Name____________________________

Find the maximum and minimum values of the function $f(x, y) = x - 3y$ subject to the constraint $x^2 + y^2 = 40$. Include in your answer the points where the maximum and minimum values occur.

Solution

The $\nabla f = \lambda g$ equation is

$$\langle 1,\ \text{-}3\rangle = \lambda \langle 2x,\ 2y\rangle ,$$

which gives the equations

$$1 = 2\lambda x \quad \text{and} \quad \text{-}3 = 2\lambda y$$

Notice that $\lambda \neq 0$ (for example $\lambda = 0$ would make the first equation $1 = 0$) so we can solve

$$x = \frac{1}{2\lambda} \quad \text{and} \quad y = \text{-}\frac{3}{2\lambda}$$

In particular

$$y = \text{-}3x$$

Substituting this into the constraint gives

$$\begin{aligned} x^2 + (\text{-}3x)^2 &= 40 \\ x^2 + 9x^2 &= 40 \\ 10x^2 &= 40 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Using $y = \text{-}3x$ this gives the points (2, -6) and (-2, 6). Calculate $f(2, \text{-}6) = 20$ and $f(\text{-}2, 6) = \text{-}20$.

So, on the constraint, the function has a maximum value of 20 at the point (2, -6), and a minimum value of -20 at the point (-2, 6).