mathop{\smile\!\!\!\!\!|}^{\mathrm{dist}}$ in the stable case.

Corollary 3.5.8. *Given a Urysohn monoid $\mathcal{R}$, the following are equivalent.*

(*i*) $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ *is simple.*

(*ii*) *For all $A, B, C \subset \mathbb{U}_{\mathcal{R}}$,*

$$A \mathop{\smile\!\!\!\!\!|}^{f}_{C} B \;\Leftrightarrow\; 2d(a, BC) = 2d(a, C) \text{ for all } a \in A.$$

Proof. (*ii*) $\Rightarrow$ (*i*): If (*ii*) holds then we clearly have that $\mathop{\smile\!\!\!\!\!|}^{f}$ implies $\mathop{\smile\!\!\!\!\!|}^{f}$, and so $\mathop{\smile\!\!\!\!\!|}^{f}$ satisfies local character by Proposition 3.3.5(*d*). Therefore $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is simple by Fact 1.3.3(*a*).

(*i*) $\Rightarrow$ (*ii*): If $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is simple then $\mathop{\smile\!\!\!\!\!|}^{f}$ coincides with $\mathop{\smile\!\!\!\!\!|}^{d_{\max}}$ by Theorem 3.5.7. Fix $A, B, C \subset \mathbb{U}_{\mathcal{R}}$. Using Lemma 3.4.1, we have

$$\begin{aligned} A \mathop{\smile\!\!\!\!\!|}^{f}_{C} B &\Leftrightarrow a \mathop{\smile\!\!\!\!\!|}^{f}_{C} B \text{ for all } a \in A \\ &\Leftrightarrow B \mathop{\smile\!\!\!\!\!|}^{f}_{C} a \text{ for all } a \in A \\ &\Leftrightarrow d_{\max}(a, a/BC) = d_{\max}(a, a/C) \text{ for all } a \in A \\ &\Leftrightarrow 2d(a, BC) = 2d(a, C) \text{ for all } a \in A. \end{aligned}$$

□

Combining previous results, we have the following picture (Figure 5) of how the four ternary relations $\perp^f$, $\perp^{d_{\max}}$, $\perp^{\otimes}$, and $\perp^{\text{dist}}$ interact in $\text{Th}(\mathcal{U}_\mathcal{R})$. (Arrows of the form "$\Rightarrow$" indicate the implication cannot be reversed; no arrow indicates no implication in either direction.)

$\text{Th}(\mathcal{U}_\mathcal{R})$ is stable:	$\perp^{\otimes}$	$\Leftrightarrow$	$\perp^{\text{dist}}$	$\Leftrightarrow$	$\perp^f$	$\Leftrightarrow$	$\perp^{d_{\max}}$
$\text{Th}(\mathcal{U}_\mathcal{R})$ is simple and unstable:	$\perp^{\otimes}$	$\Rightarrow$	$\perp^{\text{dist}}$	$\Rightarrow$	$\perp^f$	$\Leftrightarrow$	$\perp^{d_{\max}}$
$\text{Th}(\mathcal{U}_\mathcal{R})$ is not simple:	$\perp^{\otimes}$	$\Rightarrow$	$\perp^f$	$\Rightarrow$	$\perp^{d_{\max}}$		
		$\searrow$	$\perp^{\text{dist}}$				

Figure 5: Implications between ternary relations on metric spaces.

Remark 3.5.9. In order to fully justify the claims made in Figure 5, we still need to verify:

(a) If $\perp^f$ implies $\perp^{\text{dist}}$ then $\text{Th}(\mathcal{U}_\mathcal{R})$ is stable.

(b) If $\perp^{\text{dist}}$ implies $\perp^{d_{\max}}$ then $\text{Th}(\mathcal{U}_\mathcal{R})$ is simple.

Proof. Part (a). Suppose $\text{Th}(\mathcal{U}_\mathcal{R})$ is unstable, and fix $r \in R$ with $r < r \oplus r$. Let $a, b, c \in \mathbb{U}_\mathcal{R}$ be such that $d(a,b) = r = d(b,c)$ and $d(a,c) = r \oplus r$. Then $a \perp^f_c b$ and $a \not\perp^{\text{dist}}_c b$.

Part (b). Suppose $\text{Th}(\mathcal{U}_\mathcal{R})$ is not simple. Let (X,d) be the $\mathcal{R}$-metric space defined in Theorem 3.5.7[$(i) \Rightarrow (iv)$]. Then $a \perp^{\text{dist}}_c B$ and $a \not\perp^{d_{\max}}_c B$. □

The final result of this section is motivated by the distance monoid

$$\mathcal{R}_n = (\{0, 1, \ldots, n\}, +_n, \leq, 0)$$

in the case when $n \in \{1, 2\}$ (see Example 3.1.1(3)). Recall that $\mathcal{U}_{\mathcal{R}_2}$ can be viewed as the countable random graph. Moreover, $\mathcal{U}_{\mathcal{R}_1}$ is simply a countably infinite complete graph, and therefore its theory is interdefinable with the theory of infinite sets in the empty language. $\text{Th}(\mathcal{U}_{\mathcal{R}_1})$ and $\text{Th}(\mathcal{U}_{\mathcal{R}_2})$ are both classical examples in which nonforking is as uncomplicated as possible. In particular, $A \perp^f_C B$ if and only if $A \cap B \subseteq C$ (see [86, Exercise 7.3.14]). We generalize this behavior as follows.

Definition 3.5.10. A distance monoid $\mathcal{R}$ is **metrically trivial** if $r \oplus s = \sup R$ for all nonzero $r, s \in R$.

The following properties of metrically trivial monoids are easy to verify.

Proposition 3.5.11. *Let $\mathcal{R}$ be a countable distance monoid.*

(a) *$\mathcal{R}$ is metrically trivial if and only if $r \leq s \oplus t$ for all nonzero $r, s, t \in R$.*

(b) *If $\mathcal{R}$ is metrically trivial then $\mathcal{R}^*$ is metrically trivial.*

(c) *If $\mathcal{R}$ is metrically trivial then $\mathcal{R}$ is a Urysohn monoid.*

In particular, property (a) says $\mathcal{R}$ is metrically trivial if and only if $\mathcal{R}$-metric spaces coincide with graphs whose edges are arbitrarily colored by nonzero elements of R. Therefore $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is, roughly speaking, the theory of a randomly colored graph, with color set $R^{>0}$.

Theorem 3.5.12. *Given a Urysohn monoid $\mathcal{R}$, the following are equivalent.*

(i) *$\mathcal{R}$ is metrically trivial.*

(ii) *For all $A, B, C \subset \mathbb{U}_\mathcal{R}$, $A \mathop{\smile\!\!\!\!\!\!|}^f_C B$ if and only if $A \cap B \subseteq C$.*

Proof. (i) $\Rightarrow$ (ii). Suppose $\mathcal{R}$ is metrically trivial, and fix $A, B, C \subset \mathbb{U}_\mathcal{R}$. If $A \mathop{\smile\!\!\!\!\!\!|}^f_C B$ then $A \cap B \subseteq C$ (this is true in any theory). So suppose $A \not\mathop{\smile\!\!\!\!\!\!|}^f_C B$. Note that metrically trivial monoids clearly satisfy condition (iv) of Theorem 3.5.7, and so $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is simple. By Corollary 3.5.8, there is some $a \in A$ such that $2d(a, BC) < 2d(a, C)$. Since $\mathcal{R}^*$ is metrically trivial, we must have $d(a, BC) = 0$ and so, using Proposition 2.3.5(b), we must have $a \in B \backslash C$. In particular, $a \in (A \cap B)\backslash C$.

(ii) $\Rightarrow$ (i): Suppose, $\mathcal{R}$ is not metrically trivial. Then there is $r \in R^{>0}$ such that $r \oplus r < \sup R$. Fix $a, b \in \mathbb{U}_\mathcal{R}$ such that $d(a, b) = r$. Then $\{a\} \cap \{b\} = \emptyset$. On the other hand, $d(a, b) \oplus d(a, b) < d_{\max}(b, b/\emptyset)$, and so $a \not\mathop{\smile\!\!\!\!\!\!|}^f_\emptyset b$. □

Note that, up to isomorphism, there is a unique nontrivial, ultrametric, and metrically trivial distance monoid, namely, $\mathcal{R}_1$. Therefore, all other metrically trivial monoids yield simple unstable Urysohn spaces. However, there is evidence to suggest that, in a quantifiable sense, these monoids form a negligible portion of the simple unstable case. See Remark 3.7.24.

3.5.3 Non-axiomatizable Properties

Summarizing previous results, we have shown that the following properties (and thus all of their equivalent formulations) are each finitely axiomatizable as properties of **RUS**.

1. $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is stable.

2. $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is simple.

3. Nonforking in $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ coincides with equality.

In this section, we show supersimplicity and superstability are characterized as properties of $\mathcal{R}$, but not in an axiomatizable way. The idea behind this characterization is the observation that, due to Corollary 3.5.8, forking in simple Urysohn spaces is witnessed by distances in $\mathcal{R}^*$ of the form 2α, where $\alpha \in \mathcal{R}^*$. Moreover, if $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is simple then, by Theorem 3.5.7(iv) and Corollary 2.6.6, we have $2\alpha \oplus 2\alpha = 2\alpha$ for any $\alpha \in \mathcal{R}^*$. Altogether, forking in simple Urysohn spaces is witnessed by idempotent elements of $\mathcal{R}^*$.

Definition 3.5.13. Given a distance monoid $\mathcal{R}$, let $\mathrm{eq}(\mathcal{R})$ be the submonoid of idempotent elements of $\mathcal{R}$ (i.e. $r \in R$ such that $r \oplus r = r$).

We can use the submonoid of idempotents to characterize supersimplicity.

Theorem 3.5.14. *Suppose $\mathcal{R}$ is a Urysohn monoid and $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is simple. Then $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is supersimple if and only if $\mathrm{eq}(\mathcal{R})$ is well-ordered.*

Proof. Suppose first that we have $r_0 > r_1 > r_2 > \ldots$ in $\mathrm{eq}(\mathcal{R})$. Clearly, we may fix a subset $B = \{b_n : n < \omega\} \subset \mathbb{U}_\mathcal{R}$ such that $d(b_m, b_n) = r_{\min\{m,n\}}$ for all $m, n < \omega$. Moreover, we may fix $a \in \mathbb{U}_\mathcal{R}$ such that $d(a, b_n) = r_n$ for all $n < \omega$. To show $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is not supersimple, it suffices to fix a finite subset $C \subseteq B$ and show $a \not\mathop{\smile\!\!\!\!|}^f_C B$. Let $N < \omega$ be such that $C \subseteq \{b_n : n \leq N\}$. Then

$$2d(a, B) \leq 2d(a, b_{N+1}) = 2r_{N+1} = r_{N+1} < r_N = 2r_N \leq 2d(a, C).$$

By Corollary 3.5.8, we have $a \not\mathop{\smile\!\!\!\!|}^f_C B$.

Conversely, suppose $\mathrm{eq}(\mathcal{R})$ is well-ordered. Fix a finite tuple $\bar{a} \in \mathbb{U}_\mathcal{R}$ and a subset $B \subset \mathbb{U}_\mathcal{R}$. We want to find a finite $C \subseteq B$ such that $\bar{a} \mathop{\smile\!\!\!\!|}^f_C B$. Let $\bar{a} = (a_1, \ldots, a_n)$.
Claim: Given $1 \leq i \leq n$, there is $b_i \in B$ such that $2d(a, b_i) = 2d(a, B)$.
Proof: If $d(a, B) \in R$ or $d(a, B) = \sup R^*$ then, using Proposition 2.3.5(b), we can in fact find $b_i \in B$ such that $d(a, b_i) = d(a, B)$. Therefore, we may assume $d(a, B) \in R^* \backslash R$ and $d(a, B) < \sup R^*$. From the construction of $\mathcal{R}^*$, it follows that there is a decreasing sequence $(r_n)_{n<\omega}$ in R such that $d(a, B) < r_n$ for all $n < \omega$ and, for any $s \in R$, if $d(a, B) < s$ then there is some $n < \omega$ such that $d(a, B) < r_n < s$. Since $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is simple, it follows from Theorem 3.5.7(v) that $2r_n$ is idempotent for all $n < \omega$. Since $\mathrm{eq}(\mathcal{R})$ is well-ordered, there is some $N < \omega$ such that $2r_n = 2r_N$ for all $n \geq N$. By Corollary 2.6.6, we have $2r_N = 2d(a, B)$. Since $d(a, B) < r_N$, we may fix $b_i \in B$ such that $d(a, b_i) \leq r_N$, and so $2d(a, b_i) = 2d(a, B)$. $\dashv_{\text{claim}}$

Let $C = \{b_i : 1 \leq i \leq n\}$, where b_i is as given by the claim. By Corollary 3.5.8, we have $\bar{a} \mathop{\smile\!\!\!\!|}^f_C B$. □

Remark 3.5.15. In particular, if $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is simple and $\mathcal{R}$ is finite, then $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is supersimple. This conclusion also follows from a general result of Koponen [56].

Recall that, from Theorem 3.5.3, $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is stable if and only if $\mathrm{eq}(\mathcal{R}) = \mathcal{R}$. This yields the following characterization of superstability.

Theorem 3.5.16. *Suppose $\mathcal{R}$ is a Urysohn monoid and* $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ *is stable. The following are equivalent.*

(*i*) $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ *is ω-stable.*

(*ii*) $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ *is superstable.*

(*iii*) $\mathcal{R}$ *is well-ordered.*

Proof. $(ii) \Leftrightarrow (iii)$: Since $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is stable, it follows from Theorem 3.5.3(iv) and Theorem 3.5.14 that $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is supersimple if and only if $\mathcal{R}$ is well-ordered. Therefore the result follows since supersimplicty and superstability coincide for stable theories.

$(i) \Rightarrow (ii)$: See Fact 1.1.2.

$(iii) \Rightarrow (i)$: Suppose $\mathcal{R}$ is well-ordered. Consider $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ as the theory of infinitely refining equivalence relations $d(x,y) \leq r$, indexed by $(R, \leq, 0)$. It is also common to refer to this situation as "expanding equivalence relations". This example is well-known in the folklore to be ω-stable. The case $R = (\mathbb{N}, \leq, 0)$ is credited to Shelah (see e.g. [44]). Despite its place in the folklore, a complete proof of this result seems difficult to locate. Therefore, for the sake of completeness, we include a proof formulated in the context of ultrametric spaces.

First, since $\mathcal{R}$ is well-ordered it follows that, for all $\alpha \in R^*$, $\alpha \notin R$ implies $\alpha = \sup R$. In particular, $\mathcal{R}^*$ is still countable and well-ordered.

Fix $A \subset \mathbb{U}_{\mathcal{R}}$, with $|A| \leq \aleph_0$. Enumerate $A = \{a_n : n < \omega\}$. We show $|S_1(A)| \leq \aleph_0$. Note that $\mathcal{R}$ is ultrametric since $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is stable.

Fix $b \in \mathbb{U}_{\mathcal{R}}$. Given $n < \omega$, set $r_n = d(b, a_n) \in R^*$. By quantifier elimination, $\mathrm{tp}(b/A)$ is uniquely determined by

$$\bigcup_{n<\omega} p_{r_n}(x, a_n),$$

and therefore uniquely determined by $(r_n)_{n<\omega}$.

We construct a (possibly finite) subsequence of $(r_n)_{n<\omega}$ as follows. Let $i_0 = 0$ and, given i_n, let $i_{n+1} > i_n$ be minimal such that $d(a_{i_n}, a_{i_{n+1}}) \leq r_{i_n}$, if such an index exists. Next, define a function $f : \omega \longrightarrow \{0,1,2\}$ such that

$$f(k) = \begin{cases} 0 & \text{if } k \neq i_n \text{ for any } n < \omega \\ 1 & \text{if } k = i_n \text{ for some } 0 < n < \omega \text{ and } r_{i_n} = r_{i_{n-1}} \\ 2 & \text{otherwise.} \end{cases}$$

Given $k < \omega$, let $n(k)$ be maximal such that $i_{n(k)} \leq k$. Note that, if $f(k) = 0$ then $d(a_{i_{n(k)}}, a_k) > r_{i_{n(k)}}$, and so $r_k = d(a_{i_{n(k)}}, a_k)$.

Define $I = \{k < \omega : f(k) = 2\}$. If $k \in I$ and $k > 0$ then we have $r_{i_{n(k)}} \neq r_{i_{n(k)-1}}$. If $r_{i_{n(k)}} > r_{i_{n(k)-1}}$ then $d(a_{i_{n(k)-1}}, a_{i_{n(k)}}) = r_{i_{n(k)}}$, which is a contradiction. Therefore $(r_n)_{n\in I}$ is a strictly decreasing sequence in R^*, and so I is finite. Set $k_* = \max I$.

Claim 1: For all $k \geq k_*$, if $f(k) > 0$ then $d(a_{k_*}, a_k) \leq r_{k_*}$.
Proof: We proceed by induction on $k \geq k_*$, where the base case is trivial. For the induction step, fix $k > k_*$ such that $f(k) > 0$ and let $k_* \leq k' < k$ be maximal such that $f(k') > 0$. Then $d(a_{k'}, a_k) \leq r_{k'} = r_{k_*}$, and so we have $d(a_k, a_{k_*}) \leq \max\{d(a_k, a_{k'}), d(a_{k'}, a_{k_*})\} \leq r_*$ by induction. $\dashv_{\text{claim}}$
Claim 2: If $k > k_*$ then $r_k = \max\{r_{k_*}, d(a_k, a_{k_*})\}$.
Proof: Fix $k > k_*$, and note that $k_* \leq i_{n(k)}$. Since $\mathcal{R}$ is ultrametric, we may assume $r_{k_*} = d(a_k, a_{k^*})$. If $i_{n(k)} < k$ then $f(k) = 0$, and so

$$d(a_k, a_{i_{n(k)}}) > r_{i_{n(k)}} = r_{k_*} = d(a_k, a_{k^*}).$$

It follows that $d(a_{i_{n(k)}}, a_{k_*}) > r_{k_*}$, which contradicts Claim 1. Therefore $i_{n(k)} = k$, and so $f(k) = 1$, which means $r_k = r_{k_*}$. $\dashv_{\text{claim}}$

By Claim 2, $(r_k)_{k<\omega}$ is uniquely determined by $(r_k)_{k \leq k_*}$, and so $|S_1(A)| \leq |(R^*)^{<\omega}|$. Therefore $S_1(A)$ is countable, as desired. □

Corollary 3.5.17. *Supersimplicity and superstability are not axiomatizable properties of* **RUS**.

Proof. Since "superstable" is equivalent to "stable and supersimple", and stability is finitely axiomatizable, it is enough to show superstability is not axiomatizable. Suppose, toward a contradiction, there is an $\mathcal{L}_{\omega_1,\omega}$-sentence φ in $\mathcal{L}_{\text{om}}$ such that, for any Urysohn monoid $\mathcal{R}$, $\text{Th}(\mathcal{U}_{\mathcal{R}})$ is superstable if and only if $\mathcal{R} \models \varphi$. After adding constants $(c_i)_{i<\omega}$ to $\mathcal{L}_{\text{om}}$, and conjuncting with φ_{QE} along with a sentence axiomatizing distance monoids with universe $(c_i)_{i<\omega}$, we obtain an $\mathcal{L}_{\omega_1,\omega}$-sentence φ^* in $\mathcal{L}_{\text{om}}$ such that, for any $\mathcal{L}_{\text{om}}$-structure $\mathcal{R}$, $\mathcal{R} \models \varphi^*$ if and only if $\mathcal{R}$ is a countable, ultrametric, well-ordered, distance monoid. By classical results in infinitary logic (see e.g. [63, Corollary 4.28]), it follows that there is some $\mu < \omega_1$ such that any model of φ^* has order type at most μ. This is clearly a contradiction, since any ordinal can be given the structure of an ultrametric distance monoid (c.f. Example 3.1.1(6)). □

Finally, it is worth reiterating that Theorem 3.5.16 can be restated as the following classical result.

Corollary 3.5.18. *Let $(R, \leq, 0)$ be a countable linear order, with least element 0, and consider the first-order language $\mathcal{L} = \{E_r : r \in R\}$, where each E_r is a binary relation symbol. Let T be the complete $\mathcal{L}$-theory asserting that each E_r is an equivalence relation, E_0 coincides with equality, and if $0 < r < s$ then E_r refines E_s into infinitely many infinite classes. Then T is superstable if and only if T is ω-stable if and only if $(R, \leq, 0)$ is a well-order.*

3.6 Cyclic Indiscernible Sequences

So far our results have been motivated by choosing a particular kind of good behavior for $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ and then characterizing when this behavior happens. In this section, we give a uniform upper bound for the complexity of $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ for any Urysohn monoid $\mathcal{R}$. In particular, we will show that if $\mathcal{R}$ is a Urysohn monoid then $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ does not have the finitary strong order property. We will accomplish this by proving the following theorem.

Theorem 3.6.1. *Suppose $\mathcal{R}$ is a Urysohn monoid and $\mathcal{I} = (\bar{a}^l)_{l<\omega}$ is an indiscernible sequence in $\mathbb{U}_\mathcal{R}$ of tuples of possibly infinite length. If $|\mathrm{NP}(\mathcal{I})| = n < \omega$ then $\mathcal{I}$ is $(n+1)$-cyclic.*

From this and Proposition 1.4.8(b), we obtain the following corollary.

Corollary 3.6.2. *If $\mathcal{R}$ is a Urysohn monoid then* $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ *does not have* FSOP*.*

In order to prove Theorem 3.6.1, we will need to work with certain partial types, which arise when considering whether or not an indiscernible sequence is n-cyclic for some n. In particular, if $(\bar{a}^l)_{l<\omega}$ is an indiscernible sequence (in $\mathbb{U}_\mathcal{R}$ for some $\mathcal{R}$), and $p(\bar{x},\bar{y}) = \mathrm{tp}(\bar{a}^0,\bar{a}^1)$, then we will consider partial types of the form

$$p(\bar{x}^1,\bar{x}^2)\cup p(\bar{x}^2,\bar{x}^3)\cup\ldots\cup p(\bar{x}^{n-1},\bar{x}^n)\cup p(\bar{x}^n,\bar{x}^1),$$

where n is some integer. A partial type of this form still has a nice structure, in the sense that, by quantifier elimination, it is entirely determined a partial $\mathcal{R}^*$-coloring of $\bar{x}^1\cup\bar{x}^2\cup\ldots\cup\bar{x}^n$. Therefore, in order to prove Theorem 3.6.1, we need to set forth some basic ideas concerning the completion of partial colorings to total metrics.

Definition 3.6.3. Fix a distance monoid $\mathcal{R}$. Suppose X is set and $f:\mathrm{dom}(f)\subseteq X\times X\longrightarrow\mathcal{R}$ is a symmetric partial function.

1. f is a **partial $\mathcal{R}$-semimetric** if, for all $x\in X$, $(x,x)\in\mathrm{dom}(f)$ and $f(x,x)=0$. In this case, (X,f) is a **partial $\mathcal{R}$-semimetric space**. We say (X,f) is **consistent** if there is an $\mathcal{R}$-pseudometric on X extending f.

2. Given $m\geq 1$, a sequence $(x_0,x_1,\ldots,x_m)$ in X^{m+1} is an **f-sequence** if $(x_0,x_m)\in\mathrm{dom}(f)$ and $(x_i,x_{i+1})\in\mathrm{dom}(f)$ for all $0\leq i<m$.

3. Given $m\geq 1$, if $\bar{x}=(x_0,\ldots,x_m)$ is an f-sequence, then we let $f[\bar{x}]$ denote the sum $f(x_0,x_1)\oplus f(x_1,x_2)\oplus\ldots\oplus f(x_{n-1},x_m)$.

4. Given $m\geq 1$, f is **m-transitive** if $f(x_0,x_m)\leq f[\bar{x}]$ for all f-sequences $\bar{x}=(x_0,\ldots,x_m)$.

5. If $\bar{x}=(x_0,\ldots,x_m)$ is a sequence of elements of X, then a **subsequence of** $\bar{x}$ is a sequence of the form $(x_0,x_{i_1},\ldots,x_{i_k},x_m)$, for some $0<i_1<\ldots<i_k<m$. If $1\leq k\leq m-2$ then the subsequence is **proper**.

Using a standard application of the minimal-length path metric, we will obtain the following test for consistency of partial semimetric spaces.

Lemma 3.6.4. *Let $\mathcal{R}$ be a distance monoid. Then a partial $\mathcal{R}^*$-semimetric space (X,f) is consistent if and only if f is m-transitive for all $m \geq 1$.*

Proof. First, suppose (X,f) is consistent and let d be an $\mathcal{R}^*$-pseudometric on X extending f. If $m > 0$ and $\bar{x} = (x_0, x_1, \ldots, x_m)$ is an f-sequence, then $d(x_0, x_m) = f(x_0, x_m)$ and $f[\bar{x}] = d(x_0, x_1) \oplus \ldots \oplus d(x_{m-1}, x_m)$. Therefore $f(x_0, x_m) \leq f[\bar{x}]$ by the triangle inequality.

Conversely, suppose f is m-transitive for all $m \geq 1$. Given $x, y \in X$, set

$$d(x,y) = \inf\{f[\bar{x}] : \bar{x} = (x_0, \ldots, x_m) \text{ is an } f\text{-sequence with } x_0 = x \text{ and } x_m = y\}.$$

For any $x, y, z \in X$, if $\bar{x}$ is an f-sequence from x to y, and $\bar{y}$ is an f-sequence from y to z, then $\bar{x}\bar{y}$ is an f-sequence from x to z, and so $d(x,z) \leq f[\bar{x}] \oplus f[\bar{y}]$. Using Proposition 2.6.4, we have $d(x,z) \leq d(x,y) \oplus d(y,z)$, and so d is an $\mathcal{R}^*$-pseudometric.

We have left to show d extends f. Indeed, if $(x,y) \in \mathrm{dom}(f)$ then (x,y) is an f-sequence and so $d(x,y) \leq f(x,y)$. Conversely, if $\bar{x}$ is an f-sequence from x to y then $f(x,y) \leq f[\bar{x}]$ since f is m-transitive for all $m \geq 1$. Therefore, we have $f(x,y) \leq d(x,y)$. □

For the rest of the section, we fix a Urysohn monoid $\mathcal{R}$. The key tool needed to prove Theorem 3.6.1 is the following test for when an indiscernible sequence in $\mathbb{U}_{\mathcal{R}}$ is n-cyclic. This result was first proved for the complete Urysohn sphere in continuous logic, in joint work with Caroline Terry [26].

Lemma 3.6.5. *Suppose $\mathcal{R}$ is a Urysohn monoid and $(\bar{a}^l)_{l<\omega}$ is an indiscernible sequence in $\mathbb{U}_{\mathcal{R}}$. Given $i, j \in \ell(\bar{a}^0)$, set $\epsilon_{i,j} = d(a_i^0, a_j^1)$. Given $n \geq 2$, $(\bar{a}^l)_{l<\omega}$ is n-cyclic if and only if, for all $i_1, \ldots, i_n \in \ell(\bar{a}^0)$, $\epsilon_{i_n,i_1} \leq \epsilon_{i_1,i_2} \oplus \epsilon_{i_2,i_3} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}$.*

Proof. Fix an indiscernible sequence $\mathcal{I} = (\bar{a}^l)_{l<\omega}$ and some $n \geq 2$. We let $p(\bar{x}, \bar{y}) = \mathrm{tp}(\bar{a}^0, \bar{a}^1)$ and set

$$q(\bar{x}^1, \ldots, \bar{x}^n) = p(\bar{x}^1, \bar{x}^2) \cup p(\bar{x}^2, \bar{x}^3) \cup \ldots \cup p(\bar{x}^{n-1}, \bar{x}^n) \cup p(\bar{x}^n, \bar{x}^1).$$

Then $\mathcal{I}$ is n-cyclic if and only if q is consistent. Let $X = \bar{x}^1 \cup \ldots \cup \bar{x}^n$. Note that, by quantifier elimination, q is determined by a partial symmetric function $f : \mathrm{dom}(f) \subseteq X \times X \longrightarrow R^*$, where $\mathrm{dom}(f)$ is the symmetric closure of

$$\{(x_i^l, x_j^m) : i, j \in \ell(\bar{a}^0),\ l, m < \omega,\ \text{and } m \in \{l, l+1\} \text{ or } (l,m) = (1,n)\},$$

and, given $(x_i^l, x_j^m) \in \mathrm{dom}(f)$, we set $f(x_i^l, x_j^m) = d(a_i^l, a_j^m)$ if $(l,m) \notin \{(1,n),(n,1)\}$ and $f(x_i^l, x_j^m) = d(a_i^1, a_j^0)$ if $(l,m) = (1,n)$. Altogether, by Proposition 3.2.2, q is consistent if and only if f can be extended to an $\mathcal{R}^*$-pseudometric on X. By Lemma 3.6.4, it follows that q is consistent if and only if f is m-transitive for all $m \geq 1$. Altogether, we have that $\mathcal{I}$ is n-cyclic if and only if f is m-transitive for all $m > 0$. Therefore, to prove the result, we show that the following are equivalent.

(i) f is m-transitive for all $m > 0$.

(ii) For all $i_1, \ldots, i_n \in \ell(\bar{a}^0)$, $\epsilon_{i_n,i_1} \leq \epsilon_{i_1,i_2} \oplus \epsilon_{i_2,i_3} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}$.

$(i) \Rightarrow (ii)$. If f is $(n-1)$-cyclic then, for all $i_1, \ldots, i_n \in \ell(\bar{a}^0)$, we have

$$\epsilon_{i_n,i_1} = f(x^1_{i_1}, x^n_{i_n}) \leq f(x^1_{i_1}, x^2_{i_2}) \oplus \ldots \oplus f(x^{n-1}_{i_{n-1}}, x^n_{i_n}) = \epsilon_{i_1,i_2} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}.$$

$(ii) \Rightarrow (i)$. Assume (ii) holds. We prove (i) by induction on m. The case $m = 1$ follows immediately by symmetry of f. For the induction step, fix $m > 1$ and assume f is j-transitive for all $j < m$. Fix an f-sequence $\bar{u} = (u_0, \ldots, u_m)$. We want to show $f(u_0, u_m) \leq f[\bar{u}]$.

Claim: If some proper subsequence of $\bar{u}$ is an f-sequence then $f(u_0, u_m) \leq f[\bar{u}]$.

Proof: Let $\bar{v} = (v_0, \ldots, v_j)$ be a proper f-subsequence, where $j < m$, $v_0 = u_0$, and $v_j = u_m$. For $0 \leq t \leq j-1$, fix i_t such that $v_t = u_{i_t}$, and set $\bar{u}_t = (u_{i_t}, u_{i_t+1}, \ldots, u_{i_{t+1}})$. By induction,

$$f(u_0, u_m) = f(v_0, v_j) \leq f(v_0, v_1) \oplus \ldots \oplus f(v_{j-1}, v_j) \leq f[\bar{u}_0] \oplus \ldots \oplus f[\bar{u}_{j-1}] = f[\bar{u}]. \dashv_{\text{claim}}$$

Suppose $\bar{u} = (x^{e_0}_{i_0}, \ldots, x^{e_m}_{i_m})$ for some $1 \leq e_t \leq n$ and $1 \leq i_t \leq k$.

Case 1: $e_s = e_t$ for some $s < t$. We will show that either $\bar{u}$ is isometric to a triangle in $(\bar{a}^l)_{l<\omega}$, or $\bar{u}$ contains a proper f-subsequence, in which case we apply the claim.

First, if $m = 2$ then $\bar{u}$ is a triangle with at least two points in $\bar{x}^{e_s} = \bar{x}^{e_t}$ and all three edges in $\mathrm{dom}(f)$. It follows from the definition of $\mathrm{dom}(f)$ that $\bar{u}$ is isometric to a triangle in $(\bar{a}^l)_{l<\omega}$. Therefore $f(u_0, u_m) \leq f[\bar{u}]$ by the triangle inequality. So we assume $m > 2$. In the rest of the cases, we find a proper f-subsequence of $\bar{u}$.

Suppose $s = 0$ and $t = m$. Then $\bar{v} = (u_0, u_1, u_m)$ is a proper subsequence of $\bar{u}$, since $m > 2$. Moreover, since $e_0 = e_m$, $\bar{v}$ is an f-sequence by definition of $\mathrm{dom}(f)$. So we may assume that $s = 0$ implies $t < m$.

If $s + 1 < t$ then, combined with the assumption that $s = 0$ implies $t < m$, it follows that $\bar{v} = (u_0, \ldots, u_s, u_t, \ldots, u_m)$ is a proper subsequence of $\bar{u}$. Moreover, $\bar{v}$ is an f-sequence since $e_s = e_t$ implies $(u_s, u_t) \in \mathrm{dom}(f)$. So we may assume $t = s + 1$.

If $t < m$ then $\bar{v} = (u_0, \ldots, u_s, u_{t+1}, \ldots, u_m)$ is a proper subsequence of $\bar{u}$. Moreover, $\bar{v}$ is an f-sequence since $e_s = e_t$ implies $(u_s, u_{t+1}) \in \mathrm{dom}(f)$.

Finally, if $t = m$ then $\bar{v} = (u_0, \ldots, u_{m-2}, u_m)$ is a proper subsequence of $\bar{u}$. Moreover, $\bar{v}$ is an f-sequence since $e_s = e_t$ implies $(u_{m-2}, u_m) \in \mathrm{dom}(f)$.

Case 2: $e_s \neq e_t$ for $s \neq t$. Since $\bar{u}$ is an f-sequence, it follows from the definition of $\mathrm{dom}(f)$ that $m = n - 1$ and, moreover, there is a permutation $\sigma \in \mathrm{Sym}(1, \ldots, n)$, which is some power of $(1\ 2\ \ldots\ n)$, such that either $(\sigma(e_0), \ldots, \sigma(e_m)) = (1, \ldots, n)$ or $(\sigma(e_0), \ldots, \sigma(e_m)) = (n, \ldots, 1)$. Note that, if $\sigma_* : X \longrightarrow X$ is such that $\sigma_*(x^e_i) = x^{\sigma(e)}_i$ then, for all $x, y \in X$, we have $f(x, y) = f(\sigma_*(x), \sigma_*(y))$. Therefore we may assume $(e_0, \ldots, e_m)$ is either $(1, \ldots, n)$ or $(n, \ldots, 1)$.

Next, note that $f(u_0, u_m) \leq f[\bar{u}]$ if and only if $f(u_m, u_0) \leq f[(u_m, u_{m-1}, \ldots, u_0)]$. Therefore we may assume $(e_0, \ldots, e_m) = (1, \ldots, n)$, and so $\bar{u} = (x^1_{i_0}, \ldots, x^n_{i_{n-1}})$. By

(ii), we have

$$f(x^1_{i_0}, x^n_{i_{n-1}}) = \epsilon_{i_{n-1},i_0} \leq \epsilon_{i_0,i_1} \oplus \ldots \oplus \epsilon_{i_{n-2},i_{n-1}} = f[\bar{u}],$$

as desired. □

The final tools needed for Theorem 3.6.1 are the following observations concerning transitivity properties of indiscernible sequences.

Lemma 3.6.6. *Suppose $\mathcal{I} = (\bar{a}^l)_{l<\omega}$ is an indiscernible sequence in $\mathbb{U}_{\mathcal{R}}$. Given $i, j \in \ell(\bar{a}^0)$, set $\epsilon_{i,j} = d(a^0_i, a^1_j)$. Fix $n \geq 2$ and $i_1, \ldots, i_n \in \ell(\bar{a}^0)$.*

(a) $\epsilon_{i_1,i_n} \leq \epsilon_{i_1,i_2} \oplus \epsilon_{i_2,i_3} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}$.

(b) If $i_s = i_t$ for some $1 \leq s < t \leq n$, then $\epsilon_{i_n,i_1} \leq \epsilon_{i_1,i_2} \oplus \epsilon_{i_2,i_3} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}$.

(c) If $i_s \notin \mathrm{NP}(\mathcal{I})$ for some $1 \leq s \leq n$, then $\epsilon_{i_n,i_1} \leq \epsilon_{i_1,i_2} \oplus \epsilon_{i_2,i_3} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}$.

Proof. Part (a). By indiscernibility,

$$\epsilon_{i_1,i_n} = d(a^1_{i_1}, a^n_{i_n}) \leq d(a^1_{i_1}, a^2_{i_2}) \oplus \ldots \oplus d(a^{n-1}_{i_{n-1}}, a^n_{i_n}) = \epsilon_{i_1,i_2} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}.$$

Part (b). First, if $s = 1$ then, by indiscernibility and part (a), we have

$$\begin{aligned} \epsilon_{i_n,i_1} &= d(a^1_{i_n}, a^2_{i_1}) \\ &\leq d(a^1_{i_n}, a^0_{i_1}) \oplus d(a^0_{i_1}, a^2_{i_1}) \\ &= \epsilon_{i_1,i_n} \oplus \epsilon_{i_1,i_1} \\ &= \epsilon_{i_1,i_t} \oplus \epsilon_{i_t,i_n} \\ &\leq \epsilon_{i_1,i_2} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}. \end{aligned}$$

Similarly, if $s = n$ then, by indiscernibility and part (a), we have

$$\begin{aligned} \epsilon_{i_n,i_1} &= d(a^0_{i_n}, a^1_{i_1}) \\ &\leq d(a^0_{i_n}, a^2_{i_n}) \oplus d(a^2_{i_n}, a^1_{i_1}) \\ &= \epsilon_{i_n,i_n} \oplus \epsilon_{i_1,i_n} \\ &= \epsilon_{i_1,i_t} \oplus \epsilon_{i_t,i_n} \\ &\leq \epsilon_{i_1,i_2} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}. \end{aligned}$$

Finally, if $1 < s < t < n$ then, by indiscernibility and part (a), we have

$$\begin{aligned} \epsilon_{i_n,i_0} &= d(a^1_{i_n}, a^2_{i_0}) \\ &\leq d(a^1_{i_n}, a^0_{i_s}) + d(a^0_{i_s}, a^3_{i_s}) + d(a^3_{i_s}, a^2_{i_0}) \\ &= \epsilon_{i_s,i_n} + \epsilon_{i_s,i_s} + \epsilon_{i_0,i_s} \\ &= \epsilon_{i_0,i_s} + \epsilon_{i_s,i_t} + \epsilon_{i_t,i_n} \\ &\leq \epsilon_{i_0,i_1} + \epsilon_{i_1,i_2} + \ldots + \epsilon_{i_{n-1},i_n}. \end{aligned}$$

Part (c). First, if $i_s \notin \mathrm{NP}(\mathcal{I})$ then $a^0_{i_s} = a^2_{i_s}$. Therefore, for any $j \in \ell(\bar{a}^0)$, we have

$$\epsilon_{i_s,j} = d(a^0_{i_s}, a^1_j) = d(a^2_{i_s}, a^1_j) = d(a^0_j, a^1_{i_s}) = \epsilon_{j,i_s}.$$

So if $s = 1$ or $s = n$ then the result follows immediately from part (a). Suppose $1 < s < n$. Then, using part (a), we have

$$\epsilon_{i_n,i_1} \leq \epsilon_{i_n,i_s} \oplus \epsilon_{i_s,i_1} = \epsilon_{i_1,i_s} \oplus \epsilon_{i_s,i_n} = \epsilon_{i_1,i_2} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}. \qquad \square$$

We can now prove the main result of this section.

Proof of Theorem 3.6.1. Let $\mathcal{R}$ be a Urysohn monoid and fix an indiscernible sequence $\mathcal{I}$ in $\mathbb{U}_{\mathcal{R}}$, with $|\mathrm{NP}(\mathcal{I})| = n < \omega$. We want to show $\mathcal{I}$ is $(n+1)$-cyclic. We may assume $n \geq 1$ and so, by Lemma 3.6.5, it suffices to fix $i_1, \ldots, i_{n+1} \in \ell(\bar{a}^0)$ and show $\epsilon_{i_{n+1},i_1} \leq \epsilon_{i_1,i_2} \oplus \epsilon_{i_2,i_3} \oplus \ldots \oplus \epsilon_{i_n,i_{n+1}}$. By Lemma 3.6.6$(c)$, we may assume $i_s \in \mathrm{NP}(\mathcal{I})$ for all $1 \leq s \leq n+1$. Therefore, there are $1 \leq s < t \leq n+1$ such that $i_s = i_t$, and so the result follows from Lemma 3.6.6(b). $\square$

3.7 Strong Order Rank

Suppose $\mathcal{R}$ is a Urysohn monoid. Summarizing our previous results, we have shown that $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ never has FSOP and, moreover, stability and simplicity are both possible for $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$. In this section, we address the complexity between simplicity and FSOP. For general first-order theories, this complexity is stratified by Shelah's SOP_n-hierarchy, which we have formulated as *strong order rank*, denoted $\mathrm{SO}(T)$ (see Definition 1.4.6).

3.7.1 Calculating the rank

First, we observe that the results of Sections 3.5.1 and 3.5.2 can be restated as follows:

(i) $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is stable if and only if $r \oplus s = s$ for all $r \leq s$ in R.

(ii) $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is simple if and only if $r \oplus s \oplus t = s \oplus t$ for all $r \leq s \leq t$ in R.

This motivates the following definition.

Definition 3.7.1. Let $\mathcal{R}$ be a distance monoid. The **archimedean complexity of** $\mathcal{R}$, denoted $\mathrm{arch}(\mathcal{R})$, is the minimum $n < \omega$ such that, for all $r_0, r_1, \ldots, r_n \in R$, if $r_0 \leq r_1 \leq \ldots \leq r_n$ then

$$r_0 \oplus r_1 \oplus \ldots \oplus r_n = r_1 \oplus \ldots \oplus r_n.$$

If no such n exists, set $\mathrm{arch}(\mathcal{R}) = \omega$.

Let us first discuss the algebraic significance of archimedean complexity. Roughly speaking, $\operatorname{arch}(\mathcal{R})$ measures when, if ever, repeated addition in $\mathcal{R}$ begins to stabilize. In the case that $\mathcal{R}$ is archimedean, this can be stated in a more precise fashion.

Definition 3.7.2. A distance monoid $\mathcal{R}$ is **archimedean** if, for all $r, s \in R^{>0}$, there is some $n > 0$ such that $s \leq nr$.

If $\mathcal{R}$ is an archimedean distance monoid, then $\operatorname{arch}(\mathcal{R})$ provides a uniform upper bound for the number of times any given positive element of $\mathcal{R}$ must be added to itself in order to surpass any other element of $\mathcal{R}$. In other words, $\operatorname{arch}(\mathcal{R}) \geq n$ if and only if $s \leq nr$ for all $r, s \in R^{>0}$. Therefore, if $\mathcal{R}$ is an archimedean distance monoid of finite archimedean complexity $n < \omega$ then, for any $r \in R^{>0}$, nr is the maximal element of $\mathcal{R}$. See Section 3.7.4 for details.

We have shown that, for general Urysohn monoids $\mathcal{R}$, $\operatorname{Th}(\mathcal{U}_\mathcal{R})$ is stable if and only if $\operatorname{arch}(\mathcal{R}) \leq 1$ and $\operatorname{Th}(\mathcal{U}_\mathcal{R})$ is simple if and only if $\operatorname{arch}(\mathcal{R}) \leq 2$. Moreover, for a general theory T, stability is equivalent to $\operatorname{SO}(T) \leq 1$. Altogether, we have $\operatorname{SO}(\operatorname{Th}(\mathcal{U}_\mathcal{R})) \leq 1$ if and only if $\operatorname{arch}(\mathcal{R}) \leq 1$. The goal of this section is to extend this result, and show $\operatorname{SO}(\operatorname{Th}(\mathcal{U}_\mathcal{R})) = \operatorname{arch}(\mathcal{R})$ for any Urysohn monoid $\mathcal{R}$. We first note that there is no loss in considering the archimedean complexity of $\mathcal{R}^*$ over $\mathcal{R}$.

Proposition 3.7.3. *If $\mathcal{R}$ is a distance monoid then* $\operatorname{arch}(\mathcal{R}^*) = \operatorname{arch}(\mathcal{R})$.

Proof. We clearly have $\operatorname{arch}(\mathcal{R}^*) \geq \operatorname{arch}(\mathcal{R})$, so it suffices to fix $n < \omega$, assume $\operatorname{arch}(\mathcal{R}^*) > n$, and show $\operatorname{arch}(\mathcal{R}) > n$. If $\operatorname{arch}(\mathcal{R}^*) > n$ then there are $\alpha_0, \alpha_1, \ldots, \alpha_n \in R^*$ such that $\alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n$ and $\alpha_1 \oplus \ldots \oplus \alpha_n < \alpha_0 \oplus \alpha_1 \oplus \ldots \oplus \alpha_n$. By Proposition 3.2.1, we may fix $r_1, \ldots, r_n \in R$ such that $\alpha_i \leq r_i$ and

$$r_1 \oplus \ldots \oplus r_n < \alpha_0 \oplus \alpha_1 \oplus \ldots \oplus \alpha_n \leq \alpha_0 \oplus r_1 \oplus \ldots \oplus r_n.$$

Without loss of generality, we may assume $r_1 \leq \ldots \leq r_n$. Then, setting $r_0 = r_1$, we have $r_0 \leq r_1 \leq \ldots \leq r_n$ and $r_1 \oplus \ldots \oplus r_n < r_0 \oplus r_1 \oplus \ldots \oplus r_n$, as desired. ☐

Toward the proof of the main result of this section (Theorem 3.7.8), we begin by refining previous results on cyclic indiscernible sequences. Throughout the section, we fix a Urysohn monoid $\mathcal{R}$.

Definition 3.7.4. Fix $n \geq 2$ and $\alpha_1, \ldots, \alpha_n \in R^*$. Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

1. $\bar{\alpha}$ is **diagonally indiscernible** if there is an indiscernible sequence $(\bar{a}^l)_{l<\omega}$ in $\mathbb{U}_\mathcal{R}$, with $\ell(\bar{a}^0) = n$, such that $d(a^0_n, a^1_1) = \alpha_n$ and, for all $1 \leq t < n$, $d(a^0_t, a^1_{t+1}) = \alpha_t$ (see Figure 6).

2. $\bar{\alpha}$ is **transitive** if $\alpha_n \leq \alpha_1 \oplus \ldots \oplus \alpha_{n-1}$.

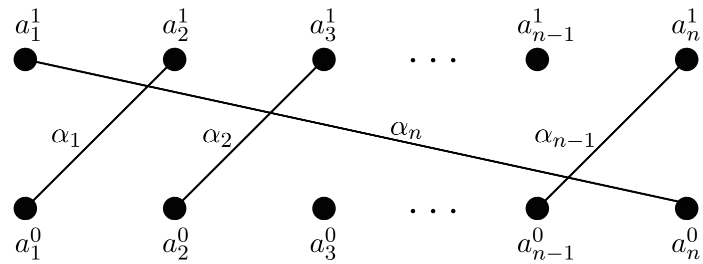

Figure 6: A diagonally indiscernible sequence $(\alpha_1, \ldots, \alpha_n)$.

Proposition 3.7.5. *Given $n > 1$, the following are equivalent.*

(i) $\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) < n$.

(ii) *Every infinite indiscernible sequence in $\mathbb{U}_{\mathcal{R}}$ is n-cyclic.*

(iii) *Every diagonally indiscernible sequence of length n in $\mathcal{R}^*$ is transitive.*

Proof. Recall that (i) and (ii) are equivalent in any theory by Proposition 1.4.8(a). Therefore, we only need to show (ii) and (iii) are equivalent.

(ii) $\Rightarrow$ (iii): Fix a diagonally indiscernible sequence $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ in $\mathcal{R}^*$, witnessed by an indiscernible sequence $(\bar{a}^l)_{l<\omega}$ in $\mathbb{U}_{\mathcal{R}}$. By (ii), $(\bar{a}^l)_{l<\omega}$ is n-cyclic and so there is some $(\bar{c}^1, \ldots, \bar{c}^n)$ such that $(\bar{c}^t, \bar{c}^{t+1}) \equiv (\bar{a}^0, \bar{a}^1) \equiv (\bar{c}^n, \bar{c}^1)$ for all $1 \leq t < n$. In particular,

$$\alpha_n = d(a_n^0, a_1^1) = d(c_1^1, c_n^n) \leq d(c_1^1, c_2^2) \oplus \ldots \oplus d(c_{n-1}^{n-1}, c_n^n) = \alpha_1 \oplus \ldots \oplus \alpha_{n-1}.$$

Therefore $\bar{\alpha}$ is transitive.

(iii) $\Rightarrow$ (ii): Suppose there is an indiscernible sequence $\mathcal{I} = (\bar{a}^l)_{l<\omega}$ in $\mathbb{U}_{\mathcal{R}}$, which is not n-cyclic. Given $i, j \in \ell(\bar{a}^0)$, let $\epsilon_{i,j} = d(a_i^0, a_j^1)$. By Lemma 3.6.5, there are $i_1, \ldots, i_n \in \ell(\bar{a}^0)$ such that $\epsilon_{i_n,i_1} > \epsilon_{i_1,i_2} \oplus \ldots \oplus \epsilon_{i_{n-1},i_n}$. By Lemma 3.6.6($b$), it follows that the map $t \mapsto i_t$ is injective. Given $l < \omega$, define $\bar{b}^l = (a_{i_1}^l, \ldots, a_{i_n}^l)$. Then $\ell(\bar{b}^0) = n$ and $\mathcal{J} = (\bar{b}^l)_{l<\omega}$ is an indiscernible sequence. Let $\alpha_n = \epsilon_{i_n,i_1}$ and, given $1 \leq t < n$, let $\alpha_t = \epsilon_{i_t,i_{t+1}}$. Then, for any $t < n$, we have $d(b_t^0, b_{t+1}^1) = d(a_{i_t}^0, a_{i_{t+1}}^1) = \alpha_t$. Moreover, $d(b_n^0, b_1^1) = d(a_{i_n}^0, a_{i_1}^1) = \alpha_n$. Therefore $\mathcal{J}$ witnesses that $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is a non-transitive diagonally indiscernible sequence. □

Next, we prove two technical lemmas.

Lemma 3.7.6. *Suppose $n > 1$ and $(\alpha_1, \ldots, \alpha_n)$ is a diagonally indiscernible sequence in $\mathcal{R}^*$. Then, for any $1 \leq i < n$, we have $\alpha_n \leq \alpha_1 \oplus \ldots \oplus \alpha_{n-1} \oplus 2\alpha_i$.*

Proof. Let $(\bar{a}^l)_{l<\omega}$ be an indiscernible sequence in $\mathbb{U}_{\mathcal{R}}$, which witnesses that $(\alpha_1, \ldots, \alpha_n)$ is diagonally indiscernible. Given $1 \leq i, j \leq n$, let $\epsilon_{i,j} = d(a_i^0, a_j^1)$. Note that, if $1 \leq i < n$ then $\epsilon_{i,i+1} = \alpha_i$ and, moreover,

$$\epsilon_{i+1,i+1} = d(a_{i+1}^1, a_{i+1}^2) \leq d(a_{i+1}^1, a_i^0) \oplus d(a_i^0, a_{i+1}^2) = 2\alpha_i.$$

If $i < n-1$ then, using Lemma 3.6.6(a), we have

$$\begin{aligned}\alpha_n &= d(a_1^2, a_n^1)\\ &\leq d(a_1^2, a_{i+1}^3) \oplus d(a_{i+1}^3, a_{i+1}^0) \oplus d(a_{i+1}^0, a_n^1)\\ &= \epsilon_{1,i+1} \oplus \epsilon_{i+1,n} \oplus \epsilon_{i+1,i+1}\\ &\leq \alpha_1 \oplus \ldots \oplus \alpha_{n-1} \oplus 2\alpha_i\end{aligned}$$

On the other hand, if $i = n-1$ then, using Lemma 3.6.6(a), we have

$$\alpha_n = d(a_1^1, a_n^0) \leq d(a_1^1, a_n^2) \oplus d(a_n^2, a_n^0) = \epsilon_{1,n} \oplus \epsilon_{n,n} \leq \alpha_1 \oplus \ldots \oplus \alpha_{n-1} \oplus 2\alpha_{n-1}. \quad \square$$

Lemma 3.7.7. *Fix $n \geq 2$ and $\alpha_1, \ldots, \alpha_n \in R^*$ such that $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$. Then*

$$(\alpha_2, \ldots, \alpha_n, \alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_n)$$

is a diagonally indiscernible sequence.

Proof. Define the sequence $(\bar{a}^l)_{l<\omega}$, such that $\ell(\bar{a}^0) = n$ and, given $k \leq l < \omega$ and $1 \leq i, j \leq n$,

$$d(a_i^k, a_j^l) = \begin{cases} \alpha_j \oplus \alpha_{j+1} \oplus \ldots \oplus \alpha_i & \text{if } k < l \text{ and } i \geq j, \text{ or } k = l \text{ and } i > j\\ \alpha_{i+1} \oplus \alpha_{i+2} \oplus \ldots \oplus \alpha_j & \text{if } k < l \text{ and } i < j.\end{cases}$$

Given $1 \leq i < n$, we have $d(a_i^0, a_{i+1}^1) = \alpha_i$ and $d(a_n^0, a_1^1) = \alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_n$. Therefore, it suffices to verify this sequence satisfies the triangle inequality. For this, given $1 \leq i \leq j \leq n$, define

$$s(i,j) = \begin{cases} 0 & \text{if } i = j\\ \alpha_{i+1} \oplus \ldots \oplus \alpha_j & \text{if } i < j.\end{cases}$$

Fix distinct a_i^l, a_j^m, a_k^r, with $l \leq m \leq r$ and $1 \leq i, j, k \leq n$. Let $d_{i,j} = d(a_i^l, a_j^m)$, $d_{j,k} = d(a_j^m, a_k^r)$, and $d_{i,k} = d(a_i^l, a_k^r)$. We need to show:

(a) $d_{i,k} \leq d_{i,j} \oplus d_{j,k}$, ($b$) $d_{i,j} \leq d_{i,k} \oplus d_{j,k}$, ($c$) $d_{j,k} \leq d_{i,j} \oplus d_{i,k}$.

Case 1: $i \geq j \geq k$. Then $d_{i,j} = s(j-1, i)$, $d_{i,k} = s(k-1, i)$, and $d_{j,k} = s(k-1, j)$.

(a) Use $s(k-1, i) = s(k-1, j-1) \oplus s(j-1, i)$.

(b) Use $s(j-1, i) \leq s(k-1, i)$.

(c) Use $s(k-1, j) \leq s(k-1, i)$.

Case 2: $i \geq k > j$. If $m = r$ then Case 1 applies. So we may assume $l \leq m < r$. Then $d_{i,j} = s(j-1, i)$, $d_{i,k} = s(k-1, i)$, and $d_{j,k} = s(j, k)$.

(a) Use $s(k-1,i) \leq s(j-1,i)$.

(b) Use $s(j-1,i) = \alpha_j \oplus s(j,i) \leq \alpha_{j+1} \oplus s(j,i) \leq s(j,k) \oplus s(k-1,i)$.

(c) Use $s(j,k) \leq s(j-1,i)$.

Case 3: $j > i \geq k$. If $l = m = r$ then Case 1 applies, and if $l = m < r$ then Case 2 applies. So we may assume $l < m \leq r$. Then $d_{i,j} = s(i,j)$, $d_{i,k} = s(k-1,i)$, and $d_{j,k} = s(k-1,j)$.

(a) Use $s(k-1,i) \leq s(k-1,j)$.

(b) Use $s(i,j) \leq s(k-1,j)$.

(c) Use $s(k-1,j) = s(k-1,i) \oplus s(i,j)$.

Case 4: $k > i \geq j$. If $l = m = r$ then Case 1 applies, and if $l < m = r$ then Case 3 applies. So we may assume $l \leq m < r$. Then $d_{i,j} = s(j-1,i)$, $d_{i,k} = s(i,k)$, and $d_{j,k} = s(j,k)$.

(a) Use $s(i,k) \leq s(j,k)$.

(b) Use $s(j-1,i) = \alpha_j \oplus s(j,i) \leq \alpha_{i+1} \oplus s(j,i) \leq s(i,k) \oplus s(j,k)$.

(c) Use $s(j,k) = s(j,i) \oplus s(i,k)$.

Case 5: $j \geq k > i$. If $l = m = r$ then Case 1 applies, and if $l = m < r$ then Case 2 applies. So we may assume $l < m \leq r$. Then $d_{i,j} = s(i,j)$, $d_{i,k} = s(i,k)$, and $d_{j,k} = s(k-1,j)$.

(a) Use $s(i,k) \leq s(i,j)$.

(b) Use $s(i,j) = s(i,k) \oplus s(k,j)$.

(c) Use $s(k-1,j) \leq s(i,j)$.

Case 6: $k > j > i$. If $l = m = r$ then Case 1 applies, if $l = m < r$ then Case 4 applies, and if $l < m = r$ then Case 5 applies. So we may assume $l < m < r$. Then $d_{i,j} = s(i,j)$, $d_{i,k} = s(i,k)$, and $d_{j,k} = s(j,k)$.

(a) Use $s(i,k) = s(i,j) \oplus s(j,k)$.

(b) Use $s(i,j) \leq s(i,k)$.

(c) Use $s(j,k) \leq s(i,k)$. □

We now have all of the pieces necessary to prove the main result of this section.

Theorem 3.7.8. *If* $\mathcal{R}$ *is a Urysohn monoid then* $\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) = \mathrm{arch}(\mathcal{R})$.

Proof. First, note that if $\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) > n$ and $\mathrm{arch}(\mathcal{R}) > n$ for all $n < \omega$ then, by Corollary 3.6.2 and our conventions, we have $\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) = \omega = \mathrm{arch}(\mathcal{R})$.

Therefore, it suffices to fix $n \geq 1$ and show $\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) \geq n$ if and only if $\mathrm{arch}(\mathcal{R}) \geq n$. Note that $\mathrm{arch}(\mathcal{R}) = 0$ if and only if $\mathcal{R}$ is the trivial monoid, in which case $\mathbb{U}_{\mathcal{R}}$ is a single point. Conversely, if $\mathcal{R}$ is nontrivial then $\mathbb{U}_{\mathcal{R}}$ is clearly infinite. So $\mathrm{arch}(\mathcal{R}) < 1$ if and only if $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ has finite models, which is equivalent to $\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) < 1$. So we may assume $n \geq 2$.

Suppose $\mathrm{arch}(\mathcal{R}) \geq n$. Then there are $r_1, \ldots, r_n \in R$ such that $r_1 \leq r_2 \leq \ldots \leq r_n$ and $r_2 \oplus \ldots \oplus r_n < r_1 \oplus r_2 \oplus \ldots \oplus r_n$. By Lemma 3.7.7, it follows that $(r_2, \ldots, r_n, r_1 \oplus \ldots \oplus r_n)$ is a non-transitive diagonally indiscernible sequence of length n. Therefore $\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) \geq n$ by Proposition 3.7.5.

Finally, suppose $\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) \geq n$. By Proposition 3.7.5 there is a non-transitive diagonally indiscernible sequence $(\alpha_1, \ldots, \alpha_n)$ in $\mathcal{R}^*$. Let $\beta_1, \ldots, \beta_{n-1}$ be an enumeration of $\alpha_1, \ldots, \alpha_{n-1}$, with $\beta_1 \leq \ldots \leq \beta_{n-1}$. Then, using Lemma 3.7.6, we have

$$\beta_1 \oplus \ldots \oplus \beta_{n-1} < \alpha_n \leq 2\beta_1 \oplus \beta_1 \oplus \ldots \oplus \beta_{n-1},$$

which implies $\beta_1 \oplus \ldots \oplus \beta_{n-1} < \beta_1 \oplus \beta_1 \oplus \ldots \oplus \beta_{n-1}$, and so $\mathrm{arch}(\mathcal{R}^*) \geq n$. By Proposition 3.7.3, $\mathrm{arch}(\mathcal{R}) \geq n$. □

Note that, as archimedean complexity is clearly a first-order property of distance monoids, we have that, for all $n < \omega$, "$\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) = n$" is a finitely axiomatizable property of **RUS**. Moreover, it follows that "$\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) = \omega$" is an axiomatizable property of **RUS**.

3.7.2 Further Remarks on Simplicity

Recall that Section 3.5.2 resulted in the equivalence: $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is simple if and only if $\mathrm{arch}(\mathcal{R}) \leq 2$. Therefore, combined with Theorem 3.7.8, we have the following corollary.

Corollary 3.7.9. *If $\mathcal{R}$ is a Urysohn monoid, and $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ is not simple, then $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ has SOP_3.*

In general, non-simple theories without SOP_3 are scarce. Indeed, there are essentially only three known examples, which are all described in [50]. The tendency for non-simple theories to have SOP_3 is also, due to a result of Evans and Wong [31], a phenomenon shared by certain Hrushovski constructions with free amalgamation.

A similar phenomenon in model theoretic dividing lines is related to the question of non-simple theories, which have neither TP_2 nor the strict order property. In particular, there are no known examples of such theories.[1] Since we have shown

[1] An example of a non-simple, NTP_2 theory, without the strict order property, is proposed in Exercise III.7.12 of [82]. However, as stated, the example does not define a complete theory. To my knowledge, all attempts at completing the theory in this example have resulted in either a simple theory or a TP_2 theory.

$\mathrm{Th}(\mathcal{U}_\mathcal{R})$ never has the strict order property, it is worth proving that any non-simple $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ has TP_2.

Theorem 3.7.10. *If $\mathcal{R}$ is a Urysohn monoid, and $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is not simple, then $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ has TP_2.*

Proof. Suppose $\mathrm{Th}(\mathcal{U}_\mathcal{R})$ is not simple. By Theorem 3.5.7, we may fix $r, s \in R$ such that $r \leq s$ and $r \oplus s < r \oplus r \oplus s$. Let $A = (a_1^{i,j}, a_2^{i,j})_{i,j<\omega}$. We define d on $A \times A$ such that

$$d(a_m^{i,j}, a_n^{k,l}) = \begin{cases} r & \text{if } m = n = 1 \text{ and } (i,j) \neq (k,l) \\ s & \text{if } m = n = 2 \text{ and } (i,j) \neq (k,l) \\ r \oplus s & \text{if } m \neq n, \text{ and } i \neq k \text{ or } j = l \\ r \oplus r \oplus s & \text{if } m \neq n,\ i = k, \text{ and } j \neq l. \end{cases}$$

To verify the triangle inequality for d, fix a non-degenerate triangle $\{a_m^{i,j}, a_n^{k,l}, a_p^{g,h}\}$ in A. Let $\alpha = d(a_m^{i,j}, a_n^{k,l})$, $\beta = d(a_m^{i,j}, a_p^{g,h})$, and $\gamma = d(a_n^{k,l}, a_p^{g,h})$. Without loss of generality, we may assume $m = n$. If $m = p$ then $\alpha = \beta = \gamma$ and so the triangle inequality holds. If $m \neq p$ then $\alpha \in \{r, s\}$ and $\beta, \gamma \in \{r \oplus s, 2r \oplus s\}$, so the triangle inequality holds.

We may assume $A \subset \mathbb{U}_\mathcal{R}$. Define the formula

$$\varphi(x, y_1, y_2) := d(x, y_1) \leq r \wedge d(x, y_2) \leq s.$$

We show A and $\varphi(x, y_1, y_2)$ witness TP_2 for $\mathrm{Th}(\mathcal{U}_\mathcal{R})$.

Fix a function $\sigma : \omega \longrightarrow \omega$ and, given $n < \omega$ and $i \in \{1, 2\}$, set $b_i^n = a_i^{n,\sigma(n)}$. Let $B = (b_1^n, b_2^n)_{n<\omega}$. To show $\{\varphi(x, b_1^n, b_2^n) : n < \omega\}$ is consistent, it suffices to show that the function $f : B \longrightarrow \{r, s\}$, such that $f(b_1^n) = r$ and $f(b_2^n) = s$, is an $\mathcal{R}^*$-Katětov map on B. In other words, we must verify the inequalities $|f(u) \ominus f(v)| \leq d(u, v) \leq f(u) \oplus f(v)$ for all $u, v \in B$. For this, we have:

- for all $n < \omega$, $|f(b_1^n) \ominus f(b_2^n)| \leq s$, $f(b_1^n) \oplus f(b_2^n) = r \oplus s$ and $d(b_1^n, b_2^n) = r \oplus s$;
- for all $m < n < \omega$, $|f(b_1^m) \ominus f(b_1^n)| = 0$, $f(b_1^m) \oplus f(b_1^n) = 2r$, and $d(b_1^m, b_1^n) = r$;
- for all $m < n < \omega$, $|f(b_2^m) \ominus f(b_2^n)| = 0$, $f(b_2^m) \oplus f(b_2^n) = 2s$, and $d(b_2^m, b_2^n) = s$;
- for all distinct $m, n < \omega$, $|f(b_1^m) \ominus f(b_2^n)| \leq s$, $f(b_1^m) \oplus f(b_2^n) = r \oplus_S s$, and $d(b_1^m, b_2^n) = s$.

Next, we fix $n < \omega$ and $i < j < \omega$ and show $\varphi(x, a_1^{n,i}, a_2^{n,i}) \wedge \varphi(x, a_1^{n,j}, a_2^{n,j})$ is inconsistent. Indeed, if c realizes this formula then we have

$$d(c, a_2^{n,i}) \oplus d(c, a_1^{n,j}) \leq r \oplus s < r \oplus r \oplus s = d(a_2^{n,i}, a_1^{n,j}). \qquad \square$$

3.7.3 Forking for Formulas

In general classification theory, many important results have been motivated by the question of when forking and dividing are the same. Our focus in this chapter has been on forking for complete types. We have not addressed the more subtle question of when forking and dividing coincide for formulas, which, for example, is the case for general simple theories (see [47]). Altogether, concerning Urysohn monoids $\mathcal{R}$, we know that forking and dividing always coincide for complete types in $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ and, moreover, if $\mathrm{arch}(\mathcal{R}) \leq 2$ then we also have equivalence at the level of formulas.

Outside of simple theories, there are few general tools concerning the equivalence forking and dividing (even for complete types). One notable result is the following theorem of Chernikov and Kaplan [20].

Theorem 3.7.11. *Suppose T is a complete first-order theory. If T is* NTP_2 *and* $\mathop{\smile\!\!\!\!\!\;|}^f$ *satisfies existence*[2], *then forking and dividing are the same for formulas in T.*

The existence axiom for a ternary relation is quite weak. Concerning $\mathop{\smile\!\!\!\!\!\;|}^f$, the failure of existence is generally considered to be very bad behavior. There are few known examples of such theories, and each one exploits some kind of dense ordering on a circle (see [20], [86, Exercise 7.1.6]). Therefore, the previous theorem is quite powerful when applied to the class of NTP_2 theories.

Regarding generalized Urysohn spaces of Urysohn monoids, we have shown NTP_2 coincides with simplicity. Therefore, Chernikov and Kaplan's results provides no further information on the equivalence of forking and dividing for formulas. Altogether, we have the following question.

Question 3.7.12. Suppose $\mathcal{R}$ is a Urysohn monoid. Are forking and dividing the same for formulas in $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$?

A general approach to this question could be to expand the reach of Chernikov and Kaplan's theorem beyond the realm of NTP_2 theories. For example, one might ask if a similar theorem could be obtained for NSOP_n theories, given a fixed $n \geq 3$. However, previous work of the author shows such a theorem is impossible for $n > 3$. In particular, fix $m \leq 3$ and let T_m be the complete theory of the generic K_m-free graph, which is obtained as the Fraïssé limit of the class of finite K_m-free graphs. These graphs were first defined by Henson in [36], and are sometimes referred to as *Henson graphs*. In [22], the author proves the following result.

Theorem 3.7.13 (Conant). *Given a fixed $m \geq 3$, forking and dividing are the same for complete types in T_m. However, forking and dividing are not the same for formulas.*

It follows from the definition of dividing that $\mathop{\smile\!\!\!\!\!\;|}^d$ satisfies existence in any theory. Therefore, if $\mathop{\smile\!\!\!\!\!\;|}^d$ and $\mathop{\smile\!\!\!\!\!\;|}^f$ coincide in some theory T, then $\mathop{\smile\!\!\!\!\!\;|}^f$ satisfies existence.

[2]A ternary relation $\mathop{\smile\!\!\!\!\!\;|}$ satisfies *existence* if $A \mathop{\smile\!\!\!\!\!\;|}_C C$ for all $A, C \subset \mathbb{M}$.

Moreover, T_m is SOP_3 and NSOP_4 for any $m \geq 3$ (see [83]). Therefore, T_m ruins the possibility an analogous version of Theorem 3.7.11 for NSOP_4 theories.

Question 3.7.14. Suppose $\mathop{\smile\!\!\!\!\!\;\mid}^f$ satisfies existence in an NSOP_3 theory T. Are forking and dividing the same for formulas?

Returning again to generalized Urysohn spaces, we have also shown NSOP_3 coincides with simplicity for $\mathrm{Th}(\mathcal{U}_\mathcal{R})$, and so a positive answer to the previous question would still not help in answering Question 3.7.12.

At this point, it is worth emphasizing the connection between metric spaces and Henson graphs. In particular, let $\mathcal{H}_m$ denote the countable model of T_m. When equipped with the path metric, $\mathcal{H}_3$ is an $\mathcal{R}_2$-metric space. Moreover, $\mathcal{H}_3$ is the Fraïssé limit of the class of finite $\mathcal{R}_2$-metric spaces omitting triangles of perimeter 3. This correspondence can be generalized to metric spaces omitting triangles of odd perimeter. See Section 4.3 for a thorough analysis.

Concerning the main question of forking and dividing for formulas, an alternate possibility is that the equivalence of forking and dividing for complete types is simply a more reasonable behavior to hope for in general. In particular, we ask the following question.

Question 3.7.15. Suppose $\mathop{\smile\!\!\!\!\!\;\mid}^f$ satisfies existence in T. Is it true that $\mathop{\smile\!\!\!\!\!\;\mid}^d$ and $\mathop{\smile\!\!\!\!\!\;\mid}^f$ coincide?

In the case of Urysohn spaces, the proof that $\mathop{\smile\!\!\!\!\!\;\mid}^f$ and $\mathop{\smile\!\!\!\!\!\;\mid}^d$ coincide relies on tools reminiscent of free amalgamation of metric spaces. A similar behavior occurs in [22], in which the author uses free amalgamation of graphs to prove $\mathop{\smile\!\!\!\!\!\;\mid}^f$ and $\mathop{\smile\!\!\!\!\!\;\mid}^d$ coincide for T_m. As free amalgamation of metric spaces and graphs are each examples of stationary independence relations, these observations motivate the following question.

Question 3.7.16. Suppose T is a complete theory with a stationary independence relation. Is it true that $\mathop{\smile\!\!\!\!\!\;\mid}^d$ and $\mathop{\smile\!\!\!\!\!\;\mid}^f$ coincide?

It is worth mentioning that a positive answer to Question 3.7.15 would imply a positive answer to Question 3.7.16. To see this, note that if $\mathop{\smile\!\!\!\!\!\;\mid}$ is a stationary independence relation for T, then the existence axiom for $\mathop{\smile\!\!\!\!\!\;\mid}$ (which follows from full existence and invariance) implies the existence axiom for $\mathop{\smile\!\!\!\!\!\;\mid}^f$ by Proposition 1.3.11.

3.7.4 Examples

In this section, we give tests for calculating the strong order rank of $\mathrm{Th}(\mathcal{U}_\mathcal{R})$, when $\mathcal{R}$ is a Urysohn monoid. We also simplify the calculation in the case when $\mathcal{R}$ is archimedean, and give conditions under which the isomorphism type of a finite distance monoid is entirely determined by cardinality and archimedean complexity.

Definition 3.7.17. Let $\mathcal{R}$ be a distance monoid.

1. Given $\alpha, \beta \in R^*$ define $\lceil \frac{\alpha}{\beta} \rceil = \inf\{n < \omega : \alpha \leq n\beta\}$, where, by convention, we let $\inf \emptyset = \omega$.

2. Given $t \in R$, define $[t]_{\mathcal{R}} = \{x \in R : x \sim_{\mathcal{R}} t\}$ (see Definition 3.5.4). Define

$$\operatorname{arch}_{\mathcal{R}}(t) = \sup\left\{\lceil \tfrac{r}{s} \rceil : r, s \in [t]_{\mathcal{R}}\right\},$$

where, by convention, we let $\sup \mathbb{N} = \omega$.

Proposition 3.7.18. *Suppose $\mathcal{R}$ is a distance monoid. Fix $t \in R$.*

(a) $[t]_{\mathcal{R}}$ is a convex subset of R, which is closed under $\oplus$.

(b) $\operatorname{arch}_{\mathcal{R}}(t) = \left\lceil \dfrac{\sup[t]_{\mathcal{R}}}{\inf[t]_{\mathcal{R}}} \right\rceil$, where the supremum and infimum are calculated in $\mathcal{R}^*$.

Proof. Part (a). Fix $u, v, w \in R$, with $u < v < w$ and $u, v \in [t]_{\mathcal{R}}$. Then there is some $n > 0$ such that $t \leq nu$ and $w \leq nt$. In particular, $v \leq nt$ and $t \leq nv$, and so $v \sim_{\mathcal{R}} t$. Also, $u \leq u \oplus w \leq (n^2+1)u$, and so $u \oplus w \sim_{\mathcal{R}} u \sim_{\mathcal{R}} t$.

Part (b). Fix $t \in R$. We may assume $t > 0$. Fix $\alpha, \beta \in R^*$ such that $\alpha = \sup[t]_{\mathcal{R}}$ and $\beta = \inf[t]_{\mathcal{R}}$. To show $\operatorname{arch}_{\mathcal{R}}(t) \leq \lceil \frac{\alpha}{\beta} \rceil$, we fix $r, s \in [t]_{\mathcal{R}}$ and show $\lceil \frac{r}{s} \rceil \leq \lceil \frac{\alpha}{\beta} \rceil$. We may clearly assume $s \leq r$. Then, for any $n < \omega$, if $\alpha \leq n\beta$ then $r \leq \alpha \leq n\beta \leq ns$, as desired.

Suppose, toward a contradiction, that $\operatorname{arch}_{\mathcal{R}}(t) < \lceil \frac{\alpha}{\beta} \rceil$. Then there is some $n < \omega$ such that $\operatorname{arch}_{\mathcal{R}}(t) \leq n$ and $n\beta < \alpha$. By Proposition 3.2.1, we may fix $s \in R$ such that $\beta \leq s$ and $ns < \alpha$. By definition of β, Proposition 2.3.5(b), and convexity of $[t]_{\mathcal{R}}$, we have $s \in [t]_{\mathcal{R}}$. By definition of α and convexity of $[t]_{\mathcal{R}}$, there is some $r \in [t]_{\mathcal{R}}$ such that $ns < r$. Therefore $\lceil \frac{r}{s} \rceil > n$, which contradicts $\operatorname{arch}_{\mathcal{R}}(t) \leq n$. □

Using this, we obtain a more direct calculation of archimedean complexity in the case that $\mathcal{R}$ is archimedean.

Proposition 3.7.19. *Suppose $\mathcal{R}$ is a distance monoid.*

(a) $\operatorname{arch}(\mathcal{R}) \geq \max\{\operatorname{arch}_{\mathcal{R}}(t) : t \in R\}$.

(b) If $\mathcal{R}$ is archimedean then, for any $t \in R^{>0}$,

$$\operatorname{arch}(\mathcal{R}) = \operatorname{arch}_{\mathcal{R}}(t) = \left\lceil \frac{\sup R^{>0}}{\inf R^{>0}} \right\rceil.$$

Proof. Part (a). It suffices to fix $t \in R$ and $r, s \in [t]_{\mathcal{R}}$, with $s < r$, and show that, if $n < \omega$ is such that $ns < r$, then $\operatorname{arch}(\mathcal{R}) > n$. Since $r, s \in [t]_{\mathcal{R}}$, there is some $m < \omega$ such that $r \leq ms$, and so we have $ns < ms$. It follows that $ns < (n+1)s$, which gives $\operatorname{arch}(\mathcal{R}) > n$.

Part (b). Fix $t \in R^{>0}$. Since $\mathcal{R}$ is archimedean, we have $[t]_{\mathcal{R}} = R^{>0}$, and so the second equality follows from Proposition 3.7.18(b). To show the first inequality, it suffices by part (a) to show $\text{arch}(\mathcal{R}) \leq \text{arch}_{\mathcal{R}}(t)$. We may assume $\text{arch}_{\mathcal{R}}(t) = n < \omega$. In particular, for any $r, s \in [t]_{\mathcal{R}}$, we have $s \leq nr$. Therefore, for any $r_0, r_1 \ldots, r_n \in R$, with $0 < r_0 \leq r_1 \leq \ldots \leq r_n$, we have $r_1, r_0 \oplus r_1 \oplus \ldots \oplus r_n \in [t]_{\mathcal{R}}$, and so $r_0 \oplus r_1 \oplus \ldots \oplus r_n \leq nr_1 \leq r_1 \oplus \ldots \oplus r_n$, as desired. ☐

Example 3.7.20.

1. Suppose $\mathcal{R}$ is a convex monoid. Fix a countable ordered abelian group $\mathcal{G}$ and a convex subset $I \subseteq G^{>0}$, witnessing the convexity of $\mathcal{R}$. If we further assume $\mathcal{G}$ is a archimedean, then $\mathcal{R}$ will be archimedean as well. Therefore, we have
$$\text{SO}(\text{Th}(\mathcal{U}_{\mathcal{R}})) = \text{arch}(\mathcal{R}) = \left\lceil \frac{\sup I}{\inf I} \right\rceil.$$
2. Using the previous example, we can calculate the model theoretic complexity of many classical examples of Urysohn spaces. In particular, using the notation of Example 3.1.1, we have
   (i) $\text{SO}(\text{Th}(\mathcal{U}_{\mathcal{Q}})) = \text{SO}(\text{Th}(\mathcal{U}_{\mathcal{Q}_1})) = \text{SO}(\text{Th}(\mathcal{U}_{\mathcal{N}})) = \omega$;
   (ii) given $n > 0$, $\text{SO}(\text{Th}(\mathcal{U}_{\mathcal{R}_n})) = n$.

   Recall that, using acronyms, rank ω is the same as NFSOP and SOP_n for all $n \geq 3$; and rank $n \geq 3$ is the same as SOP_n and NSOP_{n+1}.
3. We give an example which shows that, in Proposition 3.7.19(a), the inequality can be strict. Consider $\mathcal{S} = (\{0, 1, 2, 5, 6, 7\}, +_S, \leq, 0)$. The reader may verify $+_S$ is associative on S. Note that 1 and 5 are representatives for the two nontrivial archimedean classes in $\mathcal{S}$, and $\text{arch}_{\mathcal{S}}(1) = 2 = \text{arch}_{\mathcal{S}}(5)$. However, $1 +_S 5 < 1 +_S 1 +_S 5$, and so $\text{arch}(\mathcal{S}) \geq 3$. In fact, a direction calculation shows $\text{arch}(\mathcal{S}) = 3$.

The last counterexample shows that, given a distance monoid $\mathcal{R}$, if $\text{arch}(\mathcal{R}) \geq n$ then we cannot always expect to have some $t \in R$ with $\text{arch}_{\mathcal{R}}(t) \geq n$. On the other hand, we do have the following property.

Proposition 3.7.21. *Suppose $\mathcal{R}$ is a distance monoid. If $n < \omega$ and $\text{arch}(\mathcal{R}) \geq n$ then there is some $t \in R^{>0}$ such that $|[t]_{\mathcal{R}}| \geq n$.*

Proof. Suppose $\text{arch}(\mathcal{R}) \geq n$. We may clearly assume $n \geq 2$. Fix $r_1, \ldots, r_n \in R$, such that $r_1 \leq \ldots \leq r_n$ and $r_2 \oplus \ldots \oplus r_n < r_1 \oplus \ldots \oplus r_n$. Given $1 \leq i \leq n$, let $s_i = r_i \oplus \ldots \oplus r_n$. Since $r_1 \leq \ldots \leq r_n$, we have $s_i \in [r_n]_{\mathcal{R}}$ for all i. We prove, by induction on i, that $s_{i+1} < s_i$. The base case $s_2 < s_1$ is given, so assume $s_{i+1} < s_i$. Suppose, for a contradiction, $s_{i+1} \leq s_{i+2}$. Then
$$s_i = r_i \oplus s_{i+1} \leq r_i \oplus s_{i+2} \leq r_{i+1} \oplus s_{i+2} = s_{i+1},$$
which contradicts the induction hypothesis. Altogether, we have $|[r_n]_{\mathcal{R}}| \geq n$. ☐

Combining this result with Corollary 3.6.2, we obtain the following numeric upper bound for the strong order rank of $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$.

Corollary 3.7.22. *If $\mathcal{R}$ is a Urysohn monoid then* $\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) \leq |R^{>0}|$.

For the final result of this section, we consider a fixed integer $n > 0$. We have shown that if $\mathcal{R}$ is a distance monoid, with $|R^{>0}| = n$, then $1 \leq \mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) \leq n$. The next result addresses the extreme cases.

Theorem 3.7.23. *Fix $n > 0$ and suppose $\mathcal{R}$ is a distance monoid, with $|R^{>0}| = n$.*

(a) $\mathrm{arch}(\mathcal{R}) = 1$ *if and only if* $\mathcal{R} \cong (\{0, 1, \ldots, n\}, \max, \leq, 0)$.

(b) $\mathrm{arch}(\mathcal{R}) = n$ *if and only if* $\mathcal{R} \cong \mathcal{R}_n = (\{0, 1, \ldots, n\}, +_n, \leq, 0)$.

Proof. Part (a). We have $\mathrm{arch}(\mathcal{R}) = 1$ if and only if $\mathcal{R}$ is ultrametric, in which case the result follows. Indeed, if $\mathcal{R}$ is ultrametric, then $\mathcal{R} \cong (S, \max, \leq, 0)$ for any linear order $(S, \leq, 0)$ with least element 0 and n nonzero elements.

Part (b). We have already observed that $\mathrm{arch}(\mathcal{R}_n) = n$, so it suffices to assume $\mathrm{arch}(\mathcal{R}) = n$ and show $\mathcal{R} \cong \mathcal{R}_n$. Since $\mathrm{arch}(\mathcal{R}) = n$, it follows from Proposition 3.7.21 the there is $t \in R^{>0}$, with $|[t]_{\mathcal{R}}| \geq n$, and so $R^{>0} = [t]_{\mathcal{R}}$. Therefore $\mathcal{R}$ is archimedean and $\mathrm{arch}_{\mathcal{R}}(t) = n$. If $r = \min R^{>0}$ and $s = \max R^{>0}$ then we must have $(n-1)r < s = nr$, and so $R^{>0} = \{r, 2r, \ldots, nr\}$. From this, we clearly have $\mathcal{R} \cong \mathcal{R}_n$. □

Remark 3.7.24. Pursuing the natural line of questioning opened by Theorem 3.7.23, we fix $1 \leq k \leq n$ and define $\mathrm{DM}(n,k)$ to be the number (modulo isomorphism) of distance monoids $\mathcal{R}$ such that $|R^{>0}| = n$ and $\mathrm{arch}(\mathcal{R}) = k$ (equivalently, $\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) = k$). In particular, Theorem 3.7.23 asserts that, for all $n > 0$, $\mathrm{DM}(n,1) = \mathrm{DM}(n,n) = 1$. On the other hand, using direct calculations and induction, one may show $\mathrm{DM}(n,k) > 1$ for all $1 < k < n$. We make the following conjectures.

(a) Given a fixed $k > 1$, the sequence $(\mathrm{DM}(n,k))_{n=k}^{\infty}$ is strictly increasing.

(b) Given a fixed $n > 2$, the sequence $(\mathrm{DM}(n,k))_{k=1}^{n}$ is (strictly) unimodal.

Using exhaustive calculation, part (b) has been confirmed for $n \leq 6$ and, moreover, the maximal value of the sequence is attained at $k = 2$. Model theoretically, this is interesting since it demonstrates the existence of many more simple unstable Urysohn spaces beyond the metrically trivial ones. Indeed, for a fixed $n \geq 2$, exactly one of the $\mathrm{DM}(n,2)$ rank 2 monoids with n nontrivial elements is metrically trivial. In Chapter 5, we will pursue this line of questioning and justify the previous remarks.

3.8 Imaginaries and Hyperimaginaries

In this section, we give some partial results concerning the question that originally motivated Casanovas and Wagner [15] to consider the space $\mathcal{U}_{\mathcal{R}_n}$, which they call the *free n^{th} root of the complete graph*. At the end of Chapter 2, we replaced $\mathcal{R}_n$ with the distance monoid $\mathcal{S}_n = (\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}, +_1, \leq, 0)$, and verified that $\text{Th}(\mathcal{U}_{\mathcal{Q}_1}) = \bigcup_{n<\omega} \text{Th}(\mathcal{U}_{\mathcal{S}_n})$. We also mentioned the main result of [15], which is that $\text{Th}(\mathcal{U}_{\mathcal{Q}_1})$ does not eliminate hyperimaginaries.

We will refine and generalize the results of [15] for arbitrary Urysohn monoids, in order to obtain necessary conditions for elimination of hyperimaginaries and weak elimination of imaginaries. For the rest of this section, we fix a nontrivial Urysohn monoid $\mathcal{R}$.

Proposition 3.8.1. $\text{Th}(\mathcal{U}_{\mathcal{R}})$ *does not have elimination of imaginaries.*

Proof. We verify $\text{Th}(\mathcal{U}_{\mathcal{R}})$ satisfies the conditions of Lemma 1.5.5. First, the fact that $\text{acl}(C) = C$ for all $C \subset \mathbb{U}_{\mathcal{R}}$ follows from quantifier elimination and disjoint amalgamation in the Fraïssé class $\mathcal{K}_{\mathcal{R}}$. Next, given $n > 0$, if we fix some $r \in R^{>0}$ then there is $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{U}_{\mathcal{R}}$ such that $d(a_i, a_j) = r$ for all $i \neq j$. In particular, $\bar{a}^f \equiv \bar{a}$ for all $f \in \text{Sym}(1, \ldots, n)$. □

In order to obtain necessary conditions for weak elimination of imaginaries and elimination of hyperimaginaries, we first characterize all 0-definable *unary* equivalence relations on $\mathbb{U}_{\mathcal{R}}$.

Definition 3.8.2. Suppose $E(x, y)$ is a 0-invariant unary equivalence relation on $\mathbb{U}_{\mathcal{R}}$. Define $\Gamma(E) \subseteq R^*$ such that $\alpha \in \Gamma(E)$ if and only if there are $a, b \in \mathbb{U}_{\mathcal{R}}$ such that $E(a, b)$ and $d(a, b) = \alpha$. Let $\alpha(E) = \sup \Gamma(E)$.

Proposition 3.8.3. *Suppose $E(x, y)$ is a 0-invariant unary equivalence relation on $\mathbb{U}_{\mathcal{R}}$.*

(a) $\Gamma(E)$ is closed downwards.

(b) If $\alpha \in \Gamma(E)$ then $2\alpha \in \Gamma(E)$.

Proof. Fix $\alpha \in \Gamma(E)$ and let $a, b \in \mathbb{U}_{\mathcal{R}}$ be such that $E(a, b)$ holds and $d(a, b) = \alpha$. To prove (a) and (b), it suffices to fix $\beta \in R^*$, with $\beta \leq 2\alpha$, and show $\beta \in \Gamma(E)$. Given such a β, there is some $b' \equiv_a b$, with $d(b, b') = \beta$. We have $E(a, b)$ and $E(a, b')$, which gives $E(b, b')$. Therefore $\beta \in \Gamma(E)$, as desired. □

Lemma 3.8.4. *Suppose $E(x, y)$ is a 0-type-definable unary equivalence relation. Then, for all $a, b \in \mathbb{U}_{\mathcal{R}}$, $E(a, b)$ holds if and only if $d(a, b) \leq \alpha(E)$.*

Proof. By definition of $\alpha(E)$, we have that $E(a,b)$ implies $d(a,b) \leq \alpha(E)$. Conversely, suppose first that $a,b \in \mathbb{U}_{\mathcal{R}}$ are such that $d(a,b) = \beta < \alpha(E)$. Then $\beta \in \Gamma(E)$ by Proposition 3.8.3(a), and so there are $a',b' \in \mathbb{U}_{\mathcal{R}}$ such that $E(a',b')$ and $d(a',b') = \beta$. Then $(a,b) \equiv (a',b')$ by quantifier elimination, so $E(a,b)$ holds. Therefore, we have left to show that $d(a,b) = \alpha(E)$ implies $E(a,b)$. By quantifier elimination, it suffices to show $\alpha(E) \in \Gamma(E)$. If $\alpha(E)$ has an immediate predecessor in R^* then this is immediate. So we may assume $\alpha(E)$ has no immediate predecessor. Then, by definition of $\alpha(E)$, the type

$$E(x,y) \cup \{d(x,y) \leq r : r \in R,\ \alpha(E) \leq r\} \cup \{d(x,y) > r : r \in R,\ r < \alpha(E)\}$$

is finitely satisfiable, and so $\alpha(E) \in \Gamma(E)$. □

Definition 3.8.5.

1. Define $\mathrm{eq}^\circ(\mathcal{R}) = \{r \in \mathrm{eq}(\mathcal{R}) : 0 < r < \sup R^*\}$.
2. Define $\mathrm{heq}(\mathcal{R}) \subseteq R^*$ such that $\alpha \in \mathrm{heq}(\mathcal{R})$ if and only if $\alpha \in \mathrm{eq}(\mathcal{R}^*)\backslash R$, $\alpha < \sup R^*$, and α is not approximated from above by elements of $\mathrm{eq}(\mathcal{R})$.

Theorem 3.8.6. *Suppose $\mathcal{R}$ is a nontrivial Urysohn monoid.*

(a) *The 0-definable unary equivalence relations on $\mathbb{U}_{\mathcal{R}}$ consist precisely of equality, the trivial relation, and $d(x,y) \leq r$ for $r \in \mathrm{eq}^\circ(\mathcal{R})$.*

(b) *If $\mathrm{eq}^\circ(\mathcal{R}) \neq \emptyset$ then $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ does not have weak elimination of imaginaries.*

(c) *If $\mathrm{heq}(\mathcal{R}) \neq \emptyset$ then $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ does not have elimination of hyperimaginaries.*

Proof. Part (a). First, if $r \in \mathrm{eq}(\mathcal{R})$ then $d(x,y) \leq r$ is an 0-definable equivalence relation. Conversely, suppose $E(x,y)$ is a 0-definable equivalence relation. By Lemma 3.8.4, $E(x,y)$ is equivalent to $d(x,y) \leq \alpha(E)$. If $\alpha(E) = 0$ then E is equality, and if $\alpha(E) = \sup R^*$ then E is trivial. Therefore, we may assume $0 < \alpha(E) < \sup R^*$. We want to show $\alpha(E) \in \mathrm{eq}(\mathcal{R})$. Since $\alpha(E) \in \Gamma(E)$, we have $2\alpha(E) \in \Gamma(E)$ by Proposition 3.8.3(b). Therefore $\alpha(E)$ is an idempotent. It remains to show $\alpha(E) \in R$. But this follows since $d(x,y) \leq \alpha(E)$ is definable.

Part (b). Suppose we have $r \in \mathrm{eq}^\circ(\mathcal{R})$. Let $E_r(x,y)$ denote the definable equivalence relation $d(x,y) \leq r$. Fix $a \in \mathbb{U}_{\mathcal{R}}$ and let $e = a_{E_n}$ and $X = [a]_{E_n}$. We fix a finite real tuple $\bar{c}$ and show $\bar{c}$ is not a weak canonical parameter for e.

Case 1: There is some $c \in \bar{c} \cap X$.

For any $b \in X$, we may fix $\sigma_b \in \mathrm{Aut}(\mathbb{U}_{\mathcal{R}})$ such that $\sigma_b(c) = b$. Then $\sigma_b \in \mathrm{Aut}(\mathbb{U}_{\mathcal{R}}/e)$, and we have shown that any element of X is in the orbit of c under $\mathrm{Aut}(\mathbb{U}_{\mathcal{R}}/e)$. Since X is infinite, it follows that $\bar{c} \notin \mathrm{acl}^{\mathrm{eq}}(e)$.

Case 2: $\bar{c} \cap X = \emptyset$.

Let $\alpha = \min\{d(a,c) : c \in \bar{c}\}$. Then $r < \alpha$, by assumption of this case. Moreover, we may find $a' \in \mathbb{U}_{\mathcal{R}}$ such that $a' \equiv_{\bar{c}} a$ and $d(a,a') = \alpha$. If $\sigma \in \mathrm{Aut}(\mathbb{U}_{\mathcal{R}}/\bar{c})$ is such that $\sigma(a) = a'$ then, as $\alpha > r$, we have $\sigma(e) \neq e$. Therefore $e \notin \mathrm{dcl}^{\mathrm{eq}}(\bar{c})$.

Part (c). Suppose we have $\alpha \in \text{heq}(\mathcal{R})$. Suppose, toward a contradiction, $\text{Th}(\mathcal{U}_\mathcal{R})$ eliminates hyperimaginaries. Fix a singleton $a \in \mathbb{U}_\mathcal{R}$. Since $d(x,y) \leq \alpha$ is a 0-type-definable equivalence relation, it follows from Proposition 1.5.3 that there is a sequence $(E_i(x,y))_{i<\lambda}$ of 0-definable unary equivalence relations such that for any $b, b' \models \text{tp}(a)$, $d(b,b') \leq \alpha$ if and only if $E_i(b,b')$ holds for all $i < \lambda$. By part (a), there are $r_i \in R$, for $i < \lambda$, such that r_i is an idempotent and $E_i(x,y)$ is equivalent to $d(x,y) \leq r_i$. Since $\text{Th}(\mathcal{U}_\mathcal{R})$ has a unique 1-type over $\emptyset$, we have shown $d(x,y) \leq \alpha$ is equivalent to $d(x,y) \leq r_i$ for all $i < \lambda$. Since $\alpha \in \text{heq}(\mathcal{R})$, we may fix $\beta \in R^*$ such that $\alpha < \beta$ and, for all $s \in R$, if $s < \beta$ and s is an idempotent, then $s < \alpha$. In particular, we must have $\beta \leq r_i$ for all $i < \lambda$, which is a contradiction. □

Returning to [15], note that $0^+ \in \text{heq}(\mathcal{Q}_1)$, and so failure of elimination of hyperimaginaries for $\text{Th}(\mathcal{U}_{\mathcal{Q}_1})$ is a special case of the previous result. Note also that $0^+ \in \text{heq}(\mathcal{Q})$ and so $\text{Th}(\mathcal{U}_\mathcal{Q})$ also fails elimination of hyperimaginaries. It is worth mentioning that Casanovas and Wagner carry out an analysis of 0-definable equivalence relations of any arity in $\text{Th}(\mathcal{U}_{\mathcal{R}_n})$. From this analysis it is easy to conclude that, for all $n > 0$, $\text{Th}(\mathcal{U}_{\mathcal{R}_n})$ has weak elimination of imaginaries, which implies the same result for $\text{Th}(\mathcal{U}_{\mathcal{Q}_1})$.

For future work in this direction, we conjecture that the previously established necessary conditions for elimination of hyperimaginaries and weak elimination of imaginaries are also sufficient.

Conjecture 3.8.7. *Suppose $\mathcal{R}$ is a Urysohn monoid.*

(a) $\text{Th}(\mathcal{U}_\mathcal{R})$ *has weak elimination of imaginaries if and only if* $\text{eq}^\circ(\mathcal{R}) = \emptyset$.

(b) $\text{Th}(\mathcal{U}_\mathcal{R})$ *has elimination of hyperimaginaries if and only if* $\text{heq}(\mathcal{R}) = \emptyset$.

In particular, if the conjecture holds, then $\text{Th}(\mathcal{U}_\mathcal{R})$ has weak elimination of imaginaries for any archimedean Urysohn monoid $\mathcal{R}$. Regarding further consequences of this conjecture, we first make the following observation.

Proposition 3.8.8. *If $\mathcal{R}$ is Urysohn and* $\text{heq}(\mathcal{R}) \neq \emptyset$ *then* $\text{SO}(\text{Th}(\mathcal{U}_\mathcal{R})) = \omega$.

Proof. Suppose $\alpha \in \text{heq}(\mathcal{R})$. Fix $\beta \in R^*$ such that $\alpha < \beta$ and, for all $r \in R$, if $\alpha < r < \beta$ then $r < r \oplus r$. Fix $n > 0$. Then $n\alpha = \alpha < \beta$ so, by Proposition 3.2.1, there is some $t \in R$ such that $\alpha < t$ and $nt < \beta$. Then $nt < 2nt$, which implies $\text{arch}(\mathcal{R}) > n$. □

The purpose of Casanovas and Wagner's work in [15] is to demonstrate the existence of a theory without the strict order property that does not eliminate hyperimaginaries. Our previous work slightly sharpens this upper bound of complexity to *without the finitary strong order property.* On the other hand, if Conjecture 3.8.7(b) is true then, combined with Proposition 3.8.8(b), we would conclude that generalized Urysohn spaces provide no further assistance in decreasing the complexity of this upper bound. In particular, a consequence of Conjecture 3.8.7(b) is that if $\mathrm{SO}(\mathrm{Th}(\mathcal{U}_{\mathcal{R}})) < \omega$ then $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ eliminates hyperimaginaries. An outlandish, but nonetheless open, conjecture could be obtained from this statement by replacing $\mathrm{Th}(\mathcal{U}_{\mathcal{R}})$ with an arbitrary theory T. Concerning the converse of this statement, note that, if Conjecture 3.8.7(b) holds, then $\mathrm{Th}(\mathcal{U}_{\mathcal{N}})$ would eliminate hyperimaginaries, while still having strong order rank ω. As a side note, we have observed that $\mathrm{Th}(\mathcal{U}_{\mathcal{N}})$ is small, and so at least eliminates finitary hyperimaginaries (see [14, Theorem 18.14]).

Chapter 4

Isometry Groups of Generalized Urysohn Spaces

In this chapter, we consider the group of isometries of $\mathcal{U}_\mathcal{R}$, denoted $\text{Isom}(\mathcal{U}_\mathcal{R})$, where $\mathcal{R}$ is a countable distance monoid. We approach the study of these groups from two parallel directions of interest. First, the isometry group of the rational Urysohn space is a well-studied example in topological dynamics of Polish groups and related topics in descriptive set theory and combinatorics. Second, automorphism groups of general Fraïssé limits have played an important role in studying the interplay between model theory and the aforementioned fields.

We have chosen to focus our interest on the question of extending partial isometries in $\mathcal{R}$-metric spaces. In particular, suppose A is a finite $\mathcal{R}$-metric space. Suppose further that we wish to find an $\mathcal{R}$-metric space B, with A a subspace of B, such that any isometry between two subspaces of A extends to a total isometry of B. In this case, we of course let $B = \mathcal{U}_\mathcal{R}$. However, if we strengthen the question, and demand that B still be finite, then the existence of B becomes a more difficult issue. If such a B can always be found, for any given A, then we say the class of finite $\mathcal{R}$-metric spaces has the *Hrushovski property* (see Definition 4.1.1).

In Section 4.1, we define the Hrushovski property for general relational structures, and discuss its importance in the study of automorphism groups of countable structures. In Section 4.2, we prove that, if $\mathcal{R}$ is an archimedean distance monoid, then $\mathcal{K}_\mathcal{R}$ has the Hrushovski property. The main tool used to obtain this result (Theorem 4.2.2) is a metric space analog of a theorem of Herwig and Lascar [39] concerning extending automorphisms in classes of relational structures omitting finite substructures. The proof of Theorem 4.2.2 closely follows Solecki's proof the Hrushovski property for the class of finite metric spaces (over $(\mathbb{R}^{\geq 0}, +, \leq, 0)$. However, our formulation is applicable to cases in which one considers metric spaces forbidding certain subspaces. Such cases arise naturally when considering *metrically homogeneous graphs*, i.e., graphs that are homogeneous metric spaces when equipped with the path metric. A rather extensive catalog of such graphs is constructed by Cherlin in [16] and, in Section 4.3, we apply our results on extending isometries to obtain the Hrushovski property for the well-known example of metric spaces omitting triangles of odd perimeter. Finally, in Section 4.4, we prove the Hrushovski property for a certain class of well-behaved, possibly non-archimedean monoids, which includes the ultrametric case.

4.1 The Hrushovski Property

Definition 4.1.1. Fix a relational language $\mathcal{L}$.

1. Given an $\mathcal{L}$-structure A, a **partial isomorphism** of A is an $\mathcal{L}$-isomorphism $\varphi : A_1 \longrightarrow A_2$, where A_1 and A_2 are substructures of A.

2. Suppose $\mathcal{K}$ is a class of $\mathcal{L}$-structures and $A \in \mathcal{K}$. Then A has the **$\mathcal{L}$-extension property in $\mathcal{K}$** if there is some $B \in \mathcal{K}$ such that

   (i) A is (isomorphic to) a substructure of B;

   (ii) any partial isomorphism of A extends to a total automorphism of B.

   If, moreover, B is finite, then A has the **finite $\mathcal{L}$-extension property in $\mathcal{K}$.**

3. A class $\mathcal{K}$ of finite $\mathcal{L}$-structures has the **Hrushovski property** if every element of $\mathcal{K}$ has the (finite) $\mathcal{L}$-extension property in $\mathcal{K}$.

The significance of the Hrushovski property can be found in work of Hodges, Hodkinson, Lascar, and Shelah [41] on the small index property for automorphism groups of countable structures, which we briefly summarize.

Let $\mathcal{M}$ be a countable first-order structure and let G be its group of automorphisms, which inherits a topological group structure as a closed subgroup of S_∞. An important program of study focuses on the extent to which $\mathcal{M}$ can be reconstructed from G. This has produced fruitful and active research on interactions between model theory and topological dynamics. In [41], Hodges, Hodkinson, Lascar, and Shelah use generic automorphisms to show that, if $\mathrm{Th}(\mathcal{M})$ is ω-stable and $\aleph_0$-categorical, then G has the *small index property*, i.e., any subgroup H of G, with $[G : H] < 2^{\aleph_0}$ is open. The small index property is used to recover the topological structure of G from the group structure (see [57]). Note that, since G is separable, any open subgroup of G must have countable index. Therefore, if G has the small index property then, for any subgroup H of G, if $[G : H]$ is uncountable then $[G : H] = 2^{\aleph_0}$.

Toward establishing the small index property for automorphism groups of countable structures, the Hrushovski property can be a powerful tool. This property was shown for the class of all finite graphs (i.e. $\mathcal{K}_{\mathcal{R}_2}$) by Hrushovski [43]. The Hrushovski property for graphs is used in [41] to show the small index property for the automorphism group of the random graph (i.e. $\mathrm{Isom}(\mathcal{U}_{\mathcal{R}_2})$). Herwig [37], [38] then extended

Hrushovski's work to include many Fraïssé classes with $\aleph_0$-categorical Fraïssé limits, including the class of K_n-free graphs, for a fixed $n \geq 3$, whose Fraïssé limit was introduced by Henson in [36]. As with the random graph, the Hrushovski property is then used to obtain the small index property for the automorphism groups of these Fraïssé limits. In all of these examples, including the work in [41], the small index property is shown via an analysis of generic automorphisms.

Definition 4.1.2. Suppose G is a Polish group. For any $n > 0$ we have the action of G on G^n by conjugation, i.e., $g \cdot (h_1, \ldots, h_n) = (gh_1g^{-1}, \ldots, gh_ng^{-1})$. G has **ample generics** if, for all $n > 0$, there is an element of G^n with a comeager orbit.

In [46, Theorem 6.24, Proposition 6.27], Kechris and Rosendal show that ample generics for a Polish group G implies the small index property. Moreover, in the case that $G = \mathrm{Aut}(\mathcal{M})$, for $\mathcal{M}$ the (countable) Fraïssé limit of a Fraïssé class $\mathcal{K}$, Kechris and Rosendal characterize ample generics for G via certain amalgamation and embedding properties in $\mathcal{K}$. We will not include the exact characterization here, and instead focus on stronger conditions sufficient to prove ample generics.

Definition 4.1.3. Suppose $\mathcal{K}$ is a Fraïssé class in a relational language. Fix $n > 0$.

1. Given $n > 0$, define $\mathcal{K}^{p,n}$ to be the class of tuples $(A, \varphi_1, \ldots, \varphi_n)$, where A is in $\mathcal{K}$ and φ_i is a partial isomorphism of A. Define $\mathcal{K}^n$ to be the subclass of tuples such that each φ_i is a total isomorphism.

2. An element $(A, \varphi_1, \ldots, \varphi_n) \in \mathcal{K}^{p,n}$ **embeds** in another element $(B, \psi_1, \ldots, \psi_n)$ of $\mathcal{K}^{p,n}$ if there is a homomorphic embedding $f : A \longrightarrow B$ such that, for all $i \leq n$, $f(\mathrm{dom}(\varphi_i)) \subseteq \mathrm{dom}(\psi_i)$ and $\psi_i \circ f = f \circ \varphi_i$.

3. Suppose $\mathcal{K}$ is a subclass of $\mathcal{K}^{p,n}$.

   (*a*) $\mathcal{K}$ has the **joint embedding property**, JEP, if any two $(A, \bar{\varphi})$ and $(B, \bar{\psi})$ in $\mathcal{K}$ embed in some common $(C, \bar{\theta})$ in $\mathcal{K}$.

   (*b*) $\mathcal{K}$ has the **amalgamation property**, AP, if, given $(A, \bar{\varphi})$, $(B, \bar{\psi})$, and $(C, \bar{\theta})$ in $\mathcal{K}$ and embeddings $f_1 : (A, \bar{\varphi}) \longrightarrow (B, \bar{\psi})$ and $f_2 : (A, \bar{\varphi}) \longrightarrow (C, \bar{\theta})$, there is some $(E, \bar{\chi})$ in $\mathcal{K}$ and embeddings $g_1 : (B, \bar{\psi}) \longrightarrow (E, \bar{\chi})$ and $g_2 : (C, \bar{\theta}) \longrightarrow (E, \bar{\chi})$ such that $f_2 \circ f_1 = g_2 \circ g_1$.

   (*c*) $\mathcal{K}$ is **cofinal in** $\mathcal{K}^{p,n}$ if every $(B, \bar{\psi})$ in $\mathcal{K}^{p,n}$ embeds in some $(A, \bar{\varphi}) \in \mathcal{K}$.

4. $\mathcal{K}^{p,n}$ has the **cofinal amalgamation property**, CAP, if there is some cofinal subclass $\mathcal{K}$ of $\mathcal{K}^{p,n}$ with AP.

Theorem 4.1.4. [46, Theorem 6.2] *Let $\mathcal{K}$ be a Fraïssé class, with Fraïssé limit $\mathcal{M}$. Suppose that, for all $n > 0$, $\mathcal{K}^{p,n}$ has* JEP *and* CAP. *Then* $\mathrm{Aut}(\mathcal{M})$ *has ample generics.*

The Hrushovski property for a Fraïssé class $\mathcal{K}$ ensures $\mathcal{K}^n$ is cofinal in $\mathcal{K}^{p,n}$ for all $n > 0$. Therefore, if $\mathcal{K}$ has the Hrushovski property and $\mathcal{M}$ is the Fraïssé limit of $\mathcal{K}$, then, to show ample generics for $\mathrm{Aut}(\mathcal{M})$, it suffices to show that, for all $n > 0$, $\mathcal{K}^{p,n}$ has JEP and $\mathcal{K}^n$ has AP. We will give an example of this in Proposition 4.1.11 below.

Toward proving the Hrushovski property for a Fraïssé class $\mathcal{K}$, the following result of Herwig and Lascar [39] is quite powerful. We first summarize a few general definitions concerning isomorphisms in arbitrary relational languages.

Definition 4.1.5. Suppose $\mathcal{L}$ is a relational language.

1. A class $\mathcal{K}$ of $\mathcal{L}$-structures has the **extension property for partial automorphisms** if, for any finite $A \in \mathcal{K}$, if A has the $\mathcal{L}$-extension property in $\mathcal{K}$ then A has the finite $\mathcal{L}$-extension property in $\mathcal{K}$.

2. Suppose B is an $\mathcal{L}$-structure.

   (a) An $\mathcal{L}$-structure A **weakly embeds** in B if there is an injective function $\varphi : A \longrightarrow B$ such that, for any relation $R \in \mathcal{L}$ and $\bar{a} \in A$, if $A \models R(\bar{a})$ then $B \models R(\varphi(\bar{a}))$.

   (b) Given a class $\mathcal{F}$ of $\mathcal{L}$-structures, we say B is $\mathcal{F}$**-free** if no element of $\mathcal{F}$ weakly embeds in B.

Theorem 4.1.6. [39, Theorem 3.2] *Suppose $\mathcal{L}$ is a finite relational language and $\mathcal{F}$ is a finite class of finite $\mathcal{L}$-structures. Then the class of $\mathcal{F}$-free $\mathcal{L}$-structures has the extension property for partial automorphisms.*

Example 4.1.7. Let $\mathcal{L} = \{R\}$ be the language of graphs.

1. By Theorem 4.1.6, the class of $\mathcal{L}$-structures has has the extension property for partial $\mathcal{L}$-automorphisms. Any finite graph has the $\mathcal{L}$-extension property in the class of $\mathcal{L}$-structures (witnessed by the random graph), and therefore has finite $\mathcal{L}$-extension property in the class of $\mathcal{L}$-structures. Given a finite graph A, let B be a finite $\mathcal{L}$-structure witnessing that A has the finite $\mathcal{L}$-extension property. Endow B with a graph structure by defining edges between distinct points $a, b \in B$, such that $B \models R(a, b) \wedge R(b, a)$. Since any $\mathcal{L}$-automorphism of B is a graph automorphism, we have verified the Hrushovski property for the class of finite graphs.

2. For the class of finite K_n-graphs, where $n \geq 3$ is fixed, repeat the previous argument on the class of $\mathcal{F}$-free $\mathcal{L}$-structures, where $\mathcal{F} = \{K_n\}$, using the generic K_n-free graph in place of the random graph.

For classes of metric spaces, it is not so easy to directly apply Theorem 4.1.6. In particular, one often wants to use an infinite language, and it is not as straightforward to determine a class of structures $\mathcal{F}$ to omit. However, in a very clever argument, Solecki [85] uses Theorem 4.1.6 to prove the following result, which we will present in the context of generalized metric spaces.

Definition 4.1.8. Fix a distance monoid $\mathcal{R}$.

1. Given an $\mathcal{R}$-metric space A, a **partial isometry of** A is an isometry $\varphi : A_1 \longrightarrow A_2$, where A_1 and A_2 are subspaces of A.

2. Suppose $\mathcal{K}$ is a class of $\mathcal{R}$-metric spaces and $A \in \mathcal{K}$. Then A has the **extension property in** $\mathcal{K}$ if there is some $B \in \mathcal{K}$ such that

   (i) A is (isometric to) a subspace of B;

   (ii) any partial isometry of A extends to a total isometry of B.

   If, moreover, B is finite, then A has the **finite extension property in** $\mathcal{K}$.

3. A class $\mathcal{K}$ of finite $\mathcal{R}$-metric spaces has the **Hrushovski property** if every element of $\mathcal{K}$ has the (finite) extension property in $\mathcal{K}$.

Theorem 4.1.9. [85, Theorem 2.1] *If* $(G, +, \leq, 0)$ *is a subgroup of* $(\mathbb{R}, +, \leq, 0)$*, and* $\mathcal{R} = (G_{\geq 0}, +, \leq, 0)$*, then the class of* $\mathcal{R}$*-metric spaces has the Hrushovski property.*

As with previous examples, Solecki then shows that the Hrushovski property for the class of finite metric spaces with rational distances implies $\mathrm{Isom}(\mathcal{U}_{\mathbb{Q}})$ has ample generics (it is important here that $\mathcal{U}_{\mathbb{Q}}$ is countable).

Remark 4.1.10. The proof of Theorem 4.1.9 in [85] uses Theorem 4.1.6 in a very strong way. In turn, the proof of Theorem 4.1.6 in [39] goes through the proof of the fact, due to Ribes and Zalesskiĭ [73], that, given a finitely generated free group F, the group product of finitely many finitely generated subgroups of F is closed in the profinite topology on F. In [74], Rosendal proves a result which obtains Theorem 4.1.9 directly from Ribes and Zalesskiĭ's theorem.

In the next section, we will consider the extension property for isometries of metric spaces over general distance monoids $\mathcal{R}$. The next proposition verifies that, as with the cases considered by Solecki, the Hrushovski property is sufficient to obtain ample generics for $\mathrm{Isom}(\mathcal{U}_{\mathcal{R}})$. The proof is essentially the same as the remarks following Proposition 6.4 of [46].

Proposition 4.1.11. *Suppose* $\mathcal{R}$ *is a countable distance monoid.*

(a) *For all* $n > 0$, $\mathcal{K}^{p,n}_{\mathcal{R}}$ *has* JEP.

(b) *For all* $n > 0$, $\mathcal{K}^{n}_{\mathcal{R}}$ *has* AP.

(c) *If* $\mathcal{K}_{\mathcal{R}}$ *has the Hrushovski property then, for all* $n > 0$, $\mathcal{K}^{n}_{\mathcal{R}}$ *is cofinal in* $\mathcal{K}^{p,n}_{\mathcal{R}}$.

(d) *If* $\mathcal{K}_{\mathcal{R}}$ *has the Hrushovski property then* $\mathrm{Isom}(\mathcal{U}_{\mathcal{R}})$ *has ample generics.*

Proof. Part (a). Fix $(A,\varphi_1,\dots,\varphi_n)$ and $(B,\psi_1,\dots,\psi_n)$ in $\mathcal{K}^{p,n}_{\mathcal{R}}$. We want to find $(C,\theta_1,\dots,\theta_n)$ such that both $(A,\bar{\varphi})$ and $(B\bar{\psi})$ embed in $(C,\bar{\theta})$.

Fix bijections $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$, where A' and B' are sets, with $A' \cap B' = \emptyset$. Let $C = A' \cup B'$. Given $a_1, a_2 \in A$ let $d_C(f(a_1), f(a_2)) = d_A(a_1, a_2)$; and given $b_1, b_2 \in B$ let $d_C(f(b_1), f(b_2)) = d_B(b_1, b_2)$. Given $a \in A$ and $b \in B$, let $d_C(f(a), f(b)) = \max(\mathrm{Spec}(A) \cup \mathrm{Spec}(B))$. Define $\theta_i = (f \circ \varphi_i \circ f^{-1}) \cup (g \circ \psi_i \circ g^{-1})$. Then $(C,\theta_1,\dots,\theta_n)$ is as desired.

Part (b). Fix $(A,\bar{\varphi})$, $(B,\bar{\psi})$, $(C,\bar{\theta})$, f_1, and f_2, as in the definition of AP. Fix bijections $g_1 : B \longrightarrow B'$ and $g_2 : C \longrightarrow C'$, where B' and C' are sets, and assume $B' \cap C' = g_1 f_1(A) = g_2 f_2(A)$ and $g_1|_{f_1(A)} = g_2|_{f_2(A)}$. Let $E = B' \cup C'$ and define d_E on E as follows:

$$d_E(x,y)) = \begin{cases} d_B(g_1^{-1}(x), g_1^{-1}(y)) & \text{if } x, y \in B', \\ d_C(g_2^{-1}(x), g_2^{-1}(y)) & \text{if } x, y \in C', \\ \min_{a\in A}[d_B(g_1^{-1}(x), f_1(a)) \oplus d_C(f_2(a), g_2^{-1}(y))] & \text{if } x \in B'\backslash C',\ y \in C'\backslash B'. \end{cases}$$

Given $1 \le i \le n$, define $\chi_i = (g_1 \circ \psi_i \circ g_1^{-1}) \cup (g_2 \circ \theta_i \circ g_2^{-1})$. We first show χ_i is well-defined, which means fixing $a \in A$ and verifying

$$g_1 \psi_i f_1(a) = g_2 \theta_i f_2(a).$$

By assumption, $\psi_i \circ f_1 = f_1 \circ \varphi_i$ and $\theta_i \circ f_2 = f_2 \circ \varphi_i$, so we want to show $g_1 f_1 \varphi_i(a) = g_2 f_2 \varphi_i(a)$. Since φ_i is a permutation of A, this follows by construction.

Next, given $1 \le i \le n$, we have $\chi_i \circ g_1 = g_1 \circ \psi_i$ and $\chi_i \circ g_2 = g_2 \circ \theta_i$ by definition. Therefore, it remains to show χ_i is an isometry of E. The only nontrivial thing to show is that, given $x \in B'\backslash C'$ and $y \in C'\backslash B'$, we have $d_E(x,y) = d_E(\chi_i(x), \chi_i(y))$. By definition, $\chi_i(x) = g_1 \psi_i g_1^{-1}(x)$ and $\chi_i(y) = g_2 \theta_i g_2^{-1}(y)$. Note also that $\chi_i(x) \in B'\backslash C'$ and $\chi_i(y) \in C'\backslash B'$. Therefore,

$$\begin{aligned} d_E(\chi_i(x), \chi_i(y)) &= \min_{a\in A} \left[d_B(\psi_i g_1^{-1}(x), f_1(a)) \oplus d_C(f_2(a), \theta_i g_2^{-1}(y)) \right] \\ &= \min_{a\in A} \left[d_B(g_1^{-1}(x), \psi_i^{-1} f_1(a)) \oplus d_C(\theta_i^{-1} f_2(a), g_2^{-1}(y)) \right] \\ &= \min_{a\in A} \left[d_B(g_1^{-1}(x), f_1 \varphi_i^{-1}(a)) \oplus d_C(f_2 \varphi_i^{-1}(a), g_2^{-1}(y)) \right] \\ &= \min_{a\in A} \left[d_B(g_1^{-1}(x), f_1(a)) \oplus d_C(f_2(a), g_2^{-1}(y)) \right] \\ &= d_E(x,y). \end{aligned}$$

Part (c). Suppose $(B,\psi_1,\dots,\psi_n)$ is in $\mathcal{K}^{p,n}_{\mathcal{R}}$. By the Hrushovski property, there is some $A \in \mathcal{K}_{\mathcal{R}}$, an isometric embedding $f : B \longrightarrow A$, and total isometries $\varphi_1,\dots,\varphi_n$ of A such that $f \circ \psi_i = \varphi_i \circ f$.

Part (d). For any $n > 0$, $\mathcal{K}^{p,n}_{\mathcal{R}}$ has JEP by part (a), and CAP by parts (b) and (c). So $\mathrm{Isom}(\mathcal{U}_{\mathcal{R}})$ has ample generics by Theorem 4.1.4. □

4.2 The Extension Property for Partial Isometries

The main result of this section is a translation of Theorem 4.1.6 to the context of generalized metric spaces and isometries. The proof, which relies on Theorem 4.1.6, is a slight modification of Solecki's proof of Theorem 4.1.9 in [85]. We begin with a translation of Definition 4.1.5.

Definition 4.2.1. Fix a distance monoid $\mathcal{R}$.

1. A class $\mathcal{K}$ of $\mathcal{R}$-metric spaces has the **extension property for partial isometries** if, for any finite $A \in \mathcal{K}$, if A has the extension property in $\mathcal{K}$ then A has the finite extension property in $\mathcal{K}$.

2. Suppose B is an $\mathcal{R}$-metric space.

   (a) A partial $\mathcal{R}$-semimetric space (A, f) **weakly embeds** in B if there is an injective function $\varphi : A \longrightarrow B$ such that, for any $a, b \in A$, $d_B(\varphi(a), \varphi(b)) = f(a, b)$.

   (b) Given a class $\mathcal{F}$ of partial $\mathcal{R}$-semimetric spaces, B is $\mathcal{F}$**-free** if no element of $\mathcal{F}$ weakly embeds in B.

Theorem 4.2.2. *Suppose $\mathcal{R}$ is an archimedean distance monoid and $\mathcal{F}$ is a finite class of finite partial $\mathcal{R}$-semimetric spaces. Then the class of $\mathcal{F}$-free $\mathcal{R}$-metric spaces has the extension property for partial isometries.*

Proof. Let $\mathcal{K}$ be the class of $\mathcal{F}$-free $\mathcal{R}$-metric spaces. Suppose $A \in \mathcal{K}$ is finite and has the extension property in $\mathcal{K}$. We want to show A has the finite extension property in $\mathcal{K}$.

Let $S = \text{Spec}(A) \cup \bigcup_{Y \in \mathcal{F}} \text{Spec}(Y)$ and note that S is a finite subset of R. Define

$$\Sigma = \{(r_0, \ldots, r_n) : n > 0,\ r_i \in S,\ r_0 > r_1 \oplus \ldots \oplus r_n\}.$$

Claim 1: Σ is finite.
Proof: For each $r \in S$, let

$$\Sigma(r) = \{(r_1, \ldots, r_n) : n > 0,\ r_i \in S,\ r > r_1 \oplus \ldots \oplus r_n\}.$$

Then $\Sigma = \bigcup_{r \in S} \Sigma(r)$, so it suffices to show $\Sigma(r)$ is finite for all $r \in S$. Fix $r \in S$ and let $s = \min\{t \in S : t < r\}$. Since $\mathcal{R}$ is archimedean, there is some $m > 0$ such that $r \leq ms$. To show $\Sigma(r)$ is finite it suffices to show that if $(r_1, \ldots, r_n) \in \Sigma(r)$ then $n < m$. But if $n \geq m$ then

$$r \leq ms \leq ns \leq r_1 \oplus \ldots \oplus r_n,$$

and so $(r_1, \ldots, r_n) \notin \Sigma(r)$. $\dashv_{\text{claim}}$

Fix $\sigma = (r_0, \ldots, r_n) \in \Sigma$. We define the following partial $\mathcal{R}$-semimetric space (P_σ, f_σ):

- $P_\sigma = \{x_0, \dots, x_n\}$, with $x_i \neq x_j$ for distinct i, j,
- $f_\sigma(x_i, x_i) = 0$ for all $0 \leq i \leq n$,
- $f_\sigma(x_0, x_n) = r_0$,
- for all $1 \leq i \leq n$, $f_\sigma(x_{i-1}, x_i) = r_i$.

Let $\mathcal{F}_0 = \{P_\sigma : \sigma \in \Sigma\}$ and set $\mathcal{F}^* = \mathcal{F} \cup \mathcal{F}_0$. By the triangle inequality, any $\mathcal{R}$-metric space is $\mathcal{F}_0$-free, and so $\mathcal{K}$ is equal to the class of $\mathcal{F}^*$-free $\mathcal{R}$-metric spaces. By assumption on A, there is an $\mathcal{R}$-metric space $U \in \mathcal{K}$, with $A \subseteq U$, such that any isometry between two subspaces of A extends to an isometry of U.

In order to use Theorem 4.1.6, we must consider the entire situation in the context of a finite relational language. In particular, let $\mathcal{L} = \{d_r(x, y) : r \in S\}$, where each $d_r(x, y)$ is a binary relation. Then we may consider any partial $\mathcal{R}$-metric space (X, f) as an $\mathcal{L}$-structure where, for $r \in S$, $X \models d_r(x, y)$ if and only if $f(x, y) = r$. Let $\mathcal{K}^*$ be the class of $\mathcal{F}^*$-free $\mathcal{L}$-structures.

Consider A as an $\mathcal{L}$-structure in $\mathcal{K}^*$, and suppose φ is an $\mathcal{L}$-isomorphism between two substructures of A. Since $\operatorname{Spec}(A) \subseteq S$, it follows that φ is an isometry between two subspaces of A. By assumption, φ extends to an isometry $\hat{\varphi}$ of U, which can be viewed as an $\mathcal{L}$-automorphism of U (as an $\mathcal{L}$-structure in $\mathcal{K}^*$). Altogether, we have shown A has the $\mathcal{L}$-extension property in $\mathcal{K}^*$. By Theorem 4.1.6, there is a finite $\mathcal{L}$-structure $C \in \mathcal{K}^*$ such that every $\mathcal{L}$-isomorphism between two $\mathcal{L}$-substructures of A extends to an $\mathcal{L}$-automorphism of C. It follows that every isometry between two subspaces of A extends to an $\mathcal{L}$-automorphism of C. We now use the same strategy as in Solecki's proof of Theorem 4.1.9 to obtain, from C, a finite $\mathcal{F}$-free $\mathcal{R}$-metric space, which witnesses that A has the finite extension property in $\mathcal{K}$.

A sequence $(c_0, \dots, c_n)$ from C is a *chain from c_0 to c_n* if, for all $1 \leq i \leq n$, there is some $r \in S$ such that

$$C \models d_r(c_{i-1}, c_i) \wedge d_r(c_i, c_{i-1}).$$

Given $x, y \in C$, write $\operatorname{ch}(x, y)$ if $x = y$ or if there is a chain in C from x to y. Note that ch is an equivalence relation on C. Define

$$B = \{c \in C : \operatorname{ch}(c, a) \text{ holds for all } a \in A\}.$$

Claim 2. $A \subseteq B$.
Proof: We fix $a, c \in A$ and show $\operatorname{ch}(c, a)$ holds. If $a = c$ then this is trivial. Otherwise, there is some $r \in S$ such that $d(a, c) = r$. Since A is an $\mathcal{L}$-substructure of C, it follows that $C \models d_r(a, c) \wedge d_r(c, a)$, and so (c, a) is a chain from c to a. Therefore $\operatorname{ch}(c, a)$ holds. $\dashv_{\text{claim}}$

To prove A has the finite extension property in $\mathcal{K}$, we will equip B with an $\mathcal{R}$-metric in such a way that B is an $\mathcal{F}$-free metric space, with A a subspace of B, and any partial isometry of A extends to a total isometry of B. Given distinct

$x, y \in B$, we define $\Delta(x,y)$ to be the set of sequences $(r_1, \ldots, r_n)$ such that $r_i \in S$ for all $1 \leq i \leq n$ and there is $(c_0, \ldots, c_n)$ such that

- $(c_0, \ldots, c_n)$ is a chain in C from x to y,
- for all $1 \leq i \leq n$, $C \models d_{r_i}(c_{i-1}, c_i) \wedge d_{r_i}(c_i, c_{i-1})$.

We define a function $\delta : B \times B \longrightarrow S$ such that

$$\delta(x,y) = \begin{cases} 0 & \text{if } x = y \\ \min\{r_1 \oplus \ldots \oplus r_n : (r_1, \ldots, r_n) \in \Delta(x,y)\} & \text{if } x \neq y. \end{cases}$$

Claim 3: δ is an $\mathcal{R}$-metric on B extending d on A.
Proof: Note that if $x, y \in B$ are distinct then, for any $a \in A$, we have $\operatorname{ch}(x,a)$ and $\operatorname{ch}(y,a)$, which means we also have $\operatorname{ch}(x,y)$, and so $\Delta(x,y) \neq \emptyset$. Therefore δ is well-defined on $B \times B$. For any $r_1, \ldots, r_n \in S$, we have

$$\max\{r_1, \ldots, r_n\} \leq r_1 \oplus \ldots \oplus r_n,$$

so it follows that $\delta(x,y) = 0$ if and only if $x = y$. Moreover, if $(r_1, \ldots, r_n) \in \Delta(x,y)$ then $(r_n, \ldots, r_1) \in \Delta(y,x)$, which means δ is symmetric. For the triangle inequality, fix pairwise distinct $x, y, z \in B$. Given $(r_1, \ldots, r_m) \in \Delta(x,y)$ and $(s_1, \ldots, s_n) \in \Delta(y,z)$, let $(c_0, \ldots, c_m)$ be a chain from x to y and $(e_0, \ldots, e_n)$ a chain from y to z such that, for all $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$C \models d_{r_i}(c_{i-1}, c_i) \wedge d_{r_i}(c_i, c_{i-1}) \wedge d_{s_j}(e_{j-1}, e_j) \wedge d_{s_j}(e_j, e_{j-1}).$$

Then $(c_0, \ldots, c_{m-1}, y, e_1, \ldots, e_n)$ witnesses $(r_1, \ldots, r_m, s_1, \ldots, s_n) \in \Delta(x,z)$. Therefore

$$\delta(x,z) \leq (r_1 \oplus \ldots \oplus r_m) \oplus (s_1 \oplus \ldots \oplus s_n).$$

Altogether, we have $\delta(x,z) \leq \delta(x,y) \oplus \delta(y,z)$, and so δ is an $\mathcal{R}$-metric on B.

Next, fix distinct $a, b \in A$. We want to show $\delta(a,b) = d(a,b)$. If $d(a,b) = r$, then $(r) \in \Delta(a,b)$ and so $\delta(a,b) \leq d(a,b)$. Suppose, toward a contradiction, that $\delta(a,b) < d(a,b)$. Then there is some $(r_1, \ldots, r_n) \in \Delta(a,b)$ such that $r_1 \oplus \ldots \oplus r_n < r$, and so $(r, r_1, \ldots, r_n) \in \Sigma$. Let $(c_0, \ldots, c_n)$ be a chain from a to b witnessing $(r_1, \ldots, r_n) \in \Delta(a,b)$. Then the function $g : P_\sigma \longrightarrow C$ such that $g(x_i) = c_i$ is a weak $\mathcal{L}$-embedding, which is a contradicts that C is $\mathcal{F}^*$-free. $\dashv_{\text{claim}}$

By Claim 3, B is an $\mathcal{R}$-metric space and A is a subspace of B. Moreover, B is $\mathcal{F}$-free since $B \subseteq C$, $\mathcal{F} \subseteq \mathcal{F}^*$, and C is $\mathcal{F}^*$-free. Therefore, to finish the proof of the theorem, we have left to show that, if φ is an isometry between subspaces of A, then φ extends to an isometry of B.

Fix an isometry φ between two subspaces of A. By choice of C, we may fix an $\mathcal{L}$-automorphism $\hat{\varphi}$ of C extending φ. If φ is the empty isometry then we assume $\hat{\varphi}$ is the identity on C.

Claim 4: $\hat{\varphi}(B) = B$.
Proof: By assumption, we may assume φ is not the empty isometry. Since B is finite and $\hat{\varphi}$ is injective, it suffices to show $\hat{\varphi}(B) \subseteq B$. Fix $b \in B$. By assumption there is some $a \in \mathrm{dom}(\varphi) \subseteq C$, which, by definition of B, means $\mathrm{ch}(b, a)$ holds. Let $(c_0, \dots, c_n)$ be a chain from c_0 to c_n, with $c_0 = a$ and $c_n = b$. Let $e_i = \hat{\varphi}(c_i)$. Since $\hat{\varphi}$ is an $\mathcal{L}$-automorphism of C, it follows that $(e_0, \dots, e_n)$ is a chain witnessing that $\mathrm{ch}(\hat{\varphi}(b), \hat{\varphi}(a))$ holds. By assumption, $\hat{\varphi}(a) = \varphi(a) \in A$. Therefore, we have shown $\mathrm{ch}(\hat{\varphi}(b), a')$ holds for some $a' \in A$. Using the proof of Claim 2, it follows that $\mathrm{ch}(\hat{\varphi}(b), a')$ holds for all $a' \in A$, and so $\hat{\varphi}(b) \in B$, as desired. $\dashv_{\text{claim}}$

By Claim 4, in order to show that φ extends to an isometry of B, it suffices to show $\delta(x, y) = \delta(\hat{\varphi}(x), \hat{\varphi}(y))$ for any distinct $x, y \in B$. Given distinct $x, y \in B$, let $(r_1, \dots, r_n) \in \Delta(x, y)$ be such that

$$\delta(x, y) = r_1 \oplus \ldots \oplus r_n.$$

Let $(c_0, \dots, c_n)$ be a chain witnessing $(r_1, \dots, r_n) \in \Delta(x, y)$. Since $\hat{\varphi}$ is an $\mathcal{L}$-automorphism of C, it follows that, for any $1 \leq i \leq n$, we have

$$C \models d_{r_i}(\hat{\varphi}(c_{i-1}), \hat{\varphi}(c_i)) \wedge d_{r_i}(\hat{\varphi}(c_i), \hat{\varphi}(c_{i-1})).$$

Therefore $(\hat{\varphi}(c_0), \dots, \hat{\varphi}(c_n))$ is a chain witnessing $(r_1, \dots, r_n) \in \Delta(\hat{\varphi}(x), \hat{\varphi}(y))$, which means

$$\delta(\hat{\varphi}(x), \hat{\varphi}(y)) \leq r_1 \oplus \ldots \oplus r_n = \delta(x, y).$$

By a similar argument with $\hat{\varphi}^{-1}$, we obtain $\delta(x, y) = \delta(\hat{\varphi}(x), \hat{\varphi}(y))$, as desired. $\square$

Corollary 4.2.3. *Suppose $\mathcal{R}$ is a countable archimedean distance monoid and $\mathcal{F}$ is a finite class of finite partial $\mathcal{R}$-semimetric spaces. Let $\mathcal{K}$ be the class of finite $\mathcal{F}$-free $\mathcal{R}$-metric spaces, and assume $\mathcal{K}$ is a Fraïssé class. Then $\mathcal{K}$ has the Hrushovski property.*

Proof. By Theorem 4.2.2, the class of $\mathcal{F}$-free $\mathcal{R}$-metric spaces has the extension property for partial isomorphisms. Moreover, the Fraïssé limit of $\mathcal{K}$ witnesses that any $A \in \mathcal{K}$ has the extension property in the class of $\mathcal{F}$-free $\mathcal{R}$-metric spaces, and thus has the extension property in $\mathcal{K}$. By definition, $\mathcal{K}$ has the Hrushovski property. $\square$

Corollary 4.2.4. *Suppose $\mathcal{R}$ is an archimedean distance monoid. Then $\mathcal{K}_{\mathcal{R}}$ has the Hrushovski property.*

Proof. First, if $\mathcal{R}$ is countable then this follows from the previous result with $\mathcal{F} = \emptyset$. For general $\mathcal{R}$, simply observe that any finite $\mathcal{R}$-metric space can be viewed as a finite $\mathcal{R}_0$-metric space for some countable archimedean distance monoid $\mathcal{R}_0$ (e.g. $\mathcal{R}_0$ is the submonoid of $\mathcal{R}$ generated by $\mathrm{Spec}(A)$). $\square$

Applying Proposition 4.1.11, we obtain the next corollary.

Corollary 4.2.5. *Suppose $\mathcal{R}$ is a countable archimedean distance monoid. Then* $\mathrm{Isom}(\mathcal{U}_{\mathcal{R}})$ *has ample generics.*

It is worth observing that archimedean monoids cover the situation of Theorem 4.1.9, and so we have indeed generalized Solecki's result. It is also a proper generalization, in that there are many archimedean monoids which cannot be realized as submonoids of $(\mathcal{R}^{\geq 0}, \oplus, \leq, 0)$ (even when allowing truncated addition).

4.3 Metric Spaces Omitting Triangles of Odd Perimeter

In this section, we consider an application of Theorem 4.2.2 to a case in which $\mathcal{F}$ is nonempty. Throughout this section, we fix an odd integer $n \geq 3$. Let $n_* = \frac{n+1}{2}$, and let $\oplus$ denote $+_{n_*}$. Let $\mathcal{F}_n$ be the class of $\mathcal{R}_{n_*}$-metric spaces A (see Example 3.1.1(3)) such that $A = \{a_1, a_2, a_3\}$ and $d(a_1, a_2) + d(a_2, a_3) + d(a_3, a_1)$ is both odd and bounded by n. Let $\mathcal{K}_n$ be the class of finite $\mathcal{F}_n$-free $\mathcal{R}_{n^*}$-metric spaces.

Before beginning calculations with $\mathcal{R}_{n_*}$-metric spaces, we note that any $\mathcal{R}_{n_*}$-metric space is still a classical metric space over $(\mathbb{R}^{\geq 0}, +, \leq, 0)$. In particular, if A is an $\mathcal{R}_{n_*}$-metric space, and $x, y, z \in A$ then $d(x, z) \leq d(x, y) + d(y, z)$. We also define an equivalence relation ρ on $\mathbb{N}$ such that $\rho(k, m)$ holds if and only if k and m have the same parity.

Lemma 4.3.1. *Suppose A is an $\mathcal{F}_n$-free $\mathcal{R}_{n_*}$-metric space. Then, given $m \geq 3$ and $x_1, \ldots, x_m \in A$, if $p = d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{m-1}, x_m) + d(x_m, x_1)$ and $p \leq n$, then p is even.*

Proof. We proceed by induction on $m \geq 3$, where the base case is by assumption. Assume the result for m and fix $x_1, \ldots, x_{m+1} \in A$. Let

$$p = d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_m, x_{m+1}) + d(x_{m+1}, x_1),$$

and assume $p \leq n$. Since $d(x_m, x_1) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_1)$, it follows from the induction hypothesis that

$$d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{m-1}, x_m) + d(x_m, x_1)$$

is even. We also have $d(x_m, x_1) \leq d(x_1, x_2) + \ldots + d(x_{m-1}, x_m)$, and so $d(x_1, x_m) + d(x_m, x_{m+1}) + d(x_{m+1}, x_1)$ is even. Therefore

$$\rho(d(x_1, x_2) + \ldots + d(x_{m-1}, x_m), d(x_m, x_1)) \wedge \rho(d(x_m, x_1), d(x_m, x_{m+1}) + d(x_{m+1}, x_1)),$$

and so

$$\rho(d(x_1, x_2) + \ldots + d(x_{m-1}, x_m), d(x_m, x_{m+1}) + d(x_{m+1}, x_1)),$$

which implies p is even. $\square$

Recall that, if A and B are finite $\mathcal{R}_{n_*}$-metric spaces, with $A \cap B \neq \emptyset$, then we have the free amalgamation $A \otimes B$, as in Definition 2.7.15.

Proposition 4.3.2. *If A and B are in $\mathcal{K}_n$, with $A \cap B \neq \emptyset$, then $A \otimes B$ is in $\mathcal{K}_n$.*

Proof. We need to show $A \otimes B$ is $\mathcal{F}_n$-free. Suppose, toward a contradiction, there are $x, y, z \in A \otimes B$ such that, if $p = d(x, y) + d(y, z) + d(x, z)$, then $p \leq n$ and p is odd. Since A and B are $\mathcal{F}_n$-free, we may, without loss of generality, reduce to the following two cases.
Case 1: $x \in A \backslash B$, $y \in B \backslash A$, $z \in A \cap B$.

Suppose $d(x, y) = n_*$. Then $d(x, z) \oplus d(z, y) = n_*$, which means $d(x, z) + d(z, y) \geq n_*$. But then $p \geq 2n_* = n + 1$, which is a contradiction. Therefore $d(x, y) < n_*$. By definition, there is some $a \in A \cap B$ such that $d(x, y) = d(x, a) \oplus d(a, y)$, and so it follows that $d(x, y) = d(x, a) + d(a, y)$. Altogether,

$$p = d(x, a) + d(a, y) + d(x, z) + d(z, y).$$

Since $d(a, z) \leq \min\{d(x, a) + d(x, z), d(y, a) + d(y, z)\}$, it follows that $d(x, a) + d(x, z) + d(a, z)$ and $d(y, a) + d(y, z) + d(a, z)$ are both bounded by n. Since $x, z, a \in A$ and $y, z, a \in B$, we must have that $d(x, a) + d(x, z) + d(a, z)$ and $d(y, a) + d(y, z) + d(a, z)$ are both even. Therefore

$$\rho(d(x, a) + d(x, z), d(a, z)) \wedge \rho(d(a, z), d(y, a) + d(y, z)),$$

and so $\rho(d(x, a) + d(x, z), d(y, a) + d(y, z))$, which contradicts that p is odd.
Case 2: $x \in A \backslash B$, $y, z \in B \backslash A$.

Fix $a, b \in A \cap B$ such that $d(x, y) = d(x, a) \oplus d(a, y)$ and $d(x, z) = d(x, b) \oplus d(b, z)$. As in Case 1, we may assume $d(x, y) < n_*$, and so $d(x, y) = d(x, a) + d(a, y)$.

We first show $d(x, z) < n_*$. Indeed, if not then $d(x, a) \oplus d(a, y) \oplus d(y, z) = n_*$. But then $d(x, y) + d(y, z) = d(x, a) + d(a, y) + d(y, z) \geq n_*$, and so

$$p \geq 2n_* \geq n + 1,$$

which is a contradiction.

Therefore $d(x, z) < n_*$, and so $d(x, z) = d(x, b) + d(b, z)$, which means

$$p = d(x, a) + d(a, y) + d(y, z) + d(x, b) + d(b, z).$$

We have $d(x, a) + d(a, b) + d(b, x) \leq p$ and $a, b, x \in A$, so $d(x, a) + d(a, b) + d(b, x)$ is even. By Lemma 4.3.1, we similarly have that $d(a, b) + d(b, z) + d(z, y) + d(y, a)$ is even. But then

$$\rho(d(x, a) + d(x, b), d(a, b)) \wedge \rho(d(a, b), d(b, x) + d(z, y) + d(y, a)),$$

which contradicts that p is odd. □

Corollary 4.3.3. *$\mathcal{K}_n$ is a Fraïssé class.*

Proof. The hereditary property for $\mathcal{K}_n$ is trivial, and the amalgamation property follows from the previous proposition. Therefore, we only need to show the joint embedding property. Given $A, B \in \mathcal{K}_n$, let A' and B' be disjoint isometric copies of A and B, respectively. Let $C = A' \cup B'$ and, given $a \in A'$, $b \in B'$, set $d(a, b) = n_*$. We claim C is $\mathcal{F}_n$-free. Suppose, toward a contradiction, there are $x, y, z \in C$ such that, if $p = d(x, y)+d(y, z)+d(x, z)$, then $p \leq n$ and p is odd. Since A' and B' are each $\mathcal{F}_n$-free, we may assume $x, y \in A'$ and $z \in B'$. But then $d(x, z) + d(y, z) = 2n_* = n+1$, which contradicts $p \leq n$. □

Corollary 4.3.4. *$\mathcal{K}_n$ has the Hrushovski property. If $\mathcal{M}_n$ is the Fraïssé limit of $\mathcal{K}_n$ then* $\mathrm{Isom}(\mathcal{M}_n)$ *has ample generics.*

Proof. The Hrushovski property follows from Corollary 4.2.3. For ample generics of $\mathrm{Isom}(\mathcal{M}_n)$, follow the proof of Proposition 4.1.11 to show that, for all $m > 0$, $\mathcal{K}_n^{p,m}$ has JEP and CAP. □

4.3.1 Digression: Graphs Omitting Odd Cycles

We take this opportunity to give an exposition on the association between metric spaces omitting triangles of odd perimeter and graphs omitting cycles of odd length. In particular, given a fixed odd integer $n \geq 3$, the Fraïssé limit $\mathcal{M}_n$ of $\mathcal{K}_n$ is certainly homogeneous as a metric space. Moreover, if one considers the *unit distance graph* $(\mathcal{M}_n, E)$, where we set $E = \{(a, b) \in \mathcal{M}_n^2 : d(a, b) = 1\}$, then the path metric on $(\mathcal{M}_n, E)$ agrees with the original metric on $\mathcal{M}_n$ (this is because any distance in $\mathcal{M}_n$ is witnessed by a path of vertices with successive unit distance, see Lemma 4.3.7(a)). Altogether, $(\mathcal{M}_n, E)$ is referred to as a *metrically homogeneous graph*, and appears in Cherlin's catalog of such graphs in [16].

For odd $n \geq 3$, a well-known result of Komjáth, Mekler, and Pach [54] is the existence of a countable, universal and existentially complete $\mathcal{C}_n$-free graph, where $\mathcal{C}_n$ is the set of cycles of odd length bounded by n.[1] We use $\mathcal{G}_n$ to denote this graph. Moreover, in [18], Cherlin and Shi show that if $\mathcal{C}$ is a finite set of cycles, then there is a countable, universal and existentially complete $\mathcal{C}$-free graph if and only if $\mathcal{C} = \mathcal{C}_n$ for some odd $n \geq 3$.

In this subsection, we verify the most likely folkloric fact that $(\mathcal{M}_n, E)$ and $\mathcal{G}_n$ are the same graph. The result is not surprising, but requires a few careful considerations, and a detailed argument does not seem to appear in previous literature. In [12], Cameron considers metric spaces omitting triangles of *arbitrary* odd perimeter, which necessarily yield bipartite unit distance graphs (vs. $(\mathcal{M}_n, E)$, which contains $(n+2)$-cycles). Cameron remarks on the similarity to the constructions of Komjáth, Mekler, and Pach, but this point is not investigated further.

[1] The proof in [54] was found to have errors. A correct proof is given in [53].

The theory of $\mathcal{G}_n$ (in the graph language) is $\aleph_0$-categorical (see [17, Example 9]). However, if $n \geq 5$ then the age of $\mathcal{G}_n$, as a class of structures in the language of graphs, is not a Fraïssé class and the theory of $\mathcal{G}_n$ does not have quantifier elimination. In other words, $\mathcal{G}_n$ is not an ultrahomogeneous graph. However, one obtains quantifier elimination when expanding the language of graphs with predicates for distances up to n_*, which is the diameter of $\mathcal{G}_n$. For this reason, it is much more convenient to consider $\mathcal{G}_n$ as a metric space, and further motivates the work in this subsection.

Let us specify some conventions and terminology regarding graphs. A *path* in a graph is a sequence of vertices $(v_1, \ldots, v_m)$ such that, for all $1 \leq i < m$, there is an edge between v_i and v_{i+1}. In this case, we say the path *starts* at v_1 and *ends* at v_m. A *closed walk* is a path, which starts and ends at the same vertex. A *cycle* is a closed walk with no repeated vertices other than the starting and ending vertex. The *length* of a closed walk is the number of edges. Given an odd integer $n \geq 3$, let $\mathcal{C}_n$ denote the class of graphs, which are odd cycles of length at most n. By convention, we assume graphs have no multiple edges or loops.

Proposition 4.3.5. *Suppose Γ is a $\mathcal{C}_n$-free graph. Then any closed walk in Γ, with length bounded by n, has even length.*

Proof. We prove, by induction on odd integers $m \leq n$, that Γ does not contain any closed walks of length m. For $m = 1$ this is immediate, since we assume graphs are without loops. Fix an odd integer $m \leq n$, and assume Γ does not contain any closed walks of length k, where $k < m$ is odd. Suppose, toward a contradiction, $(a_1, \ldots, a_m, a_1)$ forms a closed walk in Γ. If $i \neq j$ for all $1 \leq i < j \leq m$, then this closed walk is a cycle, contradicting that Γ is $\mathcal{C}_n$-free. Therefore, there are $1 \leq i < j \leq m$, with $i = j$. Then we have closed walks $(x_1, \ldots, x_i, x_{j+1}, \ldots, x_m, x_1)$ and $(x_i, x_{i+1}, \ldots, x_j)$ of length $m-(j-i)$ and $(j-i)$, respectively. Since m is odd, it follows that one of $m-(j-i)$ or $(j-i)$ is odd and strictly less than m, contradicting the induction hypothesis. □

Lemma 4.3.6. *Fix an odd integer $n \geq 3$.*

(*a*) *Suppose Γ is a $\mathcal{C}_n$-free graph and d is the path metric on Γ, with distance truncated at n_*. Then (Γ, d) is an $\mathcal{F}_n$-free $\mathcal{R}_{n_*}$-metric space.*

(*b*) *Suppose A is an $\mathcal{F}_n$-free $\mathcal{R}_{n_*}$-metric space and $E = \{(a,b) \in A^2 : d(a,b) = 1\}$. Then (A, E) is a $\mathcal{C}_n$-free graph.*

Proof. Part (*a*). Suppose, toward a contradiction, there are $x, y, z \in \Gamma$ such that, if $p = d(x,y) + d(y,z) + d(x,z)$, then $p \leq n$ and p is odd. It follows that there is a sequence $\bar{x} = (x_1, x_2, \ldots, x_p, x_1)$ such that $x = x_1$ and

$$\{(x_1, x_2), (x_2, x_3), \ldots, (x_{p-1}, x_p), (x_p, x_1)\}$$

are edges in Γ. Therefore $\bar{x}$ forms a closed walk in Γ of length p, which contradicts Proposition 4.3.5.

Part (b). Suppose, toward a contradiction, $(x_1, \ldots, x_m, x_1)$ is a cycle in (A, E), where m is odd and $3 \leq m \leq n$. We have $d(x_1, x_m) = 1$, so there is some minimal $k \geq 3$ such that $d(x_1, x_k) < k - 1$. Note that the path $(x_1, \ldots, x_{k-1})$ witnesses $d(x_1, x_{k-1}) \leq k - 2$ and so, by minimality of k, we have $d(x_1, x_{k-1}) = k - 2$. Therefore

$$k - 2 = d(x_1, x_{k-1}) \leq d(x_1, x_k) + d(x_k, x_{k-1}) = d(x_1, x_k) + 1,$$

and so it follows that $d(x_1, x_k) \in \{k - 3, k - 2\}$.

If $d(x_1, x_k) = k - 3$ then $m > 3$ and

$$d(x_1, x_k) + d(x_k, x_{k+1}) + \ldots + d(x_{m-1}, x_m) + d(x_m, x_1) = (k-3) + (m-k+1) = m-2,$$

which is odd, contradicting Lemma 4.3.1.

If $d(x_1, x_k) = k - 2$ then

$$d(x_1, x_2) + \ldots + d(x_{k-1}, x_k) + d(x_k, x_1) = (k - 1) + (k - 2) = 2k - 3,$$

which is odd. From Lemma 4.3.1, it follows that $2k - 3 \geq n + 2$, and so $2k \geq n + 5$. On the other hand,

$$k - 2 = d(x_1, x_k) \leq d(x_k, x_{k+1}) + \ldots + d(x_{m-1}, x_m) + d(x_m, x_1) = m - k + 1,$$

and so $2k \leq n + 3$, which is a contradiction. □

Lemma 4.3.7. *Fix* $a, b \in \mathcal{M}_n$.

(a) *Let* $d(a, b) = m$. *Then there are* $a_0, a_1, \ldots, a_m \in \mathcal{M}_n$ *such that* $a_0 = a$, $a_m = b$, *and* $d(a_i, a_{i+1}) = 1$ *for all* $1 \leq i < m$.

(b) *Let* $m > 0$ *be such that* $m + d(a, b) \geq n + 1$. *Then there are* $a_0, a_1, \ldots, a_m \in \mathcal{M}_n$ *such that* $a_0 = a$, $a_m = b$, *and* $d(a_i, a_{i+1}) = 1$ *for all* $0 \leq i < m$.

Proof. Part (a). Define the $\mathcal{R}_{n_*}$-colored space (P_m, d_m) as follows:

- $P_m = \{a_0, a_1, \ldots, a_m\}$,
- given $0 \leq i \leq j \leq m$, $d_m(a_i, a_j) = d_m(a_j, a_i) = j - i$.

By homogeneity and universality of $\mathcal{M}_n$, it suffices to show (P_m, d_m) is an $\mathcal{F}_n$-free $\mathcal{R}_{n_*}$-metric space. Fix $1 \leq i \leq j \leq k \leq m$. Then $(k-i, j-i, k-j)$ is clearly an $\mathcal{R}_{n_*}$-triangle, and so (P_m, d_m) is an $\mathcal{R}_{n_*}$-metric space. Moreover, $d(a_i, a_j) + d(a_j, a_k) + d(a_i, a_k) = 2k - 2i$, which is even. Therefore (P_m, d_m) is $\mathcal{F}_n$-free.

Part (b). Define the $\mathcal{R}_{n_*}$-colored space (P_m, d_m) as follows:

- $P_m = \{a_0, a_1, \ldots, a_m\}$,
- given $0 \leq i \leq j \leq m$, $d_m(a_i, a_j) = d_m(a_j, a_i) = \min\{j - i, m + d(a, b) - (j - i)\}$.

By construction, we have $d_m(a_i, a_{i+1}) = 1$ for all $0 \leq i < m$. Since $d(a, b) \leq n_*$, we must have $d(a, b) \leq m$, and so we also have $d_m(a_0, a_m) = d(a, b)$. By homogeneity and universality of $\mathcal{M}_n$, it suffices to show (P_m, d_m) is an $\mathcal{F}_n$-free $\mathcal{R}_{n_*}$-metric space. Let f_m be the partial semimetric on P_m obtained by restricting d_m to $\{(x_0, x_m)\} \cup \{(x_i, x_{i+1}) : 0 \leq i < m\}$. Since $d(a, b) \leq m$, we have that f_m is k-transitive for all $k > 0$. Moreover, d_m is precisely the $\mathcal{R}_{n_*}$-metric obtained from f_m as in Lemma 3.6.4. Therefore (P_m, d_m) is an $\mathcal{R}_{n_*}$-metric space, and so it remains to verify (P_m, d_m) is $\mathcal{F}_n$-free. Let $r = d(a, b) + m$ and, given $0 \leq i \leq j \leq m$, let $d_{i,j} = d_m(a_i, a_j)$.

Fix $0 \leq i \leq j \leq k \leq m$, and set $p = d_{i,j} + d_{j,k} + d_{i,k}$. We want to show that if $p \leq n$ then p is even. There are five cases to consider.

Case 1: $d_{i,j} = j - i$, $d_{j,k} = k - j$, $d_{i,k} = k - i$.

Then $p = 2(k - i)$, which is even.

Case 2: $d_{i,j} = j - i$, $d_{j,k} = k - j$, $d_{i,k} = r - (k - i)$.

Then $p = r > n$.

Case 3: $d_{i,j} = j - i$, $d_{j,k} = r - (k - j)$, $d_{i,k} = r - (k - i)$.

Then $p = 2(r - k + j)$, which is even.

Case 4: $d_{i,j} = r - (j - i)$, $d_{j,k} = k - j$, $d_{i,k} = r - (k - i)$.

Then $p = 2(r + i - j)$, which is even.

Case 5: $d_{i,j} = r - (j - i)$, $d_{j,k} = r - (k - j)$, $d_{i,k} = r - (k - i)$.

Then $p = 3r - 2(k - i) > n$. □

Theorem 4.3.8. *Fix an odd integer $n \geq 3$.*

(a) Let $E = \{(a, b) \in \mathcal{M}_n^2 : d(a, b) = 1\}$. Then, as a graph, $(\mathcal{M}_n, E)$ is isomorphic to $\mathcal{G}_n$.

(b) Let d denote the path metric on $\mathcal{G}_n$. Then $(\mathcal{G}_n, d)$ is isometric to $\mathcal{M}_n$.

Proof. Part (a). Since $\mathrm{Th}(\mathcal{G}_n)$ (in the language of graphs) is $\aleph_0$-categorical, it suffices to show $(\mathcal{M}_n, E)$ is an existentially complete $\mathcal{C}_n$-free graph. Note that $(\mathcal{M}_n, E)$ is $\mathcal{C}_n$-free by Lemma 4.3.6. Suppose H is a $\mathcal{C}_n$-free graph, with $\mathcal{M}_n \subseteq H$. Let d_0 be the path metric on H, truncated at n_*.

Claim: Suppose $d_0|_{\mathcal{M}_n} = d$. Then $(\mathcal{M}_n, E)$ is existentially complete in H.

Proof: Fix a finite subset $A \subseteq \mathcal{M}_n$ and some $b \in H \backslash \mathcal{M}_n$. We want to find $c \in \mathcal{M}_n$ such that, given $a \in A$, $d(a, c) = 1$ if and only if $d_0(a, b) = 1$. Let $f : A \longrightarrow R_{n_*}$ such that $f(a) = d_0(a, b)$. Then, since d_0 is an $\mathcal{R}_{n_*}$-metric on H, it follows that f is an $\mathcal{R}_{n_*}$-Katětov map on (A, d_0). By assumption, this means f is an $\mathcal{R}_{n_*}$-Katětov map on (A, d), and so there is some $c \in \mathcal{M}_n$ realizing f. $\dashv_{\text{claim}}$

By the claim, it suffices to show $d_0|_{\mathcal{M}_n} = d$. So fix $a, b \in \mathcal{M}_n$. We clearly have $d_0(a, b) \leq d(a, b)$. Suppose, toward a contradiction, that $d_0(a, b) < d(a, b)$. Let

$m = n - d_0(a,b)$. Then $m + d(a,b) \geq n+1$ and so, by Lemma 4.3.7(b), there are $a_0, a_1, \ldots, a_m \in \mathcal{M}_n$ such that $a_0 = a$, $a_m = b$, and $d(a_i, a_{i+1}) = 1$ for all $0 \leq i < m$. Let γ be the closed walk in H, which travels from a to b along $(a_0, \ldots, a_m)$, and then from b back to a along a path in H of length $d_0(a,b)$. Then γ has length $m + d_0(a,b) = n$, which contradicts Proposition 4.3.5.

Part (b). Consider the unit distance graph $(\mathcal{M}_n, E)$, as in part (a). By part (a), we may fix a graph homomorphism $\varphi : \mathcal{M}_n \longrightarrow \mathcal{G}_n$. To prove part ($b$), we show φ is an isometry from $\mathcal{M}_n$ to $(\mathcal{G}_n, d)$. Fix $a, b \in \mathcal{M}_n$ and let $d(a,b) = m$. By Lemma 4.3.7(a), there are $a_0, a_1, \ldots, a_m \in \mathcal{M}_n$ such that $a_0 = a$, $a_m = b$, and $d(a_i, a_{i+1}) = 1$ for all $0 \leq i < m$. Therefore $d(\varphi(a_i), \varphi(a_{i+1})) = 1$ for all $0 \leq i < m$, and so $d(\varphi(a), \varphi(b)) \leq m$. Suppose, toward a contradiction, $d(\varphi(a), \varphi(b)) = k < m$. Then there are $c_0, c_1, \ldots, c_k \in \mathcal{G}_n$ such that $c_0 = \varphi(a)$, $c_k = \varphi(b)$, and $d(c_i, c_{i+1}) = 1$ for all $0 \leq i < m$. Therefore, $d(\varphi^{-1}(c_i), \varphi^{-1}(c_{i+1})) = 1$ for all $0 \leq i < m$, and so $d(a,b) \leq k$, which is a contradiction. □

4.4 Extending Isometries in Non-Archimedean Cases

In this section, we consider the question of the Hrushovski property for $\mathcal{K}_{\mathcal{R}}$, when $\mathcal{R}$ is non-archimedean. In this case, we have an essential failure when trying to adapt the proof of Theorem 4.2.2. In particular, if $\mathcal{R}$ is non-archimedean then, using notation as in Theorem 4.2.2, the collection Σ, and therefore the class $\mathcal{F}_0$, is necessarily infinite. For example, if $r, s \in R$, with $nr < s$ for all $n > 0$, then, for all $n > 0$, the sequence $(s, r, \ldots, r)$, where r repeats n times, is in Σ.[2]

From this observation, we have the following open question.

Question 4.4.1. Suppose $\mathcal{R}$ is a countable distance monoid. Does the class $\mathcal{K}_{\mathcal{R}}$ of finite $\mathcal{R}$-metric spaces have the Hrushovski property?

The rest of this section is dedicated to proving the Hrushovski property for $\mathcal{K}_{\mathcal{R}}$, for a certain class of "well-behaved" non-archimedean monoids. In particular, this will include ultrametric monoids. Results in this area have been shown for complete, universal Polish ultrametric spaces (e.g. the kind constructed in [33]). Specifically, in [59], Malicki uses similar methods of extending isometries to prove ample generics, automatic continuity, and the small index property for the isometry groups of Polish ultrametric Urysohn spaces. It is interesting to note that the complete Urysohn space does *not* have ample generics. However, automatic continuity has been shown for this space by Sabok [75].

[2] One may also attempt to directly recover the Hrushovski property for $\mathcal{K}_{\mathcal{R}}$ via a generalization of Rosendal's proof of Theorem 4.1.9 in [74], which uses the Ribes-Zalesskiĭ result on the profinite topology on free groups. However, this same set Σ makes an appearance in Rosendal's proof. In particular, there is a union, indexed by Σ, of closed subsets of a particular group, and the proof relies on knowing that this union is still closed.

Our focus will be on the following generalization of the class of archimedean distance monoids.

Definition 4.4.2. Suppose $\mathcal{R} = (R, \oplus, \leq, 0)$ is a distance monoid. Then $\mathcal{R}$ is **semi-archimedean** if, for all $r, s \in R$, if $r \prec_{\mathcal{R}} s$ then $r \oplus s = s$.

Note that any archimedean distance monoid is semi-archimedean. The idea behind this notion is that, while semi-archimedean distance monoids may contain multiple archimedean classes, elements from different classes have "trivial" addition.

Example 4.4.3. Any ultrametric distance monoid is semi-archimedean. Given $n > 0$, if $S = \{0, 1, 3, 5, \dots, 2n-1\}$, then $\mathcal{S} = (S, +_S, \leq, 0)$ is semi-archimedean, but not archimedean or ultrametric.

Our goal is to show that, if $\mathcal{R}$ is semi-archimedean, then $\mathcal{K}_{\mathcal{R}}$ has the Hrushovski property. The proof uses the archimedean case as a base case for an inductive argument. In the induction step, we explicitly build extensions of partial isometries by hand.

Theorem 4.4.4. *If $\mathcal{R} = (R, \oplus, \leq, 0)$ is a semi-archimedean distance monoid, then $\mathcal{K}_{\mathcal{R}}$ has the Hrushovski property.*

Proof. Fix a finite $\mathcal{R}$-metric space A. We want to find a finite $\mathcal{R}$-metric space B such that $A \subseteq B$ and any partial isometry of A extends to a total isometry of B. We may clearly replace $\mathcal{R}$ with the submonoid of $\mathcal{R}$ generated by $\operatorname{Spec}(A)$. Since $\operatorname{Spec}(A)$ is finite, $\mathcal{R}$ has only finitely many archimedean classes.

We proceed by induction on the number n of nontrivial archimedean classes of $\mathcal{R}$. If $n = 1$ then the result follows from Corollary 4.2.4. For the induction hypothesis, suppose $n > 1$ and assume that if $\mathcal{S}$ is a semi-archimedean distance monoid with $n-1$ nontrivial archimedean classes, then $\mathcal{K}_{\mathcal{S}}$ has the Hrushovski property.

By Proposition 3.7.18, we may partition $R = S_1 \cup S_2$ such that S_2 is a single nontrivial archimedean class and $r < s$ for all $r \in S_1$ and $s \in S_2$. Define the distance monoid $\mathcal{S}_1 = (S_1, \oplus, \leq, 0)$, and note that $\mathcal{S}_1$ is semi-archimedean and has $n-1$ nontrivial archimedean classes. Let $\sim$ denote the equivalence relation on A given by

$$x \sim y \;\;\Leftrightarrow\;\; d(x, y) \in S_1.$$

Let $A = A_1 \cup \ldots \cup A_m$ be the partition of A into $\sim$-classes. Note that, for all $1 \leq i \leq m$, A_i is an $\mathcal{S}_1$-metric space.

Claim 1: Given $1 \leq i < j \leq m$, there is $s_{i,j} \in S_2$ such that $d(a, b) = s_{i,j}$ for all $a \in A_i$ and $b \in A_j$.

Proof: Fix $a, a' \in A_i$ and $b, b' \in A_j$. Then $d(a, b), d(a', b') \in S_2$ and $d(a, a'), d(b, b') \in S_1$. Since $\mathcal{R}$ is semi-archimedean, it follows that

$$d(a,b) \leq d(a,a') \oplus d(a',b') \oplus d(b',b) = d(a',b') \leq d(a',a) \oplus d(a,b) \oplus d(b,b') = d(a,b).$$

Therefore $d(a,b) = d(a',b')$. $\dashv_{\text{claim}}$

Claim 2: We may assume A_i and A_j are isometric for all $i, j \leq m$.

Proof: Suppose there are $i \neq j$ such that A_i and A_j are not isometric. We extend A to an $\mathcal{R}$-metric space A_* as follows. Let d be the $\mathcal{R}$-metric on A. Fix $s \in S_1$ such that

$$s = \max(\operatorname{Spec}(A_1) \cup \ldots \cup \operatorname{Spec}(A_m)).$$

We may define an $\mathcal{S}_1$-metric d_0 on A such that, given $x, y \in A$,

$$d_0(x,y) = \begin{cases} d(x,y) & \text{if } a, b \in A_i \text{ for some } i, \\ s & \text{if } x \in A_i,\ y \in A_j \text{ for distinct } i, j. \end{cases}$$

Note that, for all $1 \leq i \leq m$, (A_i, d) is a subspace of (A_i, d_0). Therefore, we may extend each A_i to an $\mathcal{S}_1$-metric space A_i^* such that each A_i^* is isometric to (A, d_0), and $A_i^* \cap A_j^* = \emptyset$ for $i \neq j$. Now set $A^* = A_1^* \cup \ldots \cup A_m^*$. Given $1 \leq i < j \leq m$, and $x \in A_i^*$, $y \in A_j^*$, set $d(x,y) = s_{i,j}$. Then d is an $\mathcal{R}$-metric on A^*, and A is a subspace of A^*. Moreover, $\{A_1^*, \ldots, A_m^*\}$ are the $\sim$-equivalence classes of A^* and, by construction, each pair of classes is isometric. $\dashv_{\text{claim}}$

By Claim 2, we may assume A_i and A_j are isometric for all $1 \leq i, j \leq m$. Define the distance monoid $\mathcal{S}_2 = (S_2 \cup \{0\}, \oplus, \leq, 0)$, and note that $\mathcal{S}_2$ is archimedean. Let $E = \{\alpha_1, \ldots, \alpha_m\}$ be an m-element set. By Claim 1, we may define an $\mathcal{S}_2$-metric on E such that, given $1 \leq i < j \leq m$, $d(\alpha_i, \alpha_j) = s_{i,j}$. By Corollary 4.2.4, there is an $\mathcal{S}_2$-metric space F such that $E \subseteq F$ and any partial isometry of E extends to a total isometry of F.

Let $F = \{\alpha_1, \ldots, \alpha_p\}$, for some $m \leq p$. Define $A_* = A \cup A_{m+1} \cup \ldots \cup A_p$ where, for $m < i \leq p$, each A_i is a disjoint isometric copy of A_1. Extend the $\mathcal{R}$-metric on A to A_* by setting $d(a,b) = d(\alpha_i, \alpha_j)$, where $1 \leq i < j \leq p$ and $a \in A_i$, $b \in A_j$. Note that, by construction and Claim 1, this does not conflict with the original values of d on A. To verify the triangle inequality, fix $x, y, z \in A_*$. If x, y, and z are all in the same A_i, or each in a distinct A_i, then the triangle inequality follows from the fact that A_i and F are $\mathcal{R}$-metric spaces. So we may assume $x, y \in A_i$ and $z \in A_j$ for some $i \neq j$. Then $d(x,z) = d(y,z) \in S_2$ and $d(x,y) \in S_1$, and so $(d(x,y), d(y,z), d(x,z))$ is clearly an $\mathcal{R}$-triangle.

Given $1 \leq i \leq j \leq p$, fix an isometry $\theta_{i,j} : A_i \longrightarrow A_j$. By induction, there is an $\mathcal{S}_1$-metric space B_1 such that $A_1 \subseteq B_1$ and any partial isometry of A_1 extends to a partial isometry of B_1. Given $1 < i \leq p$, we define an $\mathcal{S}_1$-metric space B_i as follows.

Let $l = |B_1 \backslash A_1| \in \mathbb{N}$, and let $\{b_1, \ldots, b_l\}$ be an enumeration of $B_1 \backslash A_1$. Let B_i be a set, with $B_i = A_i \cup \{b_1^i, \ldots, b_l^i\}$. Assume $B_i \cap B_j = \emptyset$ for all $1 \leq i < j \leq m$. Extend the metric on A_i to B_i so that $d(b_u^i, b_v^i) = d(b_u, b_v)$ and, given $a \in A_i$, $d(b_u^i, a) = d(b_u, \theta_{1,i}^{-1}(a))$. In particular, if $\hat{\theta}_{1,i} = \theta_{1,i} \cup \{(b_1, b_1^i), \ldots, (b_l, b_l^i)\}$, then $\hat{\theta}_{1,i}$ is an isometry from B_1 to B_i.

Finally, we set $B = B_1 \cup \ldots \cup B_p$. We extend the metrics defined on each B_i to all of B by setting $d(x,y) = d(\alpha_i, \alpha_j)$ where $i \neq j$, $x \in B_i$ and $y \in B_j$. This gives well-defined metric on B by the same argument for A_* above. Note that A_* is a subspace of B, and so A is also a subspace of B. We fix a partial isometry φ of A, and show φ extends to a total isometry of B.

Let $I = \{1 \leq i \leq m : \mathrm{dom}(\varphi) \cap A_i \neq \emptyset\}$.

Claim 3: Given $i \in I$, there is a unique $i' \in \{1, \ldots, m\}$ such that $\varphi(A_i) \cap A_{i'} \neq \emptyset$.

Proof: First, since $\mathrm{dom}(\varphi) \cap A_i \neq \emptyset$ and φ is a partial isometry of A, there is some $i' \in \{1, \ldots, m\}$ such that $\varphi(A_i) \cap A_{i'} \neq \emptyset$. Suppose we have $a, b \in A_i \cap \mathrm{dom}(\varphi)$ such that $\varphi(a) \in A_j$ and $\varphi(b) \in A_k$ for some $j \neq k$. Then $d(\varphi(a), \varphi(b)) \in S_2$ and $d(a, b) \in S_1$, which contradicts that φ is a partial isometry. $\dashv_{\text{claim}}$

By Claim 3, we may define a function $f : I \longrightarrow \{1, \ldots, m\}$ such that $f(i)$ is the unique element of $\{1, \ldots, m\}$ satisfying the condition $\varphi(A_i) \cap A_{f(i)} \neq \emptyset$. By a similar argument as in the proof of Claim 3, it follows that f is injective. Define a partial function $\hat{f} : E \longrightarrow E$ such that $\mathrm{dom}(\hat{f}) = \{\alpha_i : i \in I\}$ and $\hat{f}(\alpha_i) = \alpha_{f(i)}$.

Claim 4: $\hat{f}$ is a partial isometry of E.

Proof: We clearly have that $\hat{f}$ is injective. Fix distinct $i, j \in I$. We want to show $d(\alpha_i, \alpha_j) = d(\alpha_{f(i)}, \alpha_{f(j)})$. We have $\varphi(a_i) \in A_{f(i)}$ and $\varphi(a_j) \in A_{f(j)}$, which means

$$d(\alpha_{f(i)}, \alpha_{f(j)}) = d(\varphi(a_i), \varphi(a_j)) = s_{i,j} = d(\alpha_i, \alpha_j),$$

as desired. $\dashv_{\text{claim}}$

By Claim 4, we may extend $\hat{f}$ to a total isometry ψ of F. Let $f_* : \{1, \ldots, p\} \longrightarrow \{1, \ldots, p\}$ such that $\psi(\alpha_i) = \alpha_{f_*(i)}$.

Claim 5: f_* is a bijection extending f.

Proof: The fact that f_* is a bijection follows from the fact that ψ is a bijection. Given $i \in I$, we have

$$\alpha_{f_*(i)} = \psi(\alpha_i) = \hat{f}(\alpha_i) = \alpha_{f(i)},$$

and so $f_*(i) = f(i)$. $\dashv_{\text{claim}}$

Let $J = \{1, \ldots, p\} \backslash I$. Given $i \in I$, let $\varphi_i = \varphi|_{A_i}$. In particular, note that $\varphi = \bigcup_{i \in I} \varphi_i$. Given $i \in J$, let $\varphi_i = \theta_{i, f_*(i)}$. Since f_* extends f, we have that, for all $1 \leq i \leq p$, φ_i is a partial isometry from A_i to $A_{f_*(i)}$. Define

$$\hat{\varphi} = \bigcup_{i=1}^{p} \varphi_i.$$

Claim 6: $\hat{\varphi}$ is a partial isometry of A_*, which extends φ.

Proof: We clearly have that $\hat{\varphi}$ extends φ. So we only need to show $\hat{\varphi}$ is a well-defined partial isometry of A_*. By construction, we have $\mathrm{dom}(\hat{\varphi}) \cup \mathrm{Im}(\hat{\varphi}) \subseteq A_*$. So it remains to show $\hat{\varphi}$ is a well-defined isometry from $\mathrm{dom}(\hat{\varphi})$ to $\mathrm{Im}(\hat{\varphi})$. Since $\{\mathrm{dom}(\varphi_1), \ldots, \mathrm{dom}(\varphi_i)\}$ is a partition of $\mathrm{dom}(\hat{\varphi})$, and each φ_i is a partial isometry, we only need to show $\mathrm{Im}(\varphi_i) \cap \mathrm{Im}(\varphi_j) \neq \emptyset$ for all $i \neq j$. This follows from the fact that f_* is a bijection. $\dashv_{\text{claim}}$

To finish the proof, we extend φ to a total isometry φ_* of B. Fix $i \in \{1, \ldots, p\}$ and define $\chi_i = \theta^{-1}_{1, f_*(i)} \circ \varphi_i \circ \theta_{1,i}$. Then χ_i is a partial isometry of A_1, and so χ_i

extends to a total isometry $\hat{\chi}_i$ of B_1. Set $\hat{\varphi}_i = \hat{\theta}_{1,f_*(i)} \circ \hat{\chi}_i \circ \hat{\theta}^{-1}_{1,i}$. Then $\hat{\varphi}_i$ is a total isometry from B_i to $B_{f_*(i)}$. Set

$$\varphi_* = \bigcup_{i=1}^{p} \hat{\varphi}_i.$$

Since f_* is a permutation of $\{1,\dots,p\}$ and $\{B_1,\dots,B_p\}$ is a partition of B, it follows that $\varphi_* : B \longrightarrow B$ is a well-defined bijection. We show φ_* is an isometry extending φ.

To verify φ_* is an isometry, fix $x, y \in B$. We may assume $x \in B_i$ and $y \in B_j$ for some $i \neq j$. Then $\varphi_*(x) \in B_{f_*(i)}$, $\varphi_*(y) \in B_{f_*(j)}$, and

$$d(x,y) = d(\alpha_i,\alpha_j) = d(\psi(\alpha_i),\psi(\alpha_j)) = d(\alpha_{f_*(i)},\alpha_{f_*(j)}) = d(\varphi_*(x),\varphi_*(y)).$$

Finally, we show φ_* extends φ. Fix $x \in \mathrm{dom}(\varphi)$. Then there is some $i \in I$ such that $x \in \mathrm{dom}(\varphi_i)$. Since $\mathrm{dom}(\varphi_i) \subseteq A_i \subseteq B_i$, we have

$$\varphi_*(x) = \hat{\varphi}_i(x) = \hat{\theta}_{1,f_*(i)}(\hat{\chi}_i(\hat{\theta}^{-1}_{1,i}(x))).$$

Since $x \in A_i = \mathrm{dom}(\theta^{-1}_{1,i})$ we have

$$\varphi_*(x) = \hat{\theta}_{1,f_*(i)}(\hat{\chi}_i(\theta^{-1}_{1,i}(x))).$$

Since $x \in \mathrm{dom}(\varphi_i)$, we have $\theta^{-1}_{1,i}(x) \in \mathrm{dom}(\chi_i)$, and so

$$\varphi_*(x) = \hat{\theta}_{1,f_*(i)}(\chi_i(\theta^{-1}_{1,i}(x))) = \hat{\theta}_{1,f_*(i)}(\theta^{-1}_{1,f_*(i)}(\varphi_i(x))) = \varphi_i(x) = \varphi(x). \qquad \square$$

Corollary 4.4.5. *If $\mathcal{R}$ is a countable semi-archimedean distance monoid, then* $\mathrm{Isom}(\mathcal{U}_\mathcal{R})$ *has ample generics.*

We now describe an "operation" on distance monoids, which will be useful for later results, and is related to how one obtains semi-archimedean monoids from archimedean ones. Roughly speaking, given a sequence of distance monoids $(\mathcal{R}_i)_{i\in I}$, where I is a linear order, we construct a new distance monoid, denoted $[\![\mathcal{R}_i]\!]_{i\in I}$, by concatenating the nonzero elements of each $\mathcal{R}_i$, and defining addition between distinct $\mathcal{R}_i$ and $\mathcal{R}_j$ to coincide with the max operation. We give the formal definition.

Definition 4.4.6. Fix a linear order I, and suppose $(\mathcal{R}_i)_{i\in I}$ is a sequence of distance monoids, with $\mathcal{R}_i = (R_i, \oplus_i, \leq_i, 0)$. Define a distance monoid $[\![\mathcal{R}_i]\!]_{i\in I} = (R, \oplus, \leq, 0)$, as follows:

(*i*) $R = \{0\} \cup \bigcup_{i\in I}(R_i^{>0} \times \{i\})$;

(*ii*) given distinct $i, j \in I$ and $r \in R_i^{>0}$, $s \in R_j^{>0}$, $(r,i) \leq (s,j)$ if and only if $i < j$ or $i = j$ and $r \leq_i s_i$.

(*iii*) given $i \in I$ and $r, s \in R_i^{>0}$, $(r, i) \oplus (s, i) = (r \oplus_i s, i)$;

(*iv*) given distinct $i, j \in I$ and $r \in R_i^{>0}$, $s \in R_j^{>0}$, $(r, i) \oplus (s, i) = \max\{(r, i), (s, i)\}$.

In particular, if $\mathcal{R}$ is semi-archimedean, then $\mathcal{R} \cong [\![\mathcal{R}_i]\!]_{i \in I}$ for some linear order I and sequence $(\mathcal{R}_i)_{i \in I}$ of archimedean monoids. From the proof of Theorem 4.4.4, we obtain the following corollary.

Corollary 4.4.7. *Suppose $\mathcal{R}_i$ and $\mathcal{R}_j$ are distance monoids such that $\mathcal{K}_{\mathcal{R}_i}$ and $\mathcal{K}_{\mathcal{R}_j}$ have the Hrushovski property. If $\mathcal{R} = [\![\mathcal{R}_i, \mathcal{R}_j]\!]$ then $\mathcal{K}_{\mathcal{R}}$ has the Hrushovski property.*

Note that the class of semi-archimedean monoids is closed under this bracketing operation. Therefore, in order to use this result to obtain the Hrushovski property for more monoids, one would need to first demonstrate a non-semi-archimedean monoid with Hrushovski property. In particular, up to isomorphism, the smallest distance monoid, which is not semi-archimedean, is $\mathcal{S} = (\{0, 1, 3, 4\}, +_{\mathcal{S}}, \leq, 0)$ (this claim is justified in Chapter 5). As such, $\mathcal{S}$ is the smallest distance monoid for which we have not settled the Hrushovski property.

Finally, we introduce the following notation. Recall that $\mathcal{R}_1$ denotes the unique distance monoid with one nontrivial element.

Definition 4.4.8. Suppose $\mathcal{R}$ is a distance monoid.

1. Let $\mathcal{R}^\infty$ denote $[\![\mathcal{R}, \mathcal{R}_1]\!]$.
2. Let $\mathcal{R}_\epsilon$ denote $[\![\mathcal{R}_1, \mathcal{R}]\!]$.

The monoids $\mathcal{R}^\infty$ and $\mathcal{R}_\epsilon$ should be thought of, respectively, as the result of adding an infinite element or a positive infinitesimal element to $\mathcal{R}$.

Chapter 5

Combinatorics of Finite Distance Monoids

The work in this chapter was originally motivated by Appendix A of [69], in which Nguyen Van Thé classifies the subsets $S \subseteq \mathbb{Z}^+$ such that $|S| \leq 4$ and $S \cup \{0\}$ satisfies the four-values condition in $(\mathbb{R}^{\geq 0}, +, \leq, 0)$. This is used to verify that, given such an S, if $\mathcal{S} = (S, +_S, \leq, 0)$ (see Example 3.1.1(5), then $\mathcal{U}_{\mathcal{S}}$ is *indivisible*, i.e., for any 2-coloring of $\mathcal{U}_{\mathcal{S}}$, there is a monochromatic subset isometric to $\mathcal{U}_{\mathcal{S}}$. The same result was later proved for any finite S by Sauer in [76].

Our interest in Nguyen Van Thé's work is motivated by questions surrounding the enumerative behavior of finite distance monoids. This includes finite distance monoids of a fixed archimedean complexity k, as the enumeration of such objects can be linked to the asymptotic model theoretic behavior of $\mathcal{U}_{\mathcal{R}}$, as the size of $\mathcal{R}$ grows.

The results of in this chapter are a mixture of fairly straightforward observations and raw data obtained from a computer program. Therefore, this chapter is mostly meant to set the stage for further combinatorial study, including the formulation of several interesting conjectures. We also note some surprising connections to other topics in additive and algebraic combinatorics.

In Section 5.1, we give an upper bound for the number of finite distance monoids by showing that finite distance *magmas* are in bijective correspondence with alternating sign matrices. We also provide a lower bound. In Section 5.2, we focus on distance monoids of the form $\mathcal{S} = (S, +_S, \leq, 0)$, where $S \subseteq \mathbb{R}^{\geq 0}$ is finite and satisfies the four-values condition in $(\mathbb{R}^{\geq 0}, +, \leq, 0)$. We prove that, without loss of generality, it suffices to assume $S \subseteq \mathbb{N}$. Moreover, we classify, up to isomorphism, the monoids $\mathcal{S}$ in which S is an arithmetic progression. In Section 5.3, we exhibit a distance monoid, which has 8 nontrivial elements and is not isomorphic to $(S, +_S, \leq, 0)$ for any $S \subseteq \mathbb{N}$. The origins of this counterexample motivate questions on the connections between monoids of the form $(S, +_S, \leq, 0)$ and totally, positively ordered commutative monoids, which are *formally integral* (see Definition 5.3.4).

The final sections of this chapter together classify all distance monoids with at most 6 nontrivial elements. We also show that, given $n \geq 3$, there are, modulo isomorphism, exactly $2n-2$ distance monoids, with n nontrivial elements and archimedean complexity $n-1$.

Throughout the chapter, we use the following notation. Given an ordered set $(X, <)$, and some finite subset $A \subseteq X$, we write $A = \{a_1, \ldots, a_n\}_<$ to mean $A = \{a_1, \ldots, a_n\}$ and $a_1 < a_2 < \ldots < a_n$.

5.1 Finite Distance Monoids

Definition 5.1.1. Fix $n > 0$.

1. Let $\mathrm{DM}(n)$ be the number, modulo isomorphism, of distance monoids with n nontrivial elements.
2. Given $k > 0$, let $\mathrm{DM}(n, k)$ be the number, modulo isomorphism, of distance monoids with n nontrivial elements and archimedean complexity k.

Recall that, in Section 3.7.4, we showed that $\mathrm{DM}(n, 1) = 1 = \mathrm{DM}(n, n)$ for any $n > 0$. Moreover, $\mathrm{DM}(n, k) = 0$ for any $k > n > 0$.

Note that the isomorphism type of a finite distance monoid $\mathcal{R}$ is completely determined by the inequalities $a \leq b \oplus c$, where $a, b, c \in R^{>0}$. Therefore, a weak upper bound for $\mathrm{DM}(n)$ is 2^{n^3}. It seems that not much has been done regarding analysis of the sequence $(\mathrm{DM}(n))_{n=1}^{\infty}$. In particular, an explicit expression is unknown. However, we can give a better upper bound through the following exact enumeration of finite distance magmas. The argument is elementary, modulo the solution to the *Alternating Sign Matrix Conjecture*, which was a famous open problem (and is now a famous theorem) in algebraic combinatorics.

Definition 5.1.2. An **alternating sign matrix** is a square matrix A such that

(*i*) each entry of A is in $\{0, 1, \text{-}1\}$,

(*ii*) the sum of the entries in any row or column of A is 1,

(*iii*) the nonzero entries in any row or column of A alternate in sign.

Alternating sign matrices, which are a generalization of permutation matrices, arise naturally in the *Dodgson condensation* method of evaluating determinants (see [11]). They were defined by Mills, Robbins, and Rumsey in [65], where the authors also state the following conjecture.

Alternating Sign Matrix Conjecture (1982 [65])**.** *Given $n > 0$, the number of $n \times n$ alternating sign matrices is*

$$R(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

This conjecture was proved to be true by Zeilberger [95] in 1992, and we will use this solution to enumerate finite distance magmas. The sequence $(R(n))_{n=1}^{\infty}$ is known as the sequence of *Robbins numbers*. Much like Catalan numbers, Robbins numbers have been shown to enumerate a rich class of combinatorial objects. Some examples include monotone triangles of order n, descending plane partitions of order n, totally symmetric self-complementary plane partitions of order $2n$, $n \times n$ corner-sum matrices, $n \times n$ tilings by "baskets and gaskets", and $n \times n$ arrays of "square ice" (see [11], [42], [71]). A particular example, which is of importance to us, is the class of "magog triangles" of order n.

Definition 5.1.3. Given $n > 0$, a **magog triangle of order** n is an $n \times n$ lower-triangular matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ such that

(*i*) $a_{i,j} \in \{0, 1, \dots, n\}$,

(*ii*) if $1 \leq j \leq i \leq n$ then $1 \leq a_{i,j} \leq j$,

(*iii*) the nonzero entries in any row or column of A are nondecreasing.

Magog triangles were named by Zeilberger, who also referred to monotone triangles as "gog triangles". Previous work had uncovered explicit bijections between $n \times n$ alternating sign matrices and gog triangles of order n, and also between totally symmetric self-complementary plane partitions of order $2n$ and magog triangles of order n. In [95], Zeilberger shows that gog triangles of order n and magog triangles of order n are both enumerated by the Robbins numbers.[1]

We will enumerate distance magmas by demonstrating an explicit bijection with magog triangles. The construction is straightforward, but we have not found this exact formulation in previous literature. The essential idea is that, after a few minor translations, magog triangles describe the addition matrices of distance magmas.

Theorem 5.1.4. *Given $n > 0$, the set of distance magmas, with n nontrivial elements, is in bijective correspondence with the set of magog triangles of order n.*

Proof. Fix $n > 0$ and let $\mathrm{Mag}(n)$ denote the set of magog triangles of order n. Given an $n \times n$ matrix X, we let $X(i,j)$ denote the (i,j) entry of X.

Suppose $\mathcal{R} = (R, \oplus, \leq, 0)$ is a distance magma with n nontrivial elements. Enumerate $R = \{0, r_1, \dots, r_n\}_<$. Define the $n \times n$ matrix $P(\mathcal{R})$ by

$$P(\mathcal{R})(i,j) = \begin{cases} 0 & \text{if } i < j \\ k & \text{if } j \leq i \text{ and } r_i \oplus r_j = r_k. \end{cases}$$

In other words, $P(\mathcal{R})$ is a particular representation of the addition matrix of $\mathcal{R}$. Using the axioms of distance magmas, it is easy to see that $P(\mathcal{R})$ is an $n \times n$ lower triangular matrix satisfying the following properties:

[1] A current open problem asks for an explicit bijection between these two families of objects.

(i) $P(\mathcal{R})(i,j) \in \{0,1,\ldots,n\}$,

(ii) if $1 \leq j \leq i \leq n$ then $j \leq P(\mathcal{R})(i,j)$,

(iii) the nonzero entries in any row or column of $P(\mathcal{R})$ are nondecreasing.

Let $\mathcal{A}(n)$ be the family of $n \times n$ lower triangular matrices satisfying properties (i) through (iii). Let $\mathcal{D}(n)$ be the family of distance magmas with n nontrivial elements (modulo isomorphism). Then we have that $P : \mathcal{D}(n) \longrightarrow \mathcal{A}(n)$ is a well-defined function, and it is easy to see P is injective. Moreover, given $X \in \mathcal{A}(n)$, define the structure $\mathcal{R} = (R, \oplus, \leq, 0)$ such that

(i) $R = \{0, r_1, \ldots, r_n\}_<$,

(ii) $r_k \oplus 0 = r_k = 0 \oplus r_k$ for all k,

(iii) $r_i \oplus r_j = r_{X(i,j)} = r_j \oplus r_i$ for all $1 \leq j \leq i \leq n$.

Then $\mathcal{R} \in \mathcal{D}(n)$ and $P(\mathcal{R}) = X$. Therefore P is a bijection.

To finish the proof, we construct a bijection from $\mathcal{A}(n)$ to $\mathrm{Mag}(n)$. Let σ be the permutation of $\{0,1,\ldots,n\}$ such that $\sigma(i) = \text{-}i \pmod{n+1}$. Define $f : \mathcal{A}(n) \longrightarrow \mathrm{Mag}(n)$ such that

$$f(X)(i,j) = \sigma(X(\sigma(j), \sigma(i))).$$

It is straightforward to verify f is a well-defined bijection. □

Corollary 5.1.5. *Given $n > 0$, the number of distance magmas with n nontrivial elements is*

$$R(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

Therefore $\mathrm{DM}(n) \leq R(n)$.

The final results of this section concern upper and lower bounds for the asymptotic growth of $\mathrm{DM}(n)$.

Definition 5.1.6. Fix functions $f : \mathbb{Z}^+ \longrightarrow \mathbb{R}^+$ and $g : \mathbb{Z}^+ \longrightarrow \mathbb{R}^+$.

1. $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

2. $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

3. $f(n) = O(g(n))$ if there are $c \in \mathbb{R}^{>0}$ and $n_0 > 0$ such that $f(n) \leq cg(n)$ for all $n > n_0$.

4. $f(n) = \Omega(g(n))$ if there are $c \in \mathbb{R}^{>0}$ and $n_0 > 0$ such that $f(n) \geq cg(n)$ for all $n > n_0$.

Quoting [42], Stirling's formula yields the logarithmic asymptotic approximation $\log R(n) \sim kn^2$, where $k = \log\left(\sqrt{\frac{27}{16}}\right) \approx 0.262$. The sequence $(R(n))_{n=1}^{\infty}$ is OEIS sequence A005130, whose entry includes an asymptotic approximation of $R(n)$ due to R. W. Gosper. In particular, $R(n) \sim a\frac{c^{n^2}}{n^b}$, where $c = \sqrt{\frac{27}{16}}$, $b = \frac{5}{36}$ and

$$a = 2^{\frac{5}{12}}\Gamma(\tfrac{1}{3})^{-\frac{2}{3}}\left(\pi e^{\zeta(1,-1)}\right)^{\frac{1}{3}}.$$

Altogether, we have the asymptotic upper bound $\mathrm{DM}(n) = O\left(\frac{c^{n^2}}{n^b}\right)$.

Concerning an explicit enumeration of $\mathrm{DM}(n)$, it appears not much is known. In particular, the sequence $\mathrm{DM}(n)$ is OEIS sequence A030453, whose entry only contains the first 13 terms. The following proposition provides a method for computing lower bounds for $\mathrm{DM}(n)$.

Proposition 5.1.7. *Given an integer $k > 0$, let $f(k) = \sqrt[k]{\mathrm{DM}(k)}$. For any fixed $k > 0$, $\mathrm{DM}(n) = \Omega(f(k)^n)$.*

Proof. We will use the observation that $\mathrm{DM}(n)$ is an increasing function. For example, given $n > 0$, the function $\mathcal{R} \mapsto \mathcal{R}^{\infty}$ is a injection from distance monoids with n nontrivial elements to distance monoids with $n+1$ nontrivial elements.

Fix an integer $k > 0$ and define $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that $g(n) = \left\lfloor \frac{n}{k} \right\rfloor$. We show that, for all $n > 0$, $\mathrm{DM}(n) \geq \mathrm{DM}(k)^{g(n)}$. Fix $n > 0$ and let $(m_i)_{i \leq g(n)}$ be a sequence of integers such that $m_i \geq k$ for all $i \leq g(n)$, and $\sum_{i \leq g(n)} m_i = n$. We consider distance monoids $\mathcal{R}$, with n nontrivial elements, such that $\mathcal{R} \cong [\![\mathcal{R}_i]\!]_{i \leq g(n)}$, and each $\mathcal{R}_i$ has m_i nontrivial elements. It is clear that distinct sequences $(\mathcal{R}_i)_{i \leq g(n)}$ will yield non-isomorphic monoids $\mathcal{R}$. Since there are $\mathrm{DM}(m_i)$ choices for each $\mathcal{R}_i$, we have $\mathrm{DM}(n) \geq \prod_{i \leq g(n)} \mathrm{DM}(m_i) \geq \mathrm{DM}(k)^{g(n)}$. Finally, note that $\mathrm{DM}(k)^{g(n)} \geq \frac{1}{\mathrm{DM}(k)} f(k)^n$, and so we have $\mathrm{DM}(n) = \Omega(f(k)^n)$. □

We can use the previous result, together with explicit calculations of $\mathrm{DM}(k)$, to increase the lower bound of $\mathrm{DM}(n)$. For example, in Section 5.6, we will explicitly show $\mathrm{DM}(4) = 22$, and so $\mathrm{DM}(n) = \Omega(b^n)$, where $b = f(4) \approx 2.16$. The largest value of $\mathrm{DM}(n)$ given in OEIS sequence A030453 is $\mathrm{DM}(13) = 382549464$. Therefore $\mathrm{DM}(n) = \Omega(b^n)$, where $b = f(13) \approx 4.57$.

Based on all of the above information, we make the following conjecture.

Conjecture 5.1.8.

(*a*) $\mathrm{DM}(n) = o(R(n))$.

(*b*) $\mathrm{DM}(n) = \Omega(b^n)$ *for any* $b > 0$.

(*c*) $\mathrm{DM}(n) = O(n^n)$.

Note that, by Proposition 5.1.7, part (b) of the previous conjecture is equivalent to the statement that the function $f(k)$ diverges as k tends to infinity. Moreover, if part (c) were true then, given the asymptotic approximation of $R(n)$ above, part (a) would follow.

5.2 Integral Distance Monoids

We now turn our focus to a subclass of distance monoids, which were among the original examples motivating the study of metric spaces over arbitrary monoids (see Example 2.1.4). Given a subset $S \subseteq \mathbb{R}^{\geq 0}$, such that $0 \in S$ and S is closed under $r +_S s = \sup\{x \in S : x \leq r + s\}$, we let $\mathcal{S}$ denote the distance magma $(S, +_S, \leq, 0)$. Although the additive structure of $\mathcal{S}$ can be unpredictable, these monoids are a natural choice of focus when considering classical metric spaces over a restricted set of distances (i.e. $\mathcal{K}^S_{\mathcal{R}}$, where $\mathcal{R} = (\mathbb{R}^{\geq 0}, +, \leq, 0)$). Indeed, note that any $\mathcal{S}$-metric space is still a metric space over $(\mathbb{R}^{\geq 0}, +, \leq, 0)$.

The first observation of this section shows that, in the case that S is finite, there is no loss in only considering subsets of integers.

Proposition 5.2.1. *Suppose $S \subseteq \mathbb{R}^{\geq 0}$ is finite, with $0 \in S$. Then there is $S' \subseteq \mathbb{N}$, with $0 \in S'$, such that $\mathcal{S}$ is isomorphic to $\mathcal{S}'$.*

Proof. It suffices to find $S' \subseteq \mathbb{Q}^{\geq 0}$ such that $0 \in S'$ and $\mathcal{S} \cong \mathcal{S}'$. Indeed, given such an S', we then replace S' with kS', where $k > 0$ is chosen so that $kS' \subseteq \mathbb{N}$.

Let $S = \{0, s_1, \ldots, s_n\}_<$. Define sets

$$I = \{(i, j, k) : s_i \leq s_j + s_k\} \text{ and } J = \{(i, j, k) : s_i > s_j + s_k\}.$$

Consider the first-order formula

$$\theta(\bar{v}) := \bigwedge_{1 \leq i \leq n} v_i > 0 \wedge \bigwedge_{1 \leq i < j \leq n} v_i < v_j \wedge \bigwedge_{(i,j,k) \in I} v_i \leq v_j + v_k \wedge \bigwedge_{(i,j,k) \in J} v_i > v_j + v_k.$$

Then $(s_1, \ldots, s_n)$ witnesses that $(\mathbb{R}, +, \leq, 0) \models \exists \bar{v} \theta(\bar{v})$. By quantifier elimination for ordered divisible abelian groups (i.e. $\mathrm{Th}(\mathbb{Q}, +, \leq, 0)$, see [62]), it follows that there are $t_1, \ldots, t_n \in \mathbb{Q}^{>0}$ such that $(\mathbb{Q}, +, \leq, 0) \models \theta(t_1, \ldots, t_n)$. Setting $S' = \{0, t_1, \ldots, t_n\}$, we then have $\mathcal{S} \cong \mathcal{S}'$, as desired. □

Definition 5.2.2. A distance magma $\mathcal{R}$ is **integral** if it is isomorphic to a magma of the form $\mathcal{S} = (S, +_S, \leq, 0)$, where $0 \in S \subseteq \mathbb{N}$. If, moreover, $+_S$ is associative, then $\mathcal{R}$ is an **integral distance monoid**.

Recall that, if $\mathcal{S} = (S, +_S, \leq, 0)$ is an integral distance magma, then $+_S$ is associative if and only if S satisfies the four-values condition in $(\mathbb{R}^{\geq 0}, +, \leq, 0)$ (see Section 2.7). As our focus will be on subsets of $S \subseteq \mathbb{R}^{\geq 0}$, and the induced operation $+_S$, the reader should assume that when we say "four-values condition", we mean with respect to $(\mathbb{R}^{\geq 0}, +, \leq, 0)$. We also note the following observation.

Proposition 5.2.3. *Suppose $S \subseteq \mathbb{R}^{\geq 0}$, with $0 \in S$. If $I \subseteq S^{>0}$ is convex then $S' := I \cup \{0\}$ satisfies the four-values condition.*

Proof. It is straightforward to see that if $+_S$ associative then $+_{S'}$ is associative. □

Next, we note that finite ultrametric monoids are integral.

Proposition 5.2.4. *Suppose $\mathcal{R} = (R, \max, \leq, 0)$, where $(R, \leq, 0)$ is a finite linear order with least element 0. Then $\mathcal{R}$ is an integral distance monoid.*

Proof. Suppose $R = \{0, r_1, \dots, r_n\}_<$. If $S = \{0, 1, 3, 7, \dots, 2^n - 1\}$, then $\mathcal{R} \cong \mathcal{S}$. □

This observation motivates an interesting line of questioning concerning strictly increasing functions $g : \mathbb{Z}^+ \longrightarrow \mathbb{R}^+$ such that $\{0\} \cup g(\mathbb{Z}^+)$ satisfies the four-values condition. More specifically, let

$$\mathrm{FV}(g(n)) = \{n > 0 : \{0, g(1), \dots, g(n)\} \text{ satisfies the four-values condition}\}.$$

Then $\mathrm{FV}(g(n))$ is an initial segment of $\mathbb{Z}^+$ by Proposition 5.2.3. Define $\mathrm{fv}(g(n)) = \max \mathrm{FV}(g(n)) \in \mathbb{Z}^+ \cup \{\infty\}$. In particular, we have already shown $\mathrm{fv}(n) = \infty$ and $\mathrm{fv}(2^n - 1) = \infty$. For amusement, the reader may try the following exercises:

1. $\mathrm{fv}(2^{n-1}) = 2$;
2. $\mathrm{fv}(n^2) = 5$;
3. Given $k > 2$,
$$\mathrm{fv}(n^k) = \left\lceil \frac{1}{2^{\frac{1}{k}} - 1} \right\rceil + 1 \sim \frac{k}{\ln 2} + 1$$

Our final remarks on this particular issue will address the case when $g(n)$ enumerates an arithmetic progression.

Definition 5.2.5. Fix $m, n, r \in \mathbb{Z}^+$ and define

$$S_n(r, m) = \{r + tm : 0 \leq t \leq n - 1\} \cup \{0\}$$
$$S(r, m) = \{r + tm : t \in \mathbb{N}\} \cup \{0\}.$$

Proposition 5.2.6. *Fix $m, r \in \mathbb{Z}^+$.*

(a) $S(r, m)$ and $S_n(r, m)$, for all $n > 0$, satisfy the four-values condition.

(b) Fix $n > 0$ and let $k = \min\{n - 1, \lfloor \frac{r}{m} \rfloor\}$.

(i) If $k = 0$ then $\mathcal{S}_n(r, m) \cong \mathcal{S}_n(1, 2)$.

(ii) If $k > 0$ then $\mathcal{S}_n(r, m) \cong \mathcal{S}_n(k, 1)$.

Proof. Part (a). By Proposition 5.2.3, it suffices to show $S(r,m)$ satisfies the four-values condition. Let $S=S(r,m)$. We verify $+_S$ is associative on S. Fix $s,t\in\mathbb{N}$. For any $u\in\mathbb{N}$,

$$r+um\le(r+sm)+(r+tm)\ \Leftrightarrow\ um\le r+(s+t)m\ \Leftrightarrow\ u\le\lfloor\tfrac{r}{m}\rfloor+s+t.$$

It follows that $(r+sm)+_S(r+tm)=r+(\lfloor\frac{r}{m}\rfloor+s+t)m$.

Therefore, given $s,t,u\in\mathbb{N}$, we have

$$\begin{aligned}((r+sm)+_S(r+tm))+_S(r+um)&=(r+(\lfloor\tfrac{r}{m}\rfloor+s+t)m)+_S(r+um)\\&=r+(2\lfloor\tfrac{r}{m}\rfloor s+t+u)m\\&=(r+sm)+_S(r+(\lfloor\tfrac{r}{m}\rfloor+t+u)m)\\&=(r+sm)+_S((r+tm)+_S(r+um)),\end{aligned}$$

as desired.

Part (b). First, suppose $k=0$. Then either $n=1$ or $r<m$. If $n=1$ then $S_n(r,m)=\{r\}$ and $S_n(1,2)=\{1\}$, which clearly implies the result. So we may assume $r<m$. Fix $0\le i,j,l\le n-1$. Then

$$\begin{aligned}r+lm\le(r+im)+(r+jm)\ &\Leftrightarrow\ l\le i+j+\tfrac{r}{m}\\&\Leftrightarrow\ l\le i+j+\tfrac{1}{2}\\&\Leftrightarrow\ 1+2l\le(1+2i)+(1+2j)\end{aligned}$$

So $\mathcal{S}_n(r,m)\cong\mathcal{S}_n(1,2)$.

Next, suppose $k>0$. Fix $0\le i,j,l\le n-1$. Then

$$r+lm\le(r+im)+(r+jm)\ \Leftrightarrow\ l\le i+j+\frac{r}{m}\ \Leftrightarrow\ l\le i+j+\left\lfloor\frac{r}{m}\right\rfloor,$$

and

$$k+l\le(k+j)+(k+i)\ \Leftrightarrow\ l\le i+j+k.$$

Therefore, in order to verify $\mathcal{S}_n(r,m)\cong\mathcal{S}_n(k,1)$, we need to show

$$l\le i+j+\left\lfloor\frac{r}{m}\right\rfloor\ \Leftrightarrow\ l\le i+j+k.$$

If $k=\lfloor\frac{r}{m}\rfloor$ then this is trivial. Otherwise, we must have $k=n-1\le\lfloor\frac{r}{m}\rfloor$, in which case both inequalities are trivially true since $l\le n-1$. □

The previous proposition says that, for any fixed $n>0$, if $S\subseteq\mathbb{N}$ is such that $S^{>0}$ is an arithmetic progression of length n, then $\mathcal{S}\cong\mathcal{S}'$ where $(S')^{>0}$ is either $\{1,3,5,\dots,2n-1\}$ or $\{k,k+1,\dots,k+n-1\}$ for some $1\le k\le n-1$. Moreover, it is easy to see that these form non-isomorphic representatives, and so we see that arithmetic progressions of length n constitute exactly n isomorphism types among the integral distance monoids with n nontrivial elements.

5.3 Formally Integral Distance Monoids

In the last section, we established several classes of naturally occurring distance monoids which, moreover, were integral. The next question we address is on non-integral distance monoids.

Definition 5.3.1. Given $n > 0$, let $\mathrm{IM}(n)$ denote the number, modulo isomorphism, of integral distance monoids with n nontrivial elements.

First, we establish that $\mathrm{DM}(n)$ and $\mathrm{IM}(n)$ do not yield the same sequence by giving an explicit example of a non-integral distance monoid. This example is taken directly from [32], although our presentation differs slightly.

Example 5.3.2. Define the distance magma $\mathcal{R} = (\{0, 1, 2, 3, 4, 5, 6, 7, 8\}, \oplus, \leq, 0)$, where $\leq$ is the natural ordering and 0 is the identity. To define $\oplus$, we let $r \oplus s = 8$ whenever $\max\{r, s\} \geq 5$ and, otherwise, we use Figure 7.

	1	2	3	4
1	4	5	6	6
2	5	5	7	8
3	6	7	7	8
4	6	8	8	8

Figure 7: Addition matrix of a non-(formally integral) monoid.

The reader may verify $\oplus$ is associative, and so $\mathcal{R}$ is a distance monoid. Using techniques from [32], we show $\mathcal{R}$ is not integral.

Suppose, toward a contradiction, there is $S \subset \mathbb{N}$, with $0 \in S$, such that $\mathcal{R} \cong \mathcal{S}$. Let $S = \{0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}_<$. Note the following inequalities, which hold in $\mathcal{R}$.

(*i*) $2 \oplus 2 = 5 < 6 = 1 \oplus 3$

(*ii*) $1 \oplus 1 \oplus 1 = 6 < 7 = 2 \oplus 3$

(*iii*) $3 \oplus 3 = 7 < 8 = 1 \oplus 1 \oplus 2$

By (*i*), we have $s_2 +_S s_2 < s_1 +_S s_3$, and so

$$s_2 + s_2 < s_1 + s_3. \qquad (\dagger)$$

By (*ii*), we have $s_1 +_S s_1 +_S s_1 < s_2 +_S s_3$, and so

$$(s_1 +_S s_1) + s_1 < s_2 + s_3. \qquad (\dagger\dagger)$$

Combining ($\dagger$) and ($\dagger\dagger$), we obtain

$$(s_1 +_S s_1) + s_1 + s_2 + s_2 < s_1 + s_2 + s_3 + s_3,$$

and so

$$(s_1 +_S s_1) + s_2 < s_3 + s_3,$$

which implies

$$s_1 +_S s_1 +_S s_2 < s_3 +_S s_3,$$

contradicting (*iii*).

Proposition 5.3.3. *If* $n \geq 8$ *then* $\mathrm{IM}(n) < \mathrm{DM}(n)$.

Proof. We clearly have $\mathrm{IM}(n) \leq \mathrm{DM}(n)$. We prove, by induction on $n \geq 8$, that there is a non-integral distance monoid with n nontrivial elements. The base case $n = 8$ follows from Example 5.3.2.

For the induction step, assume $\mathrm{IM}(n) < \mathrm{DM}(n)$. Then there is a non-integral distance monoid $\mathcal{R} = (R, \oplus, \leq, 0)$, with $R = \{0, r_1, \ldots, r_n\}_<$. Since submonoids of integral monoids are integral, it follows that the infinite expansion $\mathcal{R}^\infty$ is a non-integral distance monoid with $n + 1$ nontrivial elements. □

Let us return to the monoid in Example 5.3.2. This monoid was originally constructed in [32] to give an example of a finite distance monoid[2], which is not *formally integral*.

Definition 5.3.4. A distance monoid $\mathcal{R}$ is **formally integral** if there is an index set I, a positive monoid ordering $\leq$ on $\bigoplus_{i \in I} \mathbb{N}$, and a surjective order-preserving monoid homomorphism $\varphi : (\bigoplus_{i \in I} \mathbb{N}, +, \leq, 0) \longrightarrow \mathcal{R}$ (where $+$ denotes coordinate addition).

Put another way, a distance monoid is formally integral if and only if it is a quotient of a free abelian monoid under some positive monoid ordering. These monoids arise naturally in the study of orderings on free abelian monoids, which has applications to Gröbner bases, toric varieties, and integer programming (see [32]).

In [94], it is shown that any distance monoid, with $n \leq 7$ nontrivial elements, is formally integral. The distance monoid in Example 5.3.2 is constructed in [32] to verify that the bound of 7 is sharp.

In this next section, we will see that any distance monoid, with at most 6 nontrivial elements, is integral (this result is obtained with the help of a computer). This motivates the following conjecture, which proposes a structure theorem for formally integral monoids.

Conjecture 5.3.5. *A finite distance monoid* $\mathcal{R}$ *is integral if and only if it is formally integral.*

[2]We remind the reader that the terminology *distance monoid* is not standard. In [32], the authors would say "commutative monoid with a total, translation-invariant order".

5.4 Distance Monoids of Small Size

In this section, we count and classify distance monoids with $n \leq 6$ nontrivial elements. This undertaking can be seen as an extension of [69, Appendix A], in which Nguyen Van Thé classifies the subsets $S \subseteq \mathbb{N}$, which satisfy the four-values condition and have at most 4 nonzero elements. This is done in order to prove that, if $|S^{>0}| \leq 4$, then $\mathcal{U}_S$ is indivisible (see [69], [76]). In [76], Sauer proves that $\mathcal{U}_S$ is indivisible for any finite $S \subseteq \mathbb{N}$.

With the help of a computer, we obtain the following counts. Recall that $R(n)$ counts the number of distance magmas with n nontrivial elements; we also let $\mathrm{IS}(n)$ denote the number of *integral* distance magmas with n nontrivial elements.

Theorem 5.4.1.

n	$\mathrm{DM}(n)$	$\mathrm{IS}(n)$	$R(n)$
1	1	1	1
2	2	2	2
3	6	7	7
4	22	40	42
5	94	339	429
6	451	3965	7436

One may verify that these calculations of $\mathrm{DM}(n)$ agree with OEIS sequence A030453. Using the same computer program, we are also able to prove the following result.

Theorem 5.4.2. *If $n \leq 6$ then any distance monoid, with n nontrivial elements, is integral, i.e.,* $\mathrm{DM}(n) = \mathrm{IM}(n)$.

The verification of this theorem relies on a computer program, whose algorithm we will describe below. The running time of the algorithm is shortened by implementing the following conjecture, which addresses the problem of finding isomorphism representatives for integral distance monoids.

Conjecture 5.4.3. *Suppose $\mathcal{R}$ is an integral distance monoid and $|R^{>0}| = n$. Then there are $s_1, \ldots, s_n \in \mathbb{Z}^+$ such that*

(*i*) $\mathcal{R} \cong (S, +_S, \leq, 0)$, *where* $S = \{0, s_1, \ldots, s_n\}$;

(*ii*) $2^k - 1 \leq s_k \leq 2^n - 1$ *for all* $1 \leq k \leq n$.

In particular, note that $s_n = 2^n - 1$, which, for a fixed $n > 0$, significantly decreases the number of isomorphism representatives to check when testing whether or not a monoid is integral.

We can now describe the algorithm used to prove Theorems 5.4.1 and 5.4.2.

1. Fix $n > 0$.

2. Create an array `masterList` of sequences of n increasing integers $(t_1, \dots, t_n)$, which satisfy the constraints of Conjecture 5.4.3.

3. Create a new array `intMag` consisting of the first tuple in `masterList`. Given $k \leq$ `masterList.length`, check whether the k^{th} entry of `masterList` is isomorphic to an element of `intMag` (e.g. by comparing addition matrices). If it is not, add it it to `intMag`.

4. Create an array `allMag` consisting of addition matrices for distance magmas with n nontrivial elements (e.g. via the description in the proof of Theorem 5.1.4).

5. Create an empty array `badMag`. For each magma in `allMag`, check if this magma is isomorphic to one represented by a tuple in `intMag`. If it is not, add it to `badMag`.

6. We have `intMag` and `badMag`, which partition all distance magmas with n nontrivial elements. Each magma in `badMag` is a possible counterexample to Conjecture 5.4.3. Therefore, we check each magma in `badMag` for inequalities demonstrating the failure of integrality (as in Example 5.3.2).

7. Let $\mathrm{IS}(n) =$ `intMag.length`.

8. Create an empty array `intMon`. For each magma in `intMag`, check the addition matrix of the magma for associativity. If it passes, add it to `intMon`.

9. Let $\mathrm{IM}(n) =$ `intMon.length`.

This algorithm has been run for $n \leq 6$, which produces the results of Theorems 5.4.1 and Theorem 5.4.2, and also verifies Conjecture 5.4.3 for $n \leq 6$.

Remark 5.4.4. Conjecture 5.4.3 has a similar flavor to [55, Conjecture 1], which is a current open problem in additive combinatorics. In particular, this conjecture addresses the structure of isomorphism representatives for finite sets of integers under *Freiman isomorphism*, which identifies sets of integers having a similar additive structure. This notion is sufficiently different from distance monoid isomorphism, as it does not incorporate the ordering on integers.

In Section 5.6, we include isomorphism representatives for distance monoids of size at most 4. Sizes $n = 5$ and $n = 6$ can be found in [25]. It is worth reiterating that, for $n \leq 4$, a similar list was first determined by Nguyen Van Thé in [69].[3]

[3] In [69], 32 monoids of size 4 are produced. It can be verified that this list contains isomorphic repetitions. Recall that $\mathrm{DM}(4) = 22$.

Over the course of this chapter, we have defined four sequences $(\mathrm{IM}(n))_{n=1}^{\infty}$, $(\mathrm{DM}(n))_{n=1}^{\infty}$, $(\mathrm{IS}(n))_{n=1}^{\infty}$, and $(R(n))_{n=1}^{\infty}$, which enumerate, respectively, integral distance monoids, distance monoids, integral distance magmas, and distance magmas. We have $\mathrm{IM}(n) = \mathrm{DM}(n)$ for $n \leq 6$, and $\mathrm{IM}(n) < \mathrm{DM}(n)$ for $n \geq 8$. We have $\mathrm{IM}(n) \leq \mathrm{IS}(n)$, $\mathrm{IS}(n) \leq R(n)$, and $\mathrm{DM}(n) \leq R(n)$. We have already conjectured $\mathrm{DM}(n) = o(R(n))$, and it seems reasonable to similarly conjecture

$$\mathrm{IM}(n) = o(\mathrm{IS}(n)), \;\; \mathrm{IM}(n) = o(\mathrm{DM}(n)), \;\; \text{and} \;\; \mathrm{IS}(n) = o(R(n)).$$

A priori, there is no sensible comparison to make between $\mathrm{IS}(n)$ and $\mathrm{DM}(n)$.

5.5 Archimedean Complexity

In this section, we consider the archimedean complexity of distance monoids of small size. While this is often computed by hand, some cases can be simplified via the following partial strengthening of Proposition 3.7.19(a).

Proposition 5.5.1. *Suppose $\mathcal{R}$ is a semi-archimedean distance monoid. Then* $\mathrm{arch}(\mathcal{R}) = \max\{\mathrm{arch}_{\mathcal{R}}(t) : t \in R\}$.

Proof. Fix $n > 0$ and suppose $r_0, \ldots, r_n \in R$ are such that $r_0 \leq r_1 \leq \ldots \leq r_n$ and $r_1 \oplus \ldots \oplus r_n < r_0 \oplus r_1 \oplus \ldots \oplus r_n$. Since $\mathcal{R}$ is semi-archimedean, we must have $r_0 \sim_{\mathcal{R}} r_n$, and so $r_0, r_1, \ldots, r_n$ all lie in a single archimedean class. □

We can also easily calculate the archimedean complexity of integral distance monoids given by arithmetic progressions.

Proposition 5.5.2. *Fix $n > 1$.*

(a) *Given* $1 \leq k \leq n-1$, $\mathrm{arch}(\mathcal{S}_n(k,1)) = \left\lceil \dfrac{n-1}{k} \right\rceil + 1$.

(b) $\mathrm{arch}(\mathcal{S}_n(1,2)) = n-1$.

Proof. Part (a). Note that $\mathcal{S}_n(k,1)$ is archimedean and so, by Proposition 3.7.19(b),

$$\mathrm{arch}(\mathcal{S}_n(k,1)) = \left\lceil \frac{k+n-1}{k} \right\rceil = \left\lceil \frac{n-k}{k} \right\rceil + 1.$$

Part (b). First, we have $\mathcal{S}_{n-1}(3,2) \cong \mathcal{S}_{n-1}(1,1)$, and so $\mathrm{arch}(\mathcal{S}_{n-1}(3,2)) = n-1$ by part (a). Moreover, $\mathcal{S}_n(1,2)$ is semi-archimedean, with archimedean classes $\{1\}$ and $S_{n-1}(3,2)$. By Proposition 5.5.1, we have $\mathrm{arch}(\mathcal{S}_n(1,2)) = n-1$. □

Combined with brute force calculation, we obtain the following values for $\mathrm{DM}(n,k)$.

Theorem 5.5.3.

rank k \ size n	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	4	14	51	202
3	0	0	1	6	33	183
4	0	0	0	1	8	54
5	0	0	0	0	1	10
6	0	0	0	0	0	1
$\mathrm{DM}(n)$	1	2	6	22	94	451

Based on these numbers, we expand on conjectures made in Remark 3.7.24 concerning the asymptotic behavior of sequences produced by the values $\mathrm{DM}(n, k)$.

Conjecture 5.5.4.

(a) *Given a fixed* $k > 1$, $(\mathrm{DM}(n, k))_{n=k}^{\infty}$ *is strictly increasing.*

(b) *Given a fixed* $n > 2$, $(\mathrm{DM}(n, k))_{k=1}^{n}$ *is (strictly) unimodal.*

(c) *Given a fixed* $k > 0$, $\mathrm{DM}(n, k) = o(\mathrm{DM}(n))$.

5.5.1 $\mathrm{DM}(n, n-1) = 2n - 2$

The reader may have noticed, in Theorem 5.5.3, the pattern $\mathrm{DM}(n, n-1) = 2n-2$, when $3 \leq n \leq 6$. The goal of this subsection is to prove the following theorem.

Theorem 5.5.5. *If* $n \geq 3$ *then* $\mathrm{DM}(n, n-1) = 2n - 2$.

Unfortunately, the proof is somewhat lengthy and cumbersome, and our methods do not directly yield a general strategy for understanding the sequence $(\mathrm{DM}(n, k))_{n=1}^{\infty}$, for an arbitrary fixed k. However, the fact that such a simple pattern exists is provocative. Therefore, it is worth including a proof.

For cleaner exposition, we will fix an integer $n \geq 2$, and prove $\mathrm{DM}(n+1, n) = 2n$. The case $n = 2$ deviates from the general method, and must be calculated separately. For this, it is easiest to directly calculate that there are exactly six distance monoids with 3 nontrivial elements, and then, by inspection, observe that four of them have archimedean complexity 2. Isomorphism representatives for these monoids can be found in Section 5.6. The verification that this is an exhaustive list is also done in [69, Appendix A].

Throughout this section, we fix $n \geq 3$. Toward the proof that $\mathrm{DM}(n+1, n) = 2n$, we first show that most distance monoids, with $n + 1$ nontrivial elements and archimedean complexity n, are archimedean.

Proposition 5.5.6. *Suppose $\mathcal{R} = (R, \oplus, \leq, 0)$ is a distance monoid, with $|R^{>0}| = n+1$ and $\text{arch}(\mathcal{R}) = n$. Then exactly one of the following holds.*

(i) $\mathcal{R} \cong (\mathcal{R}_n)^\infty$ *(with $\mathcal{R}_n$ as in Example 3.1.1(3)),*

(ii) $\mathcal{R} \cong (\mathcal{R}_n)_\epsilon$*, or*

(iii) $\mathcal{R}$ *is archimedean.*

Proof. The uniqueness aspect of the claim is clear. In particular, $(\mathcal{R}_n)^\infty$ and $(\mathcal{R}_n)_\epsilon$ are clearly non-archimedean. Moreover, it is easy to see that $(\mathcal{R}_n)^\infty \cong (\mathcal{R}_n)_\epsilon$ if and only if $n = 1$. Therefore, to prove the result, it suffices to assume $\mathcal{R}$ is non-archimedean and show either (i) or (ii) holds.

By Proposition 3.7.21, there is some $S \subseteq R^{>0}$, such that S is a single archimedean class and $|S| \geq n$. Since $|R^{>0}| = n+1$ and $\mathcal{R}$ is non-archimedean, it follows that $|S| = n$ and so $R = S \cup \{0, r\}$ for some $r \notin S$. Note also that we must have either $r < \min S$ or $\max S < r$. In either case, $\{r\}$ is an archimedean class in $\mathcal{R}$, and so $r \oplus r = r$. Since S is an archimedean class, we may consider the submonoid $\mathcal{S} = (S \cup \{0\}, \oplus, \leq, 0)$ of $\mathcal{R}$.
Claim: $\text{arch}(\mathcal{S}) = n$.
Proof: Since $\text{arch}(\mathcal{R}) = n$, we may fix elements $r_0 \leq r_1 \leq \ldots \leq r_{n-1}$ in $\mathcal{R}$ such that $r_1 \oplus \ldots \oplus r_{n-1} < r_0 \oplus r_1 \oplus \ldots \oplus r_{n-1}$. If $\max S < r$ then it follows that $r_0, r_1, \ldots, r_{n-1} \in S$, and so $\text{arch}(\mathcal{S}) = n$. So we may assume $r < \min S$ and $r_k = r$ for some $0 \leq k \leq n-1$. As in the proof of Proposition 3.7.21, we have

$$r_{n-1} < r_{n-1} \oplus r_{n-2} < \ldots < r_0 \oplus r_1 \oplus \ldots \oplus r_{n-1},$$

and so it follows that $k = 0$ and $r_1 > r$. If $s = \min S$, then $s \leq r_1 \leq \ldots \leq r_{n-1}$ and

$$r_1 \oplus \ldots \oplus r_{n-1} < r_0 \oplus r_1 \oplus \ldots \oplus r_{n-1} \leq s \oplus r_1 \oplus \ldots \oplus r_{n-1}.$$

Therefore $\text{arch}(\mathcal{S}) = n$. $\dashv_{\text{claim}}$

By the claim and Theorem 3.7.23, we have $\mathcal{S} \cong \mathcal{R}_n$. Let $S = \{s, 2s, \ldots, ns\}$, where $s = \min S$. If $ns < r$ then, since S is closed under $\oplus$, we immediately obtain $\mathcal{R} \cong (\mathcal{R}_n)^\infty$. Therefore, we may assume $0 < r < s$. We have $r \oplus r = r$, and so, to show $\mathcal{R} \cong (\mathcal{R}_n)_\epsilon$, it remains to show that, for all $1 \leq k \leq n$, we have $ks \oplus r = ks$. For this, it suffices to assume $k = 1$.

Suppose, toward a contradiction, $s < r \oplus s$. Since $r \oplus s \leq 2s$, it follows that $r \oplus s = s \oplus s$. Therefore,

$$3s = s \oplus s \oplus s = r \oplus s \oplus s = r \oplus r \oplus s = r \oplus s = 2s,$$

which contradicts $n \geq 3$. □

By the previous result, in order to prove $\mathrm{DM}(n+1,n) = 2n$, it suffices to show that, modulo isomorphism, there are exactly $2n-2$ *archimedean* distance monoids $\mathcal{R}$, with $|R^{>0}| = n+1$ and $\mathrm{arch}(\mathcal{R}) = n$.

Suppose $\mathcal{R}$ is an archimedean distance monoid, with $|R^{>0}| = n+1$ and $\mathrm{arch}(\mathcal{R}) = n$. If $r = \min R^{>0}$ and $s = \max R^{>0}$, then, by Proposition 3.7.19(b), we have $nr = s$ and $r < 2r < \ldots < nr$. Therefore, $R = \{0, r, 2r, \ldots, nr\} \cup \{t\}$, for some t. Without loss of generality, we may assume $\mathcal{R} = \{0, 1, 2, \ldots, n\} \cup \{t\}$ and $\oplus$ coincides with $+_n$ on $\{0, 1, \ldots, n\}$.

Let Σ be the class of distance monoids $\mathcal{R} = (\mathcal{R}, \oplus, \leq, 0)$ satisfying the following requirements:

(i) $R = \{0, 1, \ldots, n\} \cup \{t\}$, with $t \notin \{0, 1, \ldots, t\}$;

(ii) $1 < t < n$ and, when restricted to $\{0, 1, \ldots, n\}$, the ordering $\leq$ on R agrees with the natural ordering;

(iii) for all $i, j \in \{0, 1, \ldots, n\}$, $i \oplus j = i +_n j$.

We have shown that every archimedean monoid $\mathcal{R}$, with $|R^{>0}| = n+1$ and $\mathrm{arch}(\mathcal{R}) = n$, is isomorphic to some monoid in Σ. Therefore, it suffices to show Σ contains exactly $2n-2$ pairwise non-isomorphic monoids.

Given $\mathcal{R} \in \Sigma$, we define the following distinguished elements of $\mathcal{R}$. Let $t_{\mathcal{R}} \in R$ be the unique element of R not in $\{0, 1, \ldots, n\}$. Let $i_{\mathcal{R}} \in \{1, \ldots, n-1\}$ be such that $i_{\mathcal{R}} < t_{\mathcal{R}} < i_{\mathcal{R}} + 1$. Set $u_{\mathcal{R}} = t_{\mathcal{R}} \oplus_{\mathcal{R}} 1$. Note that $i_{\mathcal{R}} + 1 \leq u_{\mathcal{R}} \leq 2t_{\mathcal{R}}$, and so $u_{\mathcal{R}}, 2t_{\mathcal{R}} \in \{i_{\mathcal{R}} + 1, \ldots, n\}$.

We will frequently use the following observations.

(P1) If $\mathcal{R} \in \Sigma$ and $k \in \{1, \ldots, n\}$, then $t_{\mathcal{R}} \oplus_{\mathcal{R}} k = u_{\mathcal{R}} +_n (k-1)$.

(P2) As a special case of (P1), if $\mathcal{R} \in \Sigma$ then $2t_{\mathcal{R}} +_n 1 = u_{\mathcal{R}} +_n (u_{\mathcal{R}} - 1)$.

Lemma 5.5.7. *Suppose $\mathcal{R}, \mathcal{S} \in \Sigma$. Then $\mathcal{R} \cong \mathcal{S}$ if and only if $i_{\mathcal{R}} = i_{\mathcal{S}}$, $u_{\mathcal{R}} = u_{\mathcal{S}}$, and $2t_{\mathcal{R}} = 2t_{\mathcal{S}}$.*

Proof. Suppose $\varphi : \mathcal{R} \longrightarrow \mathcal{S}$ is an isomorphism. To prove the claims, it suffices to show $\varphi(t_{\mathcal{R}}) = t_{\mathcal{S}}$. For this, note that $\varphi(1) = 1$ since φ is order preserving. It follows that $\varphi(k) = k$ for all $k \in \{0, 1, \ldots, n\}$, and so we must have $\varphi(t_{\mathcal{R}}) = t_{\mathcal{S}}$.

Conversely, suppose $i_{\mathcal{R}} = i_{\mathcal{S}}$, $u_{\mathcal{R}} = u_{\mathcal{S}}$, and $2t_{\mathcal{R}} = 2t_{\mathcal{S}}$. We show that the function $\varphi : \mathcal{R} \longrightarrow \mathcal{S}$ such that $\varphi(t_{\mathcal{R}}) = t_{\mathcal{S}}$ and $\varphi(i) = i$ for all $i \in \{0, 1, \ldots, n\}$ is an isomorphism. Note that $i_{\mathcal{R}} = i_{\mathcal{S}}$ implies φ is order preserving. Next, $2t_{\mathcal{R}} = 2t_{\mathcal{S}}$ implies $\varphi(t_{\mathcal{R}} \oplus_{\mathcal{R}} t_{\mathcal{R}}) = \varphi(t_{\mathcal{R}}) \oplus_{\mathcal{S}} \varphi(t_{\mathcal{R}})$. We have left to show $\varphi(t_{\mathcal{R}} \oplus_{\mathcal{R}} k) = t_{\mathcal{S}} \oplus_{\mathcal{S}} k$ for all $k \in \{1, \ldots, n\}$. By (P1),

$$\varphi(t_{\mathcal{R}} \oplus_{\mathcal{R}} k) = \varphi(u_{\mathcal{R}} +_n (k-1)) = u_{\mathcal{R}} +_n (k-1) = u_{\mathcal{S}} +_n (k-1) = t_{\mathcal{S}} \oplus_{\mathcal{S}} . \quad \square$$

Given $i, j, k \in \{1, \ldots, n\}$, let $q(i, j, k)$ denote the number, modulo isomorphism, of $\mathcal{R} \in \Sigma$ such that $i_{\mathcal{R}} = i$, $u_{\mathcal{R}} = j$, and $2t_{\mathcal{R}} = k$. By Lemma 5.5.7, we have $q(i, j, k) \in \{0, 1\}$ for all $i, j, k \in \{1, \ldots, n\}$. Given $i, j \in \{1, \ldots, n\}$, define

$$q(i, j) = \sum_{k=1}^{n} q(i, j, k) \text{ and } q(i) = \sum_{k=1}^{n} q(i, k).$$

By Lemma 5.5.7, $\sum_{i=1}^{n-1} q(i)$ counts size of Σ, modulo isomorphism. Therefore, our goal is to show $\sum_{i=1}^{n-1} q(i) = 2n-2$. We first isolate sufficient conditions for showing $q(i, j, k) = 1$, given some $i, j, k \in \{1, \ldots, n\}$.

Lemma 5.5.8. *Suppose $i, j, k \in \{1, \ldots, n\}$ satisfy the following properties:*

(i) $i < n$, $i + 1 \leq j \leq i +_n 2$, *and* $j +_n (i - 1) \leq k \leq j +_n i$;

(ii) *for all* $w \in \{1, \ldots, n\}$, $k +_n w = j +_n ((j +_n (w - 1)) - 1)$.

Then $q(i, j, k) = 1$.

Proof. Let $\mathcal{R}$ denote the $\mathcal{L}_{\mathrm{om}}$-structure $(R, \oplus, \leq, 0)$ such that:

- $R = \{0, 1, \ldots, n\} \cup \{t\}$, where t is a symbol not in $\{0, 1, \ldots, n\}$;
- $i < t < i + 1$ and, restricted to $\{0, 1, \ldots, n\}$, $\leq$ is the natural ordering;
- $\oplus$ coincides with $+_n$ on $\{0, 1, \ldots, n\}$;
- $t \oplus t = k$, $t \oplus 0 = 0 \oplus t = t$ and, given $l \in \{1, \ldots, n\}$, $t \oplus l = l \oplus t = j +_n (l - 1)$.

We want to show $\mathcal{R}$ is a distance monoid. By construction, $\oplus$ is commutative and 0 is the identity (this follows even without the extra assumptions on i, j, k). Therefore, we must show $\oplus$ is order preserving and associative.

To show $\oplus$ is order preserving, fix $u, v, w \in R^{>0}$, with $u < v$. We want to show $u \oplus w \leq v \oplus w$. If neither u nor v is equal to t, then this follows easily from the construction of $\mathcal{R}$.

Suppose $u = t$. Then $i + 1 \leq v$. If $w \neq t$ then we have $t \oplus w = j +_n (w - 1)$ and $v \oplus w = v +_n w$. Since $i + 1 \leq v$ and $j \leq i +_n 2$, we have $j +_n (w - 1) \leq v +_n w$. If $w = t$ then we want to show $k \leq j +_n (v - 1)$. This follows from $i + 1 \leq v$.

Next, suppose $v = t$. Then $u \leq i$. If $w \neq t$ then we want to show $u +_n w \leq j +_n (w - 1)$. This follows from $u \leq i$ and $i + 1 \leq j$. If $w = t$ then we want to show $j +_n (u - 1) \leq 2k$. This follows from $u \leq i$.

Finally, to show $\oplus$ is associative, fix $u, v, w \in R^{>0}$. We want to show $(u \oplus v) \oplus w = u \oplus (v \oplus w)$. We may clearly assume $t \in \{u, v, w\}$. If $u = v = w = t$ then the result is clear.

Case 1: Exactly one of u, v, w is equal to t. Without loss of generality, we may assume $u = t$ or $v = t$.

If $u = t$ then we want to show $(j +_n (v-1)) +_n w = j +_n ((v +_n w) - 1)$, which is clear. If $v = t$ then we want to show $(j +_n (u-1)) +_n w = (j +_n (w-1)) +_n u$, which is clear.

Case 2: Exactly two of u, v, w are equal to t. Without loss of generality, we may assume $u = v = t$.

We want to show $k +_n w = j +_n ((j +_n (w-1)) - 1)$, which is provided by assumption. $\square$

Let $n_* = \lceil \frac{n}{2} \rceil$. Note that $n \geq 3$ implies $n_* \geq 2$. Recall that our goal is to show $\sum_{i=1}^{n-1} q(i) = 2n - 2$. In particular, we will show $q(n-1) = 1$, $q(n_* - 1) = 3$, and $q(i) = 2$ for all $i \notin \{n_* - 1, n-1\}$. To accomplish this, we will need $q(i, j, k) = 1$ for the values in the following lemma. Given $i \in \{1, \ldots, n\}$, we use $2i$ to denote $i +_n i$.

Lemma 5.5.9.

(a) *If* $i \in \{1, \ldots, n-1\}$ *and* $i + 1 \leq j \leq i +_n 2$ *then* $q(i, j, 2(j-1)) = 1$.

(b) *If* n *is even then* $q(n_* - 1, n_* + 1, n - 1) = 1$.

(c) *If* n *is odd then* $q(n_* - 1, n_*, n) = 1$.

Proof. Part (a). If $i, j, k \in \{1, \ldots, n\}$, with $i < n$, $j \in \{i+1, i+_n 2\}$, and $k = 2(j-1)$, then it is straightforward to see that (i, j, k) satisfies the conditions of Lemma 5.5.8.

Part (b). Suppose n is even. Let $i = n_* - 1$, $j = n_* + 1$, and $k = n - 1$. We verify that (i, j, k) satisfies the conditions of Lemma 5.5.8. We clearly have $i < n$, $i + 1 \leq j \leq i +_n 2$, and $k \leq j +_n i$. Moreover, $j +_n (i-1) \leq k$ follows from the assumption that n is even. Next, suppose $w \in \{1, \ldots, n\}$ and let $v = j +_n ((j +_n (w-1)) - 1)$. Then $v \geq 2n_* = n$. Therefore $k +_n w = v$.

Part (c). Suppose n is odd. Let $i = n_* - 1$, $j = n_*$, and $k = n$. We verify that (i, j, k) satisfies the conditions of Lemma 5.5.8. We clearly have $i < n$, $i + 1 \leq j \leq i +_n 2$, and $j +_n (i-1) \leq k$. Moreover, $k \leq j +_n i$ follows from the assumption that n is odd. Next, suppose $w \in \{1, \ldots, n\}$ and let $v = j +_n ((j +_n (w-1)) - 1)$. Since n is odd, we have $v = n = k +_n l$. $\square$

Lemma 5.5.10.

(a) *Fix* $i \in \{1, \ldots, n-1\}$.

- (i) *If* $2i = n$ *or* $2(i+1) < n$, *then* $q(i, j) \leq 1$ *for all* $j \in \{1, \ldots, n\}$.
- (ii) *If* $i < n - 1$ *then* $q(i) = q(i, i+1) + q(i, i+2)$.
- (iii) *If* $i \notin \{n_* - 1, n - 1\}$ *then* $q(i) = 2$.

(b) $q(n-1) = 1$.

(c) $q(n_* - 1) = 3$.

Proof. Part (a). Fix $i \in \{1, \ldots, n-1\}$, and let $\Sigma_i = \{\mathcal{R} \in \Sigma : i_{\mathcal{R}} = i\}$.

Part $(a)(i)$. First, if $2i = n$ then, for any $\mathcal{R} \in \Sigma_i$, we have $i < t_{\mathcal{R}}$ and so $2t_{\mathcal{R}} = n$. Therefore, for any $j \in \{1, \ldots, n\}$, $q(i,j) = q(i,j,n) \leq 1$.

Next, if $2(i+1) < n$ then, for any $\mathcal{R} \in \Sigma_i$, we have $2t_{\mathcal{R}} \leq 2(i+1) < n$. By (P2), we have $2t_{\mathcal{R}} +_n 1 = u_{\mathcal{R}} +_n (u_{\mathcal{R}} - 1)$, and so we must have $2t_{\mathcal{R}} = 2(u_{\mathcal{R}} - 1)$. Therefore, for any $j \in \{1, \ldots, n\}$, $q(i,j) = q(i,j,2(j-1)) \leq 1$.

Part $(a)(ii)$. Suppose $i < n-1$. If $\mathcal{R} \in \Sigma_i$, then $i < t_{\mathcal{R}} < i+1$ implies $i+1 \leq u_{\mathcal{R}} \leq i+2$. Therefore $q(i,j) = 0$ when $j \notin \{i+1, i+2\}$.

Part $(a)(iii)$. Suppose $i \notin \{n_*-1, n-1\}$. By part $(a)(ii)$, it suffices to show $q(i,i+1) = 1 = q(i,i+2)$. If $n_* \leq i$ then $2i = n$. On the other hand, if $i < n_*$ then $2(i+1) \leq 2(n_*-1) < n$. In either case, $q(i,i+1) \leq 1$ and $q(i,i+2) \leq 1$ by part $(a)(i)$. Moreover, $q(i,i+1) \geq q(i,i+1,2i) \geq 1$ and $q(i,i+2) \geq q(i,i+2,2(i+1)) \geq 1$ by Lemma 5.5.9(a).

Part (b). If $\mathcal{R} \in \Sigma_{n-1}$ then $u_{\mathcal{R}} = n = 2t_{\mathcal{R}}$ and so $q(n-1) = q(n-1,n,n) = 1$ by Lemma 5.5.9(a).

Part (c). Note that $n \geq 3$ implies $n_* - 1 < n-1$ and so, by part $(a)(ii)$, we have

$$q(n_*-1) = q(n_*-1, n_*) + q(n_*-1, n_*+1). \qquad (\dagger)$$

Case 1: n is even.

If $\mathcal{R} \in \Sigma_{n_*-1}$ and $u_{\mathcal{R}} = n_*$ then, by (P2), $2t_{\mathcal{R}} +_n 1 = n_* +_n (n_*-1) < n$, which means $2t_{\mathcal{R}} = 2(n_*-1)$. Therefore $q(n_*-1, n_*) = q(n_*-1, n_*, 2(n_*-1)) = 1$ by Lemma 5.5.9(a).

Next, if $\mathcal{R} \in \Sigma_{n_*-1}$ and $u_{\mathcal{R}} = n_*+1$ then, by (P2), $2t_{\mathcal{R}} +_n 1 = n$, which means $n-1 \leq 2t_{\mathcal{R}} \leq n$. Therefore,

$$q(n_*-1, n_*+1) = q(n_*-1, n_*+1, n-1) + q(n_*-1, n_*+1, n).$$

Moreover, $q(n_*-1, n_*+1, n-1) = 1$ by Lemma 5.5.9(b), and $q(n_*-1, n_*+1, n) = 1$ by Lemma 5.5.9(a). Altogether, by $(\dagger)$, we have $q(n_*-1) = 3$.

Case 2: n is odd.

If $\mathcal{R} \in \Sigma_{n_*-1}$ and $u_{\mathcal{R}} = n_*+1$ then

$$n = (n_*+1) +_n (n_*-2) = u_{\mathcal{R}} +_n (n_*-2) = t_{\mathcal{R}} \oplus_{\mathcal{R}} (n_*-1) \leq 2t_{\mathcal{R}},$$

and so $q(n_*-1, n_*+1) = q(n_*-1, n_*+1, n) = 1$ by Lemma 5.5.9(a).

Next, if $\mathcal{R} \in \Sigma_{n_*-1}$ and $u_{\mathcal{R}} = n_*$ then, by (P2), $2t_{\mathcal{R}} +_n 1 = n$, and so $n-1 \leq 2t_{\mathcal{R}} \leq n$. Therefore,

$$q(n_*-1, n_*) = q(n_*-1, n_*, n-1) + q(n_*-1, n_*, n).$$

Moreover, $q(n_*-1, n_*, n-1) = 1$ by Lemma 5.5.9(a), and $q(n_*-1, n_*, n) = 1$ by Lemma 5.5.9(c). Altogether, by $(\dagger)$, we have $q(n_*-1) = 3$. □

From the previous result, we have $\sum_{i=1}^{n-1} q(i) = 2n-2$, finishing the proof of Theorem 5.5.5.

5.6 Tables

This section consists of tables of isomorphism representatives for distance monoids with $n \leq 4$ nontrivial elements. Similar tables for $n = 5$ and $n = 6$ can be found in [25]. All monoids are of the form $\mathcal{S} = (S, +_S, \leq, 0)$, for some $S \subseteq \mathbb{N}$, with $0 \in S$. Therefore, we will associate the monoid $\mathcal{S}$ with the set S. The following is a key for reading the tables.

1. Column "$\mathcal{S}$" lists $S^{>0}$, where S an isomorphism representative with $\min S^{>0}$ minimal.
2. Column "arch($\mathcal{S}$)" gives the archimedean complexity of $\mathcal{S}$. Recall that this coincides with the model theoretic complexity of $\text{Th}(\mathcal{U}_{\mathcal{S}})$ (see Theorem 3.7.8).
3. Column "notation" gives a description of $\mathcal{S}$, if one exists, using the notation of Definition 4.4.6. When possible, we also incorporate the notation for arithmetic progressions (see Section 5.2). We let $S_1 = \{0, 1\}$.
4. Column "details" gives a classification of $\mathcal{S}$, if one exists, according to previous notions.
5. Column "alt. rep." gives a representative for $\mathcal{S}$, which fits the constraints of Conjecture 5.4.3.

Table I: Distance monoids with 1 nontrivial element.

$\mathcal{S}$	arch($\mathcal{S}$)	notation	details	alt. rep.
$\{1\}$	1	S_1	ultrametric	$\{1\}$

Table II: Distance monoids with 2 nontrivial elements.

$\mathcal{S}$	arch($\mathcal{S}$)	notation	details	alt. rep.
$\{1, 2\}$	2	$S_2(1, 1)$	met. trivial	$\{2, 3\}$
$\{1, 3\}$	1	$(S_1)^\infty$	ultrametric	$\{1, 3\}$

Table III: Distance monoids with 3 nontrivial elements.

$\mathcal{S}$	arch($\mathcal{S}$)	notation	details	alt. rep.
$\{1,2,3\}$	3	$S_3(1,1)$	archimedean	$\{3,5,7\}$
$\{1,3,5\}$	2	$S_3(1,2)$	semi-arch.	$\{1,5,7\}$
$\{1,2,5\}$	2	$S_2(1,1)^\infty$	semi-arch.	$\{2,3,7\}$
$\{2,3,4\}$	2	$S_3(2,1)$	met. trivial	$\{5,6,7\}$
$\{1,3,4\}$	2			$\{2,6,7\}$
$\{1,3,7\}$	1	$((S_1)^\infty)^\infty$	ultrametric	$\{1,3,7\}$

Table IV: Distance monoids with 4 nontrivial elements.

$\mathcal{S}$	arch($\mathcal{S}$)	notation	details	alt. rep.
$\{1,2,3,4\}$	4	$S_4(1,1)$	archimedean	$\{5,9,11,15\}$
$\{1,3,5,7\}$	3	$S_4(1,2)$	semi-arch.	$\{2,6,9,15\}$
$\{1,2,3,7\}$	3	$S_3(1,1)^\infty$	semi-arch	$\{3,4,7,15\}$
$\{2,3,4,5\}$	3	$S_4(2,1)$	archimedean	$\{6,9,12,15\}$
$\{2,3,4,6\}$	3		archimedean	$\{6,8,12,15\}$
$\{2,4,5,6\}$	3		archimedean	$\{6,9,13,15\}$
$\{3,4,6,9\}$	3		archimedean	$\{6,7,9,15\}$
$\{3,4,5,6\}$	2	$S_4(3,1)$	met. trivial	$\{8,9,10,15\}$
$\{2,3,4,9\}$	2	$S_3(2,1)^\infty$	semi-arch.	$\{4,5,7,15\}$
$\{1,4,6,8\}$	2	$S_3(2,1)_\epsilon$	semi-arch.	$\{2,8,12,15\}$
$\{1,2,5,11\}$	2	$(S_2(1,1)^\infty)^\infty$	semi-arch.	$\{2,3,7,15\}$
$\{1,3,5,11\}$	2	$S_3(1,2)^\infty$	semi-arch.	$\{1,5,7,15\}$
$\{1,3,7,11\}$	2	$S_3(1,2)_\epsilon$	semi-arch.	$\{1,3,8,15\}$
$\{1,3,4,9\}$	2	$\{1,3,4\}^\infty$		$\{2,5,7,15\}$
$\{1,3,7,9\}$	2	$\{1,3,4\}_\epsilon$		$\{2,5,12,15\}$
$\{1,2,5,8\}$	2	$[\![S_2(1,2),S_2(1,2)]\!]$	semi-arch.	$\{2,4,9,15\}$
$\{1,2,5,6\}$	2			$\{3,6,14,15\}$
$\{1,3,4,6\}$	2			$\{3,8,11,15\}$
$\{1,3,5,6\}$	2			$\{1,8,14,15\}$
$\{1,3,7,8\}$	2			$\{2,6,13,15\}$
$\{2,5,6,7\}$	2			$\{6,13,14,15\}$
$\{1,3,7,15\}$	1	$(((S_1)^\infty)^\infty)^\infty$	ultrametric	$\{1,3,7,15\}$

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Index

Curriculum Vitae

Gabriel Conant

University of Illinois at Chicago

Education

Summer 2015	**Ph.D. in Pure Mathematics** University of Illinois at Chicago, Chicago, IL
Spring 2010	**MS in Pure Mathematics** University of Illinois at Chicago, Chicago, IL
Spring 2008	**BA, Mathematics** Colgate University, Hamilton, NY

Preprints

An axiomatic approach to free amalgamation, arXiv 1505.00762 [math.LO]
Neostability in countable homogeneous metric spaces, arXiv 1504.02427 [math.LO]
Distance structures for generalized metric spaces, arXiv 1502.05002 [math.LO]
Forking and dividing in Henson graphs, arXiv 1401.1570 [math.LO]
Model theoretic properties of the Urysohn sphere, with Caroline Terry, arXiv 1401.2132 [math.LO]

Invited Talks & Seminars

Oct. 2015	**AMS Sectional Meeting** Loyola University, Chicago, IL
June 2015	**BLAST** University of North Texas, Denton, TX
Feb. 2015	**UIUC Logic Seminar** University of Illinois at Urbana-Champaign, Urbana, IL
Oct. 2014	**Southern Wisconsin Logic Colloquium** University of Wisconsin - Madison, Madison, WI
Oct. 2014	**Model Theory Seminar** University of Notre Dame, Notre Dame, IN
Sept. 2014	**The McMaster Model Theory Seminar** McMaster University, Hamilton, Ontario
April 2014	**15th Graduate Student Conference in Logic** University of Wisconsin - Madison, Madison, WI

Feb. 2014 **UIUC Logic Seminar**
University of Illinois at Urbana-Champaign, Urbana, IL

Sept. 2013 **UIC Logic Seminar**
University of Illinois at Chicago, Chicago, IL

April 2013 **Graduate Student Colloquium**
University of Illinois at Chicago, Chicago, IL

April 2013 **14th Graduate Student Conference in Logic**
University of Illinois at Urbana-Champaign, Urbana, IL

April 2012 **13th Graduate Student Conference in Logic**
University of Notre Dame, Notre Dame, IN

Contributed Talks

Oct. 2015 **ASL North American Annual Meeting**
University of Illinois at Urbana-Champaign, Urbana, IL

June 2015 **AMS Session on Mathematical Logic**
Joint Mathematics Meetings, San Antonio, TX

Conferences & Workshops

July 2015 **BIRS Workshop on Neostability Theory**
Casa Matemática Oaxaca, Oaxaca, Mexico

July 2014 **16th Latin American Symposium on Mathematical Logic**
Facultad de Ciencias Económicas, Buenos Aires, Argentina

May 2014 **Model Theory in Geometry and Arithmetic**
Mathematical Sciences Research Institute, Berkeley, CA

Feb. 2014 **Model Theory, Arithmetic Geometry, and Number Theory**
Mathematical Sciences Research Institute, Berkeley, CA

Oct. 2013 **Workshop on Homogeneous Structures**
Hausdorff Institute, Bonn, Germany

June 2013 **Model Theory Meeting**
Ravello, Italy

May 2013 **Carol Wood Retirement Conference**
Wesleyan University, Middletown, CT

July 2012 **Summer Workshop in Model Theory**
Mathematical Sciences Research Institute, Berkeley, CA

April 2012 **ASL North American Annual Meeting**
University of Wisconsin - Madison, Madison, WI

Oct. 2011 **Mid-Atlantic Mathematical Logic Seminar**
Rutgers University, Piscataway, NJ

Sept. 2011	**Ward Henson Retirement Conference** University of Illinois at Urbana-Champaign, Urbana, IL
May 2011	**Midwest Model Theory Meeting** The Ohio State University, Columbus, OH

Awards

2014-2015	**RTG Pre-doctoral Fellow** University of Illinois at Chicago, Chicago, IL
2008	**Edwin Downie Mathematics Award** Colgate University, Hamilton, NY
2006	**Sisson Prize in Mathematics** Colgate University, Hamilton, NY

Academic Involvement

2013 - present	Webmaster of `forkinganddividing.com`
Spring 2014	Co-organizer of the 16th Graduate Student Conference in Logic University of Illinois at Chicago, Chicago, IL
2013 - 2015	Member of UIC chapter of the Association for Women in Mathematics
2011-2013	Co-organizer of the UIC Louise Hay Logic Seminar
Spring 2013	Co-organizer of the UIC Graduate Student Colloquium
Fall 2013	Co-organizer of the UIC Model Theory Seminar
Fall 2011	UIC Teaching Assistant Coordinator
2009-2010	Co-president of the UIC Mathematics Graduate Student Association

Teaching

2012-2014	**Lecturer**, University of Illinois at Chicago, Chicago, IL MATH 300 Writing for Mathematics; MATH 210 Calculus 3; MATH 090 Intermediate Algebra (Summer Enrichment Workshop)
2012-2014	**Lecturer**, Loyola University, Chicago, IL COMP 163 Discrete Structures
2008-2013	**Teaching assistant**, University of Illinois at Chicago, Chicago, IL MATH 181 Calculus 2; MATH 180 Calculus 1; MATH 160 Finite Math for Business; MATH 121 Precalculus; MATH 090 Intermediate Algebra