ULTRAMETRIC SPACES IN CONTINUOUS LOGIC

GABRIEL CONANT

Abstract. We investigate the continuous model theory of ultrametric spaces of diameter ≤ 1. There is no universal Polish ultrametric space of diameter 1; but there is a Polish ultrametric space, $U_{\max}$, taking distances in $\mathbb{Q}\cap[0,1]$, which is universal for all such Polish ultrametric spaces. We show that in the continuous theory of $U_{\max}$, nonforking is characterized by a stable independence relation, which is a continuous version of "forking by equality" in first-order logic. Finally, we show that the theory of $U_{\max}$ is strictly stable.

1. Introduction

The recent development of continuous logic has allowed for a more controlled study of the model theory of bounded metric structures. An in-depth introduction to this field, as well as many examples, can be found in [3]. One example of such a metric structure is the *Uryoshn sphere*, i.e the universal Polish metric space of diameter 1. In a rather naive sense, this structure is somewhat analogous to the model completion of the empty theory in first-order logic, which is of course just an infinite set. However, the Urysohn sphere turns out to be far more complicated than an infinite set. Model theoretic results about the Urysohn sphere in continuous logic can be found in [6] and [8]. The characterization of forking and dividing for the Urysohn sphere, as well as a classification in the spectrum of unstable theories, can be found in [4].

In this paper, we study a more well-behaved continuous stucture: $U_{\max}$, the universal Polish ultrametric space with distance set $\mathbb{Q}\cap[0,1]$. We consider the continuous theory $T_{\max}$ of $U_{\max}$ in the same language as that of the Urysohn sphere: the empty language containing only a symbol for the metric d. We first show that arbitrary ultrametric spaces of diameter 1 will embed in a sufficiently saturated model $T_{\max}$. Then we show that $T_{\max}$ has a stable independence relation, which can be seen as a continuous version of independence in the first-order theory of the infinite set. Consequently, $T_{\max}$ is stable, and we further show that it is strictly stable (in the sense of continuous model theory). Finally we note that $T_{\max}$ is, in some sense, the continuous version of the model completion of infinitely refining equivalence relations in first order logic.

2. Ultrametric Spaces

Definition 2.1. *Let (X,d) be a metric space.*

Date: February 26, 2014.

*(1) The **spectrum of** X, $\operatorname{Spec}(X)$, is the set of distances realized by points in X, i.e.* $\operatorname{Spec}(X) = d(X \times X)$.

*(2) The **density character of** X, $\chi(X)$, is the least cardinal κ such that X has a dense subset of size κ.*

*(3) (X, d) is an **ultrametric space** if for all $x, y, z \in X$, $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.*

Proposition 2.2. *Suppose (X, d) is an ultrametric space and $Y \subseteq X$ is dense. Then* $\operatorname{Spec}(X) = \operatorname{Spec}(Y)$.

Proof. We clearly have $\operatorname{Spec}(Y) \leq \operatorname{Spec}(X)$. Conversely, suppose $r \in \operatorname{Spec}(X)$ and $x_1, x_2 \in X$ are such that $d(x_1, x_2) = r$. We may assume $r > 0$. For $i \in \{1, 2\}$, let $y_i \in Y$ such that $d(x_i, y_i) < r$. Then $d(x_2, y_2) < d(x_1, x_2)$ so $d(x_1, y_2) = d(x_1, x_2)$. Then $d(x_1, y_1) < d(x_1, y_2)$, so $d(y_1, y_2) = d(x_1, y_2) = r$. Therefore $r \in \operatorname{Spec}(Y)$. □

Corollary 2.3. *If (X, d) is an infinite ultrametric space then* $|\operatorname{Spec}(X)| \leq \chi(X)$.

Proof. If $Y \subseteq X$ is a dense subset of size $\chi(X)$, then $|\operatorname{Spec}(X)| = |\operatorname{Spec}(Y)| \leq |Y \times Y| = \chi(X)$. □

It follows that there is no Polish ultrametric space that embeds every Polish ultrametric space, since such a space would have uncountable spectrum and therefore fail to be separable. However, one can construct $U_{\max}$, a homogeneous Polish ultrametric space, with spectrum $\mathbb{Q} \cap [0, 1]$, which embeds every Polish ultrametric space X such that $\operatorname{Spec}(X) \subseteq \mathbb{Q} \cap [0, 1]$. The following construction of $U_{\max}$, and subsequent results in this section, are taken from [5].

Definition 2.4. *Let $I = \mathbb{Q} \cap (0, 1]$ and $\mathcal{U}_{\max} = \omega^I$. Given $f, g \in \mathcal{U}_{\max}$, let*

$$d(f, g) = \sup\{x \in I : f(x) \neq g(x)\},$$

where $\sup \emptyset = 0$.

Proposition 2.5. *$(\mathcal{U}_{\max}, d)$ is an ultrametric space, with* $\operatorname{Spec}(\mathcal{U}_{\max}) = [0, 1]$.

Proof. Fix $f, g \in \mathcal{U}_{\max}$. We clearly have $d(f, g) = d(g, f)$. Moreover,

$$d(f, g) = 0 \quad \Leftrightarrow \quad \{x \in I : f(x) \neq g(x)\} = \emptyset \quad \Leftrightarrow \quad f = g.$$

Next, fix $f, g, h \in \mathcal{U}_{\max}$. Without loss of generality, assume $d(f, g) \geq d(f, h)$. If $x \in I$ is such that $x > d(f, g)$ then $g(x) = f(x) = h(x)$. Therefore $d(f, g) \geq d(g, h)$. If $d(f, g) = d(f, h)$ then the result follows, so we may assume $d(f, g) > d(f, h)$. For any $x \in I$, with $d(f, h) < x < d(f, g)$, we have $g(x) \neq f(x) = h(x)$. Therefore $d(g, h) \geq d(f, g)$, and so we have $d(f, h) < d(f, g) = d(g, h)$, shows that $\mathcal{U}_{\max}$ is an ultrametric space.

Finally, fix $r \in [0, 1]$ and let $f : I \cap \omega$ be the characteristic function of $I \cap (0, r]$. Let $g : I \longrightarrow \omega$ be the constant 0 function. Then $d(f, g) = \sup\{x \in I : x \leq r\} = r$. Therefore $\operatorname{Spec}(\mathcal{U}_{\max}) = [0, 1]$. □

Definition 2.6. *Given $r \in I$, let $I_r = \mathbb{Q} \cap [r, 1]$. Define*

$$U_{\max} = \{f \in \mathcal{U}_{\max} : \textit{for all } r \in I,\ I_r \cap \operatorname{supp}(f) \textit{ is finite}\}.$$

Proposition 2.7. $\operatorname{Spec}(U_{\max}) = \mathbb{Q} \cap [0, 1]$.

Proof. Given $r \in I$, let $f_r : I \longrightarrow \omega$ be the characteristic function of $\{r\}$. If g is the constant 0 function, then $d(f_r, g) = \sup\{r\} = r$. Therefore $\mathbb{Q} \cap [0, 1] \subseteq \operatorname{Spec}(U_{\max})$.

Next, fix distinct $f, g \in U_{\max}$ and let $r = \sup\{x \in I : f(x) \neq g(x)\} \in (0, 1]$. We want to show $r \in \mathbb{Q}$. Fix $q \in \mathbb{Q}$ such that $0 < q < r$. Then

$$\{x \in I : x \geq q,\ f(x) \neq g(x)\} \subseteq (I_q \cap \operatorname{supp}(f)) \cup (I_q \cap \operatorname{supp}(g)),$$

and so $\{x \in I : x \geq q,\ f(x) \neq g(x)\}$ is finite. It follows that

$$r = \sup\{x \in I : f(x) \neq g(x)\} = \sup\{x \in I : x \geq q,\ f(x) \neq g(x)\} \in \mathbb{Q}. \qquad \square$$

Proposition 2.8. *$\mathcal{U}_{\max}$ and $U_{\max}$ are complete.*

Proof. Let $(f_n)_{n<\omega}$ be a Cauchy sequence in $\mathcal{U}_{\max}$. For each $k < \omega$, there is some $N_k < \omega$ such that for all $m, n \geq N_k$, $d(f_m, f_n) < \frac{1}{k}$. Assume $N_k < N_{k+1}$ for all $k < \omega$.

Given $k < \omega$, and $m, n \geq N_k$, if $x \in I$ is such that $x \geq \frac{1}{k}$, then

$$x \geq \tfrac{1}{k} > d(f_m, f_n) = \sup\{x \in I : f_n(x) \neq f_m(x)\},$$

and so $f_n(x) = f_m(x)$. Therefore we may define $f : I \longrightarrow \omega$ such that if $x \in I$, with $x \geq \frac{1}{k}$, then $f(x) = f_{N_k}(x)$.

We first show that $\lim_{n\to\infty} f_n = f$. Indeed, given $\epsilon > 0$, if $k < \omega$ is such that $\frac{1}{k} < \epsilon$ then for any $n \geq N_k$, we have

$$d(f_n, f) = \sup\{x \in I : f_n(x) \neq f(x)\} \leq \tfrac{1}{k} < \epsilon,$$

since for any $x \geq \frac{1}{k}$, we have $f(x) = f_{N_k}(x) = f_n(x)$.

This shows that $\mathcal{U}_{\max}$ is complete. To show that $U_{\max}$ is complete, we assume $f_n \in U_{\max}$ for all $n < \omega$ and show that the function f constructed above is also in $U_{\max}$. For this, fix $r \in I$. We want to show that $\{x \in I : x \geq r,\ f(x) \neq 0\}$ is finite. Let $k < \omega$ be such that $\frac{1}{k} \leq r$. Then if $x \in I$ is such that $x \geq r$, we have $f(x) = f_{N_k}(x)$, and so $I_r \cap \operatorname{supp}(f) = I_r \cap \operatorname{supp}(f_{N_k})$, which is finite by assumption. $\square$

Proposition 2.9. *$U_{\max}$ is separable.*

Proof. Given $r \in I$, set $X_r = \{t \in \omega^{I_r} : \mathrm{supp}(t) \text{ is finite}\}$. Note that each X_r is countable. Given $t \in X_r$, define $f_t : I \longrightarrow \omega$ such that

$$f_t(x) = \begin{cases} t(x) & \text{if } x \geq r \\ 0 & \text{if } x < r. \end{cases}$$

Let $Y = \{f_t : t \in X_r,\ r \in I\}$, which is a countable subset of $U_{\max}$. We show that Y is dense in $U_{\max}$. Indeed, fix $f \in U_{\max}$ and $\epsilon > 0$. Let $r \in I$ be such that $r < \epsilon$. Then $t := f|_{I_r} \in X_r$ and

$$d(f, f_t) = \sup\{x \in I : f(x) \neq f_t(x)\} \leq r < \epsilon. \qquad \square$$

Theorem 2.10. [5]

(a) $U_{\max}$ *is a Polish ultrametric space with spectrum* $\mathbb{Q} \cap [0,1]$.

(b) *If* (X,d) *is a Polish ultrametric space, with* $\mathrm{Spec}(X) \subseteq \mathbb{Q} \cap [0,1]$, *then* (X,d) *is isometric to a subspace of* $U_{\max}$.

(c) *Any isometry between two compact subspaces of* $U_{\max}$ *extends to an isometry of* $U_{\max}$.

Note that Cantor space, 2^ω, with the metric $d(f,g) = \frac{1}{\min\{n<\omega : f(n) \neq g(n)\}+1}$ is a Polish ultrametric space, with $\mathrm{Spec}(2^\omega) \subseteq \mathbb{Q} \cap [0,1]$. Therefore $|U_{\max}| = 2^{\aleph_0}$. Obviously, we also have $|\mathcal{U}_{\max}| = 2^{\aleph_0}$. On the other hand, $\chi(U_{\max}) = \aleph_0$ and $\chi(\mathcal{U}_{\max}) = 2^{\aleph_0}$.

3. Continuous Model Theory of $U_{\max}$

We assume the reader is familiar with the treatment of metric structures in continuous logic (see [3] for a full introduction). Let $T_{\max} = \mathrm{Th}(U_{\max})$ in the language containing only the metric d. Let $\mathbb{U}_{\max}$ be a sufficiently saturated monster model of $T_{\max}$.

Given $r \in [0,1]$, we let $d_r(x,y)$ denote the following formula:

$$\max\{d(x,y) \dot{-} r, r \dot{-} d(x,y)\}.$$

We will also write $d(x,y) = r$ for the condition $d_r(x,y) = 0$.

Theorem 3.1. *If* (X,d) *is an ultrametric space, with* $\mathrm{Spec}(X) \subseteq [0,1]$, *and* $\mathbb{U}_{\max}$ *is* $\chi(X)$*-saturated then* X *is isometric to a subspace of* $\mathbb{U}_{\max}$.

Proof. We may replace X with a dense subset and assume that $\mathbb{U}_{\max}$ is $|X|$-saturated. By compactness and saturation, it suffices to assume that X is finite. Let $X = \{a_1, \ldots, a_n\}$ and, given $1 \leq i,j \leq n$, let $r_{i,j} = d(a_i, a_j) \in [0,1]$. Define

$$\varphi(x_1, \ldots, x_n) = \max\{d_{r_{i,j}}(x_i, x_j) : 1 \leq i < j \leq n\}.$$

We want to show that "$\inf_{x_1,\dots,x_n} \varphi(x_1,\dots,x_n) = 0$" $\in T_{\max}$, i.e, we fix $\epsilon > 0$ and find $c_1,\dots,c_n \in U_{\max}$ such that $\varphi(c_1,\dots,c_n) < \epsilon$. Given $1 \leq i < j \leq n$, let $s_{i,j}, t_{i,j} \in \mathbb{Q} \cap [0,1]$ such that $r_{i,j} - \frac{\epsilon}{2} \leq s_{i,j} < r_{i,j} < t_{i,j} \leq r_{i,j} + \frac{\epsilon}{2}$.

Consider $(\mathbb{Q},<)$ and $(\mathbb{R},<)$ as first order structures in the language $\mathcal{L} = \{<\}$. Let $\bar{v} = (v_{i,j})_{1\leq i<j\leq n}$ and define the following first order $\mathcal{L}$-formula:

$$\begin{aligned}\theta(\bar{v}) := &\bigwedge_{1\leq i<j\leq n} s_{i,j} < v_{i,j} < t_{i,j} \\ &\wedge \bigwedge_{1\leq i<j<k\leq n} (v_{i,j} \leq \max\{v_{i,k}, v_{j,k}\} \wedge v_{j,k} \leq \max\{v_{i,j}, v_{i,k}\} \wedge v_{i,k} \leq \max\{v_{i,j}, v_{j,k}\})\end{aligned}$$

(where $u \leq \max\{v,w\}$ is shorthand for $(v \leq w \to u \leq w) \wedge (w \leq v \to u \leq v)$). If $\bar{r} = (r_{i,j})_{1\leq i<j\leq n}$ then $\mathbb{R} \models \theta(\bar{r})$. Since $(\mathbb{Q},<) \prec (\mathbb{R},<)$, it follows that $\mathbb{Q} \models \exists \bar{v}\theta(\bar{v})$. Let $q_{i,j} \in \mathbb{Q}$ be such that $\mathbb{Q} \models \theta(\bar{q})$.

Define the space $Y = \{c_1,\dots,c_n\}$ with $d(c_i,c_j) = q_{i,j}$, for $1 \leq i < j \leq n$. Then Y is an ultrametric space, with $\mathrm{Spec}(Y) \subseteq \mathbb{Q} \cap [0,1]$. So we may assume $Y \subseteq U_{\max}$. For any $1 \leq i < j \leq n$, we have $r_{i,j} - \frac{\epsilon}{2} < d(c_i,c_j) < r_{i,j} + \frac{\epsilon}{2}$. Therefore $d_{r_{i,j}}(c_i,c_j) < \epsilon$, and so $\varphi(c_1,\dots,c_n) < \epsilon$, as desired. □

Next, we prove that $T_{\max}$ has quantifier elimination. The proof is essentially the same as quantifier elimination for the Urysohn sphere (see [8]).

Theorem 3.2. *$T_{\max}$ has quantifier elimination.*

Proof. Given $\bar{a} = (a_1,\dots,a_n) \in \mathbb{U}_{\max}$ and $C \subset \mathbb{U}_{\max}$, note that the quantifier-free type of $\bar{a}$ over C is entirely determined by the following quantifier-free type:

$$\{d(x_i,x_j) = d(a_i,a_j) : 1 \leq i,j \leq n\} \cup \{d(x_i,c) = d(a_i,c) : 1 \leq i \leq n,\ c \in C\}.$$

We use quantifier elimination techniques outlined in [3]. In particular, fix a quantifier-free formula $\varphi(x,y_1,\dots,y_n)$. We want to show that the formula $\inf_x \varphi(x,\bar{y})$ is approximable in $T_{\max}$ by quantifier-free formulas. Fix $M,N \models T_{\max}$, substructures $M_0 \subseteq M$ and $N_0 \subseteq N$, an isomorphism Φ from M_0 onto N_0, and elements $a_1,\dots,a_n \in M_0$. It suffices to show that for any $\epsilon > 0$,

$$\inf_x^N \varphi(x,\Phi(a_1),\dots,\Phi(a_n)) < \inf_x^M \varphi(x,a_1,\dots,a_n) + \epsilon.$$

Let $b \in M$ be such that $\varphi^M(b,\bar{a}) < \inf_x^M \varphi(x,\bar{a})$. Note that, since Φ is an isometry, the space $X = \{x,\Phi(a_1),\dots,\Phi(a_n)\}$ with $d(\Phi(a_i),\Phi(a_j)) = d(a_i,a_j)$ and $d(x,\Phi(a_i)) = d(b,a_i)$ is an ultrametric space. Therefore the type

$$\{d(x,\Phi(a_i)) = d(b,a_i) : 1 \leq i \leq n\}$$

is realized by some $c' \in \mathcal{N}$, where $\mathcal{N}$ is a saturated elementary extension of N. We clearly have that Φ extends to an isomorphism from $\{a_1,\dots,a_n,b\}$ to $\{\Phi(a_1),\dots,\Phi(a_n),b\}$ in the obvious way. Therefore

$\varphi(c', \Phi(\bar{a})) = \varphi(b, \bar{a})$, since $\varphi(x, \bar{y})$ is quantifier-free. It follows that

$$\inf_x^N \varphi(x, \Phi(\bar{a})) = \inf_X^{\mathcal{N}} \varphi(x, \Phi(\bar{a})) \leq \varphi(c', \Phi(\bar{a})) = \varphi(b, \bar{a}) < \inf_x^M \varphi(x, \bar{a}) + \epsilon,$$

as desired. □

By quantifier elimination, given $C \subset \mathbb{U}_{\max}$ and $\bar{a} = (a_1, \ldots,_n) \in \mathbb{U}_{\max}$ the complete type $\mathrm{tp}(\bar{a}/C)$ is completely determined by its quantifier free type:

$$\{d(x_i, x_j) = d(a_i, a_j) : 1 \leq i, j \leq n\} \cup \{d(x_i, c) = d(a_i, c) : 1 \leq i \leq n,\ c \in C\}.$$

Moreover, by quantifier elimination it follows that $U_{\max} \prec \mathcal{U}_{\max}$. Note that if $r \in [0, 1]$ then the type $\{d(x, y) = r\}$ determines a complete type in $S_2(T_{\max})$. If $r \notin \mathbb{Q}$ then $\{d(x, y) = r\}$ is omitted in $U_{\max}$. It follows that $T_{\max}$ is not separably categorical. Moreover, by Corollary 2.3, $T_{\max}$ does not have a separable saturated model.

4. Stability in Continous Model Theory

Let T be a complete continuous theory, and $\mathbb{M}$ a monster model of T. The following definitions and results are quoted from [3].

Definition 4.1. *Given $A \subset \mathbb{M}$, we define the **d-metric on** $S_n(A)$ as follows: given $p, q \in S_n(A)$, define*

$$d(p, q) = \inf \left\{ \max_{1 \leq i \leq n} d(b_i, c_i) : (b_1, \ldots, b_n) \models p,\ (c_1, \ldots, c_n) \models q \right\}.$$

Definition 4.2. *Let λ be an infinite cardinal.*

*(1) T is **λ-stable with respect to the discrete metric** if for all $A \subset \mathbb{M}$, if $|A| \leq \lambda$ then $|S_1(A)| \leq \lambda$.*

*(2) T is **λ-stable** if for all $A \subset \mathbb{M}$, if $|A| \leq \lambda$ then the density character of $S_1(A)$, with respect to the d-metric, is at most λ.*

*T is **stable** if it is λ-stable for some λ.*

Theorem 4.3. [3] *If T is stable then T is λ-stable with respect to the discrete metric for any λ such that $\lambda^{|T|} = \lambda$.*

Definition 4.4. *Let T be a complete continuous theory and $\mathbb{M}$ a monster model of T. A ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ is a **stable independence relation** if it satisfies the following properties:*

*(i) **(invariance)** For all $A, B, C \subset \mathbb{M}$ and $\sigma \in \mathrm{Aut}(\mathbb{M})$, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $\sigma(A) \mathop{\smile\!\!\!\!\!\!|}_{\sigma(C)} \sigma(B)$.*

*(ii) **(symmetry)** For all $A, B, C \subset \mathbb{M}$, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $B \mathop{\smile\!\!\!\!\!\!|}_C A$.*

*(iii) **(full transitivity)** For all $A, B, C, D \subset \mathbb{M}$, $A \mathop{\smile\!\!\!\!\!\!|}_C BD$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$.*

*(iv) **(finite character)** For all $A, B, C \subset \mathbb{M}$, $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if $A_0 \mathop{\smile\!\!\!\!\!\!|}_C B_0$ for all finite $A_0 \subseteq A$ and $B_0 \subseteq B$.*

(v) ***(full existence)*** *For all $A, B, C \subset \mathbb{M}$ there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!\:|}_C B$.*

(vi) ***(local character)*** *For all $A \subset \mathbb{M}$ there is a cardinal $\kappa(A)$ such that for all $B \subset \mathbb{M}$ there is $C \subseteq B$, with $|C| \leq \kappa(A)$, such that $A \mathop{\smile\!\!\!\!\!\!\:|}_C B$.*

(vii) ***(stationarity over models)*** *For all $A, A', B \subset \mathbb{M}$ and models $M \subset \mathbb{M}$, if $A \mathop{\smile\!\!\!\!\!\!\:|}_M B$, $A' \mathop{\smile\!\!\!\!\!\!\:|}_M B$, and $A \equiv_M A'$, then $A \equiv_{BM} A'$.*

Theorem 4.5. [3] *Let T be a complete continuous theory and $\mathbb{M}$ a monster model of T. Then T is stable if and only if T has a stable independence relation. Moreover, if T has a stable independence relation $\mathop{\smile\!\!\!\!\!\!\:|}$, then $\mathop{\smile\!\!\!\!\!\!\:|} = \mathop{\smile\!\!\!\!\!\!\:|}^f = \mathop{\smile\!\!\!\!\!\!\:|}^d$.*

5. Stability and Independence in $T_{\max}$

Using Theorem 4.5, we can show that $T_{\max}$ is stable and characterize forking independence.

Theorem 5.1. *Given $A, B, C \subset \mathbb{U}_{\max}$,*

$$A \mathop{\smile\!\!\!\!\!\!\:|}^f_C B \quad \Leftrightarrow \quad \text{for all } a \in A,\ d(a, BC) = d(a, C)$$

Proof. Define the ternary relation $\mathop{\smile\!\!\!\!\!\!\:|}$ such that $A \mathop{\smile\!\!\!\!\!\!\:|}_C B$ if and only if for all $a \in A$, $d(a, BC) = d(a, C)$. We show that $\mathop{\smile\!\!\!\!\!\!\:|}$ satisfies the necessary properties to characterize forking.

Invariance, Full Transitivity: Trivial.

Symmetry: Suppose $A \mathop{\smile\!\!\!\!\!\!\:|}_C B$ and fix $b \in B$. For a contradiction, suppose $d(b, AC) < d(b, C)$. Then there is some $a \in A$ such that $d(a, b) < d(b, C)$. Given $c \in C$, we have $d(a, b) < d(b, c)$, so it follows that $d(a, c) = d(b, c)$. Therefore,

$$d(a, BC) \leq d(a, b) < d(b, C) = d(a, C),$$

which contradicts $A \mathop{\smile\!\!\!\!\!\!\:|}_C B$.

Finite Character: Suppose $A \mathop{\smile\!\!\!\!\!\!\:|}_C B$. Given $A_0 \subseteq A$ finite, we have $A_0 \mathop{\smile\!\!\!\!\!\!\:|}_C B$ by definition of $\mathop{\smile\!\!\!\!\!\!\:|}$. Therefore $A_0 \mathop{\smile\!\!\!\!\!\!\:|}_C B_0$ by monotonicity. Conversely, suppose $A_0 \mathop{\smile\!\!\!\!\!\!\:|}_C B_0$ for all finite $A_0 \subseteq A$ and $B_0 \subseteq B$. Given $a \in A$ if $d(a, BC) < d(a, C)$ then there is $b \in B$ such that $d(a, b) < d(a, C)$, which contradicts $a \mathop{\smile\!\!\!\!\!\!\:|}_C b$.

Full Existence: Given A, B, C, we may find $A' \equiv_C A$ such that for all $a' \in A'$ and $b \in B$,

$$d(a', b) = \inf_{c \in C} \max\{d(a, c), d(b, c)\}$$

(where $\inf \emptyset = 1$). In other words, A' is constructed by taking the usual free amalgamation of AC and BC over C. We show that $A' \mathop{\smile\!\!\!\!\!\!\:|}_C B$. Fix $a' \in A'$. For any $b \in B$, we have

$$d(a', C) = d(a, C) = \inf_{c \in C} d(a, c) \leq \inf_{c \in C} \max\{d(a, c), d(b, c)\} = d(a', b).$$

Therefore $d(a', BC) = d(a', C)$.

Local Character: Fix A, B. Given $a \in A$, let $(b^a_n)_{n<\omega}$ be a sequence from B such that $(d(a, b^a_n)) \to d(a, B)$. Now define

$$C = \bigcup_{a \in A} (b^a_n)_{n<\omega},$$

and note that $|C| \leq |A| + \aleph_0$. We claim that $A \mathop{\smile\!\!\!\!\!\!|}_C B$. Indeed, given $a \in A$ and $\epsilon > 0$, find $n < \omega$ such that $d(a, b^a_n) < d(a, B) + \epsilon$. Then $d(a, C) \leq d(a, b^a_n) < d(a, B) + \epsilon$. It follows that $d(a, B) = d(a, C)$.

Stationarity Over Models: We will actually show stationarity over all sets (i.e. M need not be a model). Fix A, A', B, C such that $A \equiv_C A'$, $A \mathop{\smile\!\!\!\!\!\!|}_C B$, and $A' \mathop{\smile\!\!\!\!\!\!|}_C B$. To prove $A' \equiv_{BC} A$, it suffices to show that $d(a, b) = d(a', b)$ for all $b \in B$. So suppose, towards a contradiction, that $d(a', b) < d(a, b)$ for some $b \in B$. Note that this means $d(a, a') = d(a, b)$. On the other hand, $d(a', C) = d(a', BC) < d(a, b)$, and so there is some $c \in C$ such that $d(a', c) < d(a, b) = d(a, a')$. Therefore $d(a, c) = d(a, a') > d(a', c)$, which contradicts $A \equiv_C A'$. $\square$

Corollary 5.2. *$T_{\max}$ is stable.*

This characterization of forking independence in $T_{\max}$ can be seen as a continuous version of forking independence in infinite sets: $A \mathop{\smile\!\!\!\!\!\!|}^f_C B$ if and only if $A \cap B \subseteq C$ (this is also the characterization of forking in the random graph). In these theories forking only happens as a result of changes in equality. In $T_{\max}$ forking only happens as a result of changes in distance.

Theorem 5.1 also follows from a straightforward generalization of the characterization of forking and dividing in the Urysohn sphere from [4], giving a proof that does not rely on Theorem 4.5.

In the proof of Theorem 5.1, we utilize the free amalgamation of ultrametric spaces. In the theory of the Urysohn sphere, free amalgamation of metric spaces yields a ternary relation: $A \mathop{\smile\!\!\!\!\!\!|}^{\otimes}_B C$ if and only if for all $a \in A$ and $b \in B$, $d(a, b) = \inf_{c \in C}(d(a, c) + d(b, c))$. This is a *stationary independence relation*, i.e. a ternary relation satisfying all of the axioms for a stable independence relation except possibly local character (see [7]). In any theory T, if $\mathop{\smile\!\!\!\!\!\!|}$ is a stationary independence relation then one may show that $\mathop{\smile\!\!\!\!\!\!|} \Rightarrow \mathop{\smile\!\!\!\!\!\!|}^f$ (e.g. using methods of [1]). If T is unstable (e.g. the Urysohn sphere), this implication must be strict since nonforking cannot satisfy stationarity.

In $T_{\max}$, we can consider the stationary independence relation given by free amalgamation of *ultrametric* spaces, and show that it coincides with forking independence.

Corollary 5.3. *For all $A, B, C \subset \mathbb{U}_{\max}$,*

$$A \mathop{\smile\!\!\!\!\!\!|}^f_C B \quad \Leftrightarrow \quad \text{for all } a \in A,\ b \in B,\ d(a, b) = \inf_{c \in C} \max\{d(a, c), d(b, c)\}.$$

This can be verified directly using the characterization of $\mathop{\smile\!\!\!\!\!\!|}^f$ in Theorem 5.1. Alternatively, one could show that in ultrametric spaces, free amalgamation is a stable independence relation.

Next, we show that $T_{\max}$ is *strictly stable.*

Theorem 5.4. *Given an infinite cardinal λ, $T_{\max}$ is λ-stable if and only if $\lambda^{\aleph_0} = \lambda$.*

Proof. The reverse direction is by Theorem 4.3. For the forward direction, suppose $\lambda^{\aleph_0} > \lambda$. Define $f : \mathbb{N} \longrightarrow [0,1] \cap \mathbb{Q}$ such that

$$f(n) = \frac{1}{n+2} + \frac{1}{2},$$

and note that f is strictly decreasing. Next, we define the following space (A_λ, d). Let $A_\lambda = (a_\eta)_{\eta \in \lambda^{<\omega}}$ and, for $\eta \neq \mu$, define

$$d(a_\eta, a_\mu) = f(|\nu|),$$

where ν is the meet of η and μ.

Claim 1: (A_λ, d) is an ultrametric space.

Proof: Fix distinct $\mu_1, \mu_2, \mu_3 \in \lambda^{<\omega}$. For distinct $1 \le i < j \le 3$, let $\nu_{i,j}$ be the meet of μ_i and μ_j. Without loss of generality, it suffices to assume $|\nu_{1,2}| > |\nu_{1,3}|$ and show that $|\nu_{2,3}| = |\nu_{1,3}|$. But in this case, since $\nu_{1,2} \subseteq \mu_1$ and $\nu_{1,3} \subseteq \mu_1$, it follows that $\nu_{1,3} \subset \nu_{1,2}$. Therefore $\nu_{2,3} = \nu_{1,3}$, and so the desired result follows. $/\!\!/$

We may assume $\mathbb{U}_{\max}$ is λ^+-saturated, and so (A_λ, d) is a subspace of $\mathbb{U}_{\max}$. Fix $\sigma \in \lambda^\omega$ and define

$$p_\sigma = \{d(x, a_{\sigma|_n}) = f(n) : n < \omega\} \cup \{d(x, a_{\tau|_n}) = f(m) : \tau \in \lambda^\omega,\ \tau \neq \sigma,\ m = \max\{i < \omega : \sigma|_i = \tau|_i\},\ n > m\}.$$

Claim 2: p_σ is a consistent 1-type over A_λ.

Proof: It suffices to show that $(A_\lambda \cup \{x\}, d)$ is an ultrametric space. In particular, we must check the triangle inequality in the following cases.

Case 1: $\{x, a_{\sigma|_m}, a_{\sigma|_n}\}$, for some $m < n$. We have $d(a_{\sigma|_m}, a_{\sigma|_n}) = f(m)$, $d(x, a_{\sigma|_m}) = f(m)$, and $d(x, a_{\sigma|_n}) = f(n)$.

Case 2: $\{x, a_{\sigma|_m}, a_{\tau|_n}\}$, where $\tau \in \lambda^\omega$, $\tau \neq \sigma$, $n > k := \max\{i < \omega : \sigma|_i = \tau|_i\}$. If $m \le k$ then we have $d(a_{\sigma|_m}, a_{\tau|_m}) = f(m)$, $d(x, a_{\sigma|_m}) = f(m)$, and $d(x, a_{\tau|_n}) = f(k)$. If $m > k$ then we have $d(a_{\sigma|_m}, a_{\tau|_m}) = f(k)$, $d(x, a_{\sigma|_m}) = f(m)$, and $d(x, a_{\tau|_n}) = f(k)$.

Case 3: $\{x, a_{\rho|_m}, a_{\tau|_n}\}$, where $\rho, \tau \in \lambda^\omega \backslash \{\sigma\}$, $k := \max\{i < \omega : \sigma|_i = \rho|_i\}$, $l := \max\{i < \omega : \sigma|_i = \tau|_i\}$, $k \le l$, $k < m$, and $l < n$. We have $d(x, a_{\rho|_m}) = f(k)$ and $d(x, a_{\tau|_n}) = f(l)$. If $k < l$ then $\rho|_k$ is the meet of $\rho|_m$ and $\tau|_n$, so $d(a_{\rho|_m}, a_{\tau|_n}) = f(k)$. If $k = l$ then the meet of $\rho|_m$ and $\tau|_n$ has length $r > k$ and $d(a_{\rho|_m}, a_{\tau|_n}) = f(r)$. $/\!\!/$

Next, fix distinct $\sigma, \tau \in \lambda^\omega$ and let $n = \max\{i < \omega : \sigma|_i = \tau|_i\}$. Let $b \models p_\sigma$ and $c \models p_\tau$. Then $d(b, a_{\sigma|_{n+1}}) = f(n+1)$ and $d(c, a_{\sigma|_{n+1}}) = f(n)$. Therefore $d(b,c) = f(n)$, and so we have $d(p_\sigma, p_\tau) = f(n) \ge \frac{1}{2}$. If $D \subseteq S_1(A_\lambda)$ is dense then, for each $\sigma \in \lambda^\omega$, there is some $q_\sigma \in D \cap B_{\frac{1}{2}}(p_\sigma)$. Therefore, if $\sigma, \tau \in \lambda^\omega$ are distinct, we have $d(q_\sigma, q_\tau) \ge \frac{1}{2}$. In particular, $|D| \ge \lambda^{\aleph_0} > \lambda$. Altogether, $\chi(S_1(A_\lambda)) > |A_\lambda|$. □

As a final remark, we note the connection between ultrametric spaces and refining equivalence relations. In particular, if (X, d) is an ultrametric space, then for any $r \in [0,1]$, $d(x,y) \le r$ is an equivalence relation.

Conversely, suppose we have equivalence relations $(E_r)_{r\in D}$, where $D \subseteq [0,1]$, and we consider a structure M on which these equivalence relations refine according the ordering (with E_0 equality). Then M can be considered as a (pseudo)ultrametric space when equipped with $d(a,b) = \inf\{r \in D : aE_rb\}$. For example Baire space can be considered as a model of infinitely refining equivalence relations indexed by $D = \{\frac{1}{n} : n \in \mathbb{N}\}$. Theories of refining equivalence relations are standard examples in stability theory (see e.g. [2]). Indeed, much of the previous work is guided by the behavior found in these examples.

References

[1] Hans Adler, *A geometric introduction to forking and thorn-forking*, J. Math. Log. **9** (2009), no. 1, 1–20. MR 2665779 (2011i:03026)

[2] John T. Baldwin, *Fundamentals of stability theory*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1988. MR 918762 (89k:03002)

[3] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 350, Cambridge Univ. Press, Cambridge, 2008, pp. 315–427. MR 2436146 (2009j:03061)

[4] Gabriel Conant and Caroline Terry, *Model theory of the Urysohn sphere*, arXiv:1401.2132 [math.LO], 2014.

[5] Su Gao and Chuang Shao, *Polish ultrametric Urysohn spaces and their isometry groups*, Topology Appl. **158** (2011), no. 3, 492–508. MR 2754373 (2011m:54026)

[6] Isaac Goldbring, *Definable functions in Urysohn's metric space*, Illinois J. Math. **55** (2011), no. 4, 1423–1435 (2013). MR 3082876

[7] Katrin Tent and Martin Ziegler, *On the isometry group of the Urysohn space*, J. Lond. Math. Soc. (2) **87** (2013), no. 1, 289–303. MR 3022717

[8] Alexander Usvyatsov, *Generic separable metric structures*, Topology Appl. **155** (2008), no. 14, 1607–1617. MR 2435152 (2009g:54063)

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 S Morgan St, 322 SEO, Chicago, IL, 60607, USA;

E-mail address: `gconan2@uic.edu`