# MAGMAS AND MAGOG TRIANGLES

GABRIEL CONANT

In this note, we construct an explicit bijection between the family of positively ordered abelian unital magmas with $n$ nontrivial elements and the family of magog triangles of order $n$. The size of the latter family was calculated explicitly in [4] as part of the solution to a famous problem in algebraic combinatorics called the *Alternating Sign Matrix Conjecture*.

**Definition 1.** A **unital magma** is a set, together with a binary operation and an identity element.

We will only consider abelian unital magmas, written additively $(A, \oplus, 0)$.

**Definition 2.** Suppose $(A, \oplus, 0)$ is a unital magma. $A$ is **positively ordered** if there is a linear order $\leq$ on $A$ such that

$(i)$ for all $a, b, c, d \in A$, if $a \leq b$ and $c \leq d$ then $a \oplus c \leq b \oplus d$,
$(ii)$ $0 \leq a$ for all $a \in A$.

**Definition 3.** Fix $n > 0$. Let $\mathrm{Mag}_0^+(n)$ denote the number (up to isomorphism) of positively ordered abelian unital magmas with $n$ nontrivial elements.

**Definition 4.** Given $n > 0$, a **magog triangle of order** $n$ is an $n \times n$ lower triangular matrix $A = (a_{i,j})_{i,j \leq n}$ such that

$(i)$ $a_{i,j} \in \{0, 1, \ldots, n\}$,
$(ii)$ if $1 \leq j \leq i \leq n$ then $1 \leq a_{i,j} \leq j$,
$(iii)$ the nonzero entries in any row or column of $A$ are nondecreasing.

**Theorem 5.** [4] *Given $n > 0$, the number of magog triangles of order $n$ is*

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

The expression in Theorem 5 (sequence A005130 in the OEIS) has its roots in a famous problem in algebraic combinatorics called the *Alternating Sign Matrix Conjecture*. In particular, this expression counts the number of $n \times n$ alternating sign matrices, which are a generalization of permutation matrices, and are used in the *Dodgeson concentration method* of calculating determinants. The formula for alternating sign

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matrices was first conjectured in [2], and first proved in [4]. This formula turns out to count a multitude of interesting combinatorial objects including magog triangles of order $n$, monotone triangles of order $n$, descending plane partitions of order $n$, totally symmetric self-complementary plane partitions of order $2n$, $n \times n$ corner-sum matrices, $n \times n$ tilings by "baskets and gaskets", and $n \times n$ arrays of "square ice". Despite the fact that these families of objects are all known to have the same size, the existence of explicit bijections between many of them remains open. For more on these objects, as well as details of the story of this problem, see [1] and [3].

**Corollary 6.** *Given $n > 0$,*

$$\mathrm{Mag}_0^+(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

*Proof.* We construct an explicit bijection between magog triangles of order $n$ and positively ordered abelian unital magmas with $n$ nontrivial elements. Given an $n \times n$ matrix $X$, we let $X(i,j)$ denote the $(i,j)$ entry of $X$.

Suppose $(A, \oplus, \leq, 0)$ is a positively ordered abelian unital magma with $n$ nontrivial elements. Enumerate $A = \{0, a_1, \ldots, a_n\}_<$. Define the $n \times n$ matrix $P(A)$ by

$$P(A)(i,j) = \begin{cases} 0 & \text{if } i < j \\ k & \text{if } j \leq i \text{ and } a_i \oplus a_j = a_k. \end{cases}$$

One may verify that $P(A)$ is an $n \times n$ lower triangular matrix satisfying the following properties:

($i$) $P(A)(i,j) \in \{0, 1, \ldots, n\}$,

($ii$) if $1 \leq j \leq i \leq n$ then $j \leq P(A)(i,j)$,

($iii$) the nonzero entries in any row or column of $P(A)$ are nondecreasing.

Let $\mathcal{C}(n)$ be the family of $n \times n$ lower triangular matrices satisfying properties ($i$) through ($iii$). Let $\mathcal{P}(n)$ be the family of positively ordered abelian unital magmas with $n$ nontrivial elements (modulo isomorphism). Then we have that $P : \mathcal{P}(n) \longrightarrow \mathcal{C}(n)$ is a well-defined function, and it is easy to see that $P$ is injective. Moreover, if $X \in \mathcal{C}(n)$, then define the structure $(A, \oplus, \leq, 0)$ where $A = \{0, a_1, \ldots, a_n\}_<$, $a_k \oplus 0 = a_k = 0 \oplus a_k$ for all $k$, and $a_i \oplus a_j = a_{X(i,j)} = a_j \oplus a_i$ for all $1 \leq j \leq i \leq n$. Then $A \in \mathcal{P}(n)$ and $P(A) = X$. Therefore $P$ is a bijection.

Finally, let $\mathcal{M}(n)$ be the family of magog triangles of order $n$. Let $\sigma$ be the permutation of $\{0, 1, \ldots, n\}$ such that $\sigma(i) = -i \pmod{n+1}$. Define $f : \mathcal{C}(n) \longrightarrow \mathcal{M}(n)$ such that

$$f(X)(i,j) = \sigma(X(\sigma(j), \sigma(i))).$$

It is straightforward to verify that $f$ is a well-defined bijection. □

## References

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Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 S Morgan St, 322 SEO, Chicago, IL, 60607, USA;

*E-mail address*: gconan2@uic.edu