Center of Mass

Two Dimensions: $R$is a region of density $\rho(x,y)$. The center of mass is $(\bar{x},\bar{y})$ where

$$\bar{x}=\frac{1}{m}\iint\limits_R x\rho(x,y)\ dA \qquad \bar{y}=\frac{1}{m}\iint\limits_R y\rho(x,y)\ dA \qquad m=\iint\limits_R \rho(x,y)\ dA$$

Three Dimensions: D is a region of density $\rho(x,y,z)$. The center of mass is $(\bar{x},\bar{y},\bar{z})$ where

$$\bar{x}=\frac{1}{m}\iiint\limits_D x\rho(x,y,z)\ dV \qquad \bar{y}=\frac{1}{m}\iiint\limits_D y\rho(x,y,z)\ dV$$

$$\bar{x}=\frac{1}{m}\iiint\limits_D z\rho(x,y,z)\ dV \qquad m=\iiint\limits_D \rho(x,y,z)\ dV$$

Change of Variables

Two Dimensions: Given a one-to-one transformation $T: x=g(u,v),\ y=h(u,v)$, which maps the region S onto the region R,

$$\iint\limits_R f(x,y)\ dA=\iint\limits_S f(g(u,v),h(u,v))|J(u,v)|\ dA$$

Three Dimensions: Given a one-to-one transformation $T: g(u,v,w),\ y=h(u,v,w),\ z=p(u,v,w)$, which maps the region S onto the region D,

$$\iiint\limits_D f(x,y,z)\ dV = \iiint\limits_S f(g(u,v,w),h(u,v,w),p(u,v,w))|J(u,v,w)|\ dV$$

Scalar Line Integrals

If f is a continuous on the smooth curve C, given by $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle$ for $a\le t\le b$, then

$$\int_C f\ ds=\int_a^b f(x(t),y(t),z(t))|\mathbf{r}'(t)|\ dt$$

If f and C are two-dimensional, let $z(t)=0$.

Line Integrals of Vector Fields

If $\mathbf{F}=\langle f,g,h\rangle$ is a vector field and C has parameterization $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle$ for $a\le t\le b$, then

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt$$

where $\mathbf{F}(\mathbf{r}(t))=\langle f(x(t),y(t),z(t)),\ g(x(t),y(t),z(t)),\ h(x(t),y(t),x(t))\rangle$.

If F and C are two dimensional, let $h=0$ and $z(t)=0$.

Work in Force Fields
Let $\mathbf{F}$ be a continuous force field and C a smooth curve. The work done in moving an object along C in the positive direction is

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

Circulation and Flux
The circulation of $\mathbf{F}$ on a closed, smooth curve is

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

The flux of $\mathbf{F} = \langle f, g \rangle$ on a closed, smooth curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \leq t \leq b$ is

$$\int_C \mathbf{F} \cdot \mathbf{n} = \int_a^b f(\mathbf{r}(t))y'(t) - g(\mathbf{r}(t))x'(t)\ dt$$

Conservative Vector Fields
A vector field $\mathbf{F}$ is conservative if $\mathbf{F} = \nabla \varphi$ for some scalar function φ, called a potential function for $\mathbf{F}$.

Test for Conservative Vector Fields: $\mathbf{F}$ is conservative if and only if

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x} \qquad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$$

If $\mathbf{F}$ is two-dimensional, let $h = 0$.

Finding potential functions:

1. Integrate $\varphi_x = f$ with respect to x to obtain φ (include arbitrary $c(y, z)$).
2. Compute φ_y and set it equal to g to obtain $c_y(y, z)$.
3. Integrate $c_y(y, z)$ with respect to y to obtain $c(y, z)$ (include arbitrary $d(z)$).
4. Compute φ_z and set it equal to h to obtain $d(z)$.

Fundamental Theorem for Line Integrals
$\mathbf{F} = \nabla\varphi$ if and only if

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

for all points A and B, and smooth oriented curves C from A to B.

Green's Theorem
Circulation Form: If $\mathbf{F} = \langle f, g \rangle$ then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dA$$

Flux Form:

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint\limits_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \, dA$$

where $\mathbf{n}$ is the outward unit normal vector on C.

Divergence and Curl
Let $\nabla = \left\langle \frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ \frac{\partial}{\partial z} \right\rangle$.
Given $\mathbf{F} = \langle f, g, h \rangle$, the divergence of $\mathbf{F}$ is

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

The curl of $\mathbf{F}$ is

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$

Surface Integrals, parameterized
Suppose S is a surface parameterized by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ for (u, v) in a region R.
Scalar Functions:

$$\iint\limits_S f(x, y, z) \, dS = \iint\limits_R f(x(u, v), y(u, v), z(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \, dA$$

Vector Fields:

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint\limits_R \mathbf{F}(\mathbf{r}(t)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \, dA$$

Surface Integrals, explicit
Suppose S is a surface given explicitly by $z = p(x, y)$ for (x, y) in a region R.
Scalar Functions:

$$\iint\limits_S f(x, y, z)\; dS = \iint\limits_R f(x, y, p(x, y))\sqrt{z_x^2 + z_y^2 + 1}\; dA$$

Vector Fields:

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\; dS = \iint\limits_R \text{-}f(x, y, p(x, y))z_x - g(x, y, p(x, y))z_y + h(x, y, p(x, y))\; dA$$

Stokes' Theorem
Suppose S is a smooth oriented surface in $\mathbb{R}^3$ with a smooth closed boundary C whose orientation agrees with S. If $\mathbf{F} = \langle f, g, h\rangle$ then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\; dS$$

where $\mathbf{n}$ is the unit normal vector determined by the orientation of S.