Chain Rule

One Independent Variable

$z = z(x, y), \quad x = x(t), \quad y = y(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Two Independent Variables

$z = z(x, y), \quad x = x(s, t), \quad y = y(s, t)$

$$\frac{\partial z}{ds} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Implicit Differentiation

If $F(x, y) = 0$ and $F_y \neq 0$ then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Gradient

Two Dimensions

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

Three Dimensions

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$

Directional Derivative

f differentiable at (a, b) and $\mathbf{u}$ a unit vector,

$$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u}$$

Tangent Plane

of level surface $F(x, y, z) = 0$ **at** (a, b, c)

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(x - b) + F_z(a, b, c)(x - c) = 0$$

of function $z = f(x, y)$ **at point** $(a, b, f(a, b))$

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

Second Derivative Test
Let $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$.

1. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local max at (a,b).
2. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local min at (a,b).
3. If $D(a,b) < 0$, then f has a saddle point at (a,b).
4. If $D(a,b) = 0$, then the test is inconclusive.

Lagrange Multipliers
f is optimization function, g is the constraint, f and g are differentiable with $\nabla g(x,y) \neq 0$ on the curve $g(x,y) = 0$. To locate maximum and minimum values of f subject to the constraint $g(x,y) = 0$:

1. Solve for x, y and λ:

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{and} \quad g(x,y) = 0$$

2. Among the values (x,y) from step 1, calculate the max and min.

Note: For three dimensions, solve $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = 0$.

Double Integrals in Rectangular Coordinates
R is region bounded below and above by $y = g(x)$ and $y = h(x)$, respectively, and on the left and right by the lines $x = a$ and $x = b$, respectively. If f is continuous on R then

$$\iint\limits_R f(x,y)\ dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\ dy\,dx$$

R is region bounded on the left and right by $x = g(y)$ and $x = h(y)$, respectively, and below and above by the lines $y = c$ and $y = d$, respectively. If f is continuous on R then

$$\iint\limits_R f(x,y)\ dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\ dx\,dy$$

Double Integral in Polar Coordinates
f is continuous in $R = \{(r,\theta) : 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\}$. Then

$$\iint_R f(r,\theta)\ dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Triple Integral in Rectangular Coordinates
f continuous in $D = \{(x,y,z) : a \leq x \leq b,\ g(x) \leq y \leq h(x),\ G(x,y) \leq z \leq H(x,z)\}$

$$\iiint_D f(x,y)\ dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)\ dx\, dy\, dz$$

Triple Integral in Cylindrical Coordinates
f continuous in $D = \{(r,\theta,z) : g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta,\ G(x,y) \leq z \leq H(x,z)\}$

$$\iiint_D f(x,y)\ dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r,\theta,z)\ dz\, r\, dr\, d\theta$$

Triple Integral in Spherical Coordinates
f continuous in $D = \{(\rho,\varphi,\theta) : g(\varphi,\theta) \leq \rho \leq h(\varphi,\theta),\ a \leq \varphi \leq b,\ \alpha \leq \theta \leq \beta\}$

$$\iiint_D f(x,y)\ dV = \int_\alpha^\beta \int_a^b \int_{g(\varphi,\theta)}^{h(\varphi,\theta)} f(\rho,\varphi,\theta)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

Note:

$$\rho^2 = x^2 + y^2 + z^2 \qquad x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$