EXERCISES

UPDATED: 15 MAY

The following exercises are listed in order of how they appear throughout the course. I am more than happy to discuss the exercises via email or in Zoom appointments outside of lecture. Solutions to the exercises will also be made available upon request. Feedback on the exercises is also welcome (and even turned in solutions).

Most of the exercises below will be used at some point during the course lectures. Some exercises (such as Exercise 0-2) are about basic properties of various notions from the course. On the other hand, some of the exercises have been written so that I can leave details of certain steps in proofs "to the reader". Thus one might want to pay more attention to these exercises, and at least digest the statements and put some thought into the solutions. The exercises of this kind have been marked with $(*)$. I have also added the following annotations:

($\dagger$) This denotes an exercise for which I know of a solution, but it might require extra work or additional tools/tricks. On the other hand, there may be another (possibly simpler or more "obvious") solution that I don't know about. So, in either case, I especially welcome feedback and/or turned in solutions on these exercises.

($\sharp$) This denotes an exercise that is not needed for the main results of the course.

Revision Log: I have made (and will undoubtedly continue to make) some corrections and/or amendments to previously posted exercises. In order to keep the numbering consistent with the lectures, I am making these changes as inconspicuously as possible. So I will briefly list the more serious revisions here to make them easier to find.

- *Exercise 7 (29 April):* I added part $(a.ii)$ after someone pointed out a missing ingredient.
- *Exercise 8 (30 April):* Part (a) has been added to help clarify our use of definable functions with finite image.
- *Exercise 15 (4 May):* The remark has been updated with a new result from the literature.
- *Exercises 5 & 6 (7 May):* Some small typos fixed.

General Comment: I am posting these exercises according the schedule of when they appear in lecture. However, some days (e.g. lecture 2) contain many more exercises than one can reasonably digest in a day or two. So one should not feel pressured to keep up with the schedule; and it is perfectly reasonable to revisit earlier exercises later on when they become more relevant to the material. The rate of "exercises per lecture" will also decrease significantly in the second half of the term.

Exercises from Lecture 1 (24 April)

Exercise 0. Let I be a nonempty set.

(a) ($\sharp$) Show that if I is finite then any ultrafilter $\mathcal{U}$ on I is principal, i.e., there is some $i \in I$ such that $\mathcal{U} = \{X \subseteq I : i \in X\}$.

(b) Show that if I is infinite then an ultrafilter $\mathcal{U}$ on I is nonprincipal if and only if no finite subset of I is in $\mathcal{U}$.

(c) Let $\mathcal{F} \subseteq \mathcal{P}(I)$ be a collection of nonempty subsets of I, and assume $\mathcal{F}$ is closed under finite intersections. Show that there is an ultrafilter $\mathcal{U}$ on I such that $\mathcal{F} \subseteq \mathcal{U}$.
Hint: Use Zorn's Lemma.

(d) Suppose I is infinite. Let $\mathcal{F} \subseteq \mathcal{P}(I)$ be a collection of infinite subsets of I, and assume $\mathcal{F}$ is closed under finite intersections. Show that there is a nonprincipal ultrafilter $\mathcal{U}$ on I such that $\mathcal{F} \subseteq \mathcal{U}$.

Exercise 1. Let (C, d) be a compact metric space, and let $\mathcal{U}$ be an ultrafilter on a set I.

(a) Show that for any sequence $(r_i)_{i\in I}$ from C, there is a unique element $s \in C$ such that $\lim_{\mathcal{U}} r_i = s$.

(b) ($\sharp$) Show that if $\mathcal{U}$ is a principal ultrafilter based on $i_* \in I$ then for any sequence $(r_i)_{i\in I}$, $\lim_{\mathcal{U}} r_i = r_{i_*}$.

(c) ($\sharp$) Suppose $I = \mathbb{N}$ and $\mathcal{U}$ is a nonprincipal ultrafilter. Show that if a sequence $(r_n)_{n=0}^{\infty}$ converges to $s \in C$, then $\lim_{\mathcal{U}} r_n = s$.

Exercise 2. Let $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ where each $\mathcal{M}_i$ is a finite L-structure, and $\mathcal{U}$ is an ultrafilter on some index set I.

(a) Show that any definable subset of $\mathcal{M}$ is internal.

(b) Show that the normalized pseudofinite counting measure μ is a finitely additive probability measure on the Boolean algebra of internal subsets of $\mathcal{M}$.

Exercises from Lecture 2 (27 April)

Exercise 3. Let $\mathcal{R}$ denote the ordered group $(\mathbb{R}, +, <)$ with constants for all elements of $\mathbb{R}$, and suppose $\mathcal{R}^* \succeq \mathcal{R}$. Show that for any finite $x \in \mathcal{R}^*$ there is a unique $r \in \mathbb{R}$ such that $x - r$ is infinitesimal.

Exercise 4. Let $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ where each $\mathcal{M}_i$ is a finite L-structure and $\mathcal{U}$ is an ultrafilter on an index set I. Let μ be the pseudofinite counting measure on internal subsets of $\mathcal{M}$, and let $\widetilde{\mathcal{M}}$ be the expansion of $\mathcal{M}$ in the language for μ. Show that for any L-formula $\phi(x; y_1, \ldots, y_n)$ and $b_1, \ldots, b_n \in \mathcal{M}$,

$$\mu(\phi(\mathcal{M}; b_1, \ldots, n_n)) = \operatorname{st}\left(f_\phi^{\widetilde{\mathcal{M}}}(b_1, \ldots, b_n)\right).$$

Exercise 5. Suppose $\mathcal{M}$ is a sufficiently saturated structure.

(a) ($*$) Suppose $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$ are collections of definable subsets of $\mathcal{M}^{\bar{x}}$, with I and J small. Assume $\bigcap_{i\in I} X_i \subseteq \bigcup_{j\in J} Y_j$. Show that there are finite $I_0 \subseteq I$ and $J_0 \subseteq J$ such that $\bigcap_{i\in I_0} X_i \subseteq \bigcup_{j\in J_0} Y_j$. (In particular, any definable set containing $\bigcap_{i\in I} X_i$ must contain $\bigcap_{i\in I_0} X_i$ for some finite $I_0 \subseteq I$.)

(b) ($*$) Suppose $X \subseteq \mathcal{M}^{\bar{x}}$ is such that X and its complement are both type-definable. Show that X is definable.

(c) Let $p(\bar{x}, \bar{y})$ be a type over a small set $A \subseteq \mathcal{M}$, where $\bar{x}$ and $\bar{y}$ are tuples of variables of bounded length. Show that the set

$$X = \{\bar{a} \in \mathcal{M}^{\bar{x}} : p(\bar{a}, \bar{b}) \text{ holds for some } \bar{b} \in \mathcal{M}^{\bar{y}}\}$$

is type-definable over A.

(d) Suppose $X \subseteq \mathcal{M}^{\bar{x}}$ is type-definable and A-invariant for some small set $A \subseteq \mathcal{M}$, where $\bar{x}$ has bounded length. Show that X is type-definable over A.

Hint: Consider a type $p(\bar{x},\bar{y}) \cup q(\bar{y})$ over A, where $p(\bar{x},\bar{c})$ defines X for some $\bar{c}$ and $q = \mathrm{tp}(\bar{c}/A)$.

Exercise 6. Let G be a sufficiently saturated structure expanding a group, and suppose Γ is a type-definable subgroup of G.

(a) ($*$) Let $\Gamma = \bigcap_{i\in I} X_i$, where I is small, each X_i is definable, and $\{X_i : i \in I\}$ is closed under finite intersections. Show that for any $i \in I$ there is $j \in I$ such that $X_j^2 \subseteq X_i$.
(b) ($*$) Suppose Γ has bounded index and X is a definable set containing Γ. Show that there are finite sets $E, F \subseteq G$ such that $G = EX = XF$.
(c) Suppose Γ has bounded index and is an intersection of λ definable sets, where λ is small. Show that Γ has index at most $2^{\lambda+\aleph_0}$.
(d) ($\sharp$) Show that Γ has finite index if and only if it is definable.

Exercise 7. Suppose G is a sufficiently saturated expansion of a group, and Γ is a type-definable normal subgroup of G of bounded index.

(a) (i) Show that the logic topology is a well-defined compact Hausdorff topology on G/Γ.
(ii) Show that G/Γ is a topological group.
Suggestion: Parts (d) below is helpful for proving (ii) (as well as similar ideas as in Exercises 5(a) and 6(a)).
(b) ($*$) Show that $K \subseteq G/\Gamma$ is clopen if and only if $\pi^{-1}(K)$ is definable.
(c) ($*$) Show that if $X \subseteq G$ is type-definable then $\pi(X)$ is closed.
(d) ($*$) Show that if $X \subseteq G$ is definable and $U = \{C \in G/\Gamma : C \subseteq X\}$, then U is open and $\pi^{-1}(U) \subseteq X$.
(e) Show that G/Γ is second countable if and only if Γ is countably definable.
(f) Show that G/Γ is profinite if and only if Γ is an intersection of definable finite-index normal subgroups of G. (Recall that a compact Hausdorff group is profinite if and only if it has a neighborhood basis at the identity consisting of clopen normal subgroups.)

Exercise 8. Let $\mathcal{M}$ be a structure, and C be a compact Hausdorff space.

(a) Suppose $f\colon \mathcal{M} \to C$ is a function with finite image. Show that f is definable if and only if all fibers of f are definable subsets of $\mathcal{M}$.
(b) Assume $\mathcal{M}$ is sufficiently saturated and C is bounded. Show that a function $f\colon \mathcal{M} \to C$ is definable if and only if $f^{-1}(K)$ is type-definable for any closed $K \subseteq C$.

Exercises from Lecture 3 (29 April)

Exercise 9. Let C be a compact Hausdorff group with a bi-invariant metric d. Fix $\delta > 0$ and suppose that for any $0 < \epsilon \leq \delta$ and any ϵ-approximate homomorphism $f\colon G \to C$ from a finite group G to C, there is a homomorphism $\tau\colon G \to C$ such that $d(f(x), \tau(x)) < 2\epsilon$. Fix $n \geq 1$ and let d_n be the product metric on C^n induced by d. Show that for any $0 < \epsilon \leq \delta$ and any ϵ-approximate homomorphism $f\colon G \to C^n$ from a finite group G to C^n, there is a homomorphism $\tau\colon G \to C^n$ such that $d_n(f(x), \tau(x)) < 2\epsilon$.

Exercise 10. ($*$) Let C be a compact Hausdorff group with a bi-invariant metric d. Suppose that for all $n \geq 1$, H_n is a subgroup of C, which is also a $\frac{1}{n}$-net in C (with respect to d). Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{Z}^+$ and set $H = \prod_{\mathcal{U}} H_n$ (view each H_n in the group language). Define $\sigma\colon H \to C$ such that $\sigma(x) = \lim_{\mathcal{U}} x_n$ (where $(x_n)_{n\in\mathbb{Z}^+}$ is a representative for x). Show that σ is a well-defined surjective homomorphism.

Exercise 11. ($\sharp$) Call a bipartite graph $\Gamma = (V, W; E)$ **k-NIP** if it omits $([k], \mathcal{P}([k]); \in)$ (as an induced subgraph). Show that if Γ omits some finite bipartite graph $\Gamma_0 = (V_0, W_0; E_0)$, then Γ is k-NIP for some $k \leq |V_0| + \lceil \log_2 |W_0| \rceil$.

Exercise 12. Suppose G is a group, H is a subgroup of G, and B is a (δ, n)-Bohr set in H.

(a) Show that $B = B^{-1}$, $1 \in B$, and if H is normal then $gB = Bg$ for any $g \in G$.

(b) ($\dagger$) Show that if H is finite then $|B| \geq (1/\delta)^n |H|$.

Hint: View $\mathbb{T}^n$ as a metric group under the usual "product of arclength" metric. Let η be the normalized Haar measure on $\mathbb{T}^n$. Given a homomorphism $\tau \colon H \to \mathbb{T}^n$ and some $x \in H$, let f_x be the characteristic function of the open ball of radius $\delta/2$ around $\tau(x)$ (which has Haar measure $(1/\delta)^n$). Show that $\sum_{x \in H} f_x(t) \geq (1/\delta)^n |H|$ for some $t \in \mathbb{T}^n$. Let $S = \{x \in H : d(\tau(x), t) < \delta/2\}$. Show that $Sa^{-1} \subseteq B$ for any $a \in S$.

Exercise 13. Let X be a set.

(a) Suppose $\mathcal{S} \subseteq \mathcal{P}(X)$ is such that $\mathrm{VC}(\mathcal{S}) = d$, and let $\mathcal{S}' = \{X \backslash S : S \in \mathcal{S}\}$. Show that $\mathrm{VC}(\mathcal{S}') = d$.

(b) ($\dagger$) Suppose $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{P}(X)$ are such that $\mathrm{VC}(\mathcal{S}_i) \leq d$, and let $\mathcal{S} = \{S_1 \cap S_2 : S_i \in \mathcal{S}_i\}$. Show that $\mathrm{VC}(\mathcal{S}) < 10d$.

Hint: Given an arbitrary set system $\mathcal{S}$, define the *shatter fuction* $\pi_{\mathcal{S}} \colon \mathbb{N} \to \mathbb{N}$ such that $\pi_{\mathcal{S}}(n) = \max\{|\{A \cap S : S \in \mathcal{S}\}| : A \subseteq X,\ |A| = n\}$. The *Sauer-Shelah Lemma* asserts that if $\mathrm{VC}(\mathcal{S}) \leq d$ then $\pi_{\mathcal{S}}(n) \leq \sum_{i=0}^{d} \binom{n}{i}$ (which is at most $(en/d)^d$).

(c) Suppose $\mathcal{S} \subseteq \mathcal{P}(X)$ is such that $\mathrm{VC}(\mathcal{S}) = d$, and let $\mathcal{S}^* = \{\mathcal{S}_x : x \in X\}$ where, given $x \in X$, $\mathcal{S}_x = \{S \in \mathcal{S} : x \in S\}$ (so $\mathcal{S}^* \subseteq \mathcal{P}(\mathcal{S})$). Show that $\mathrm{VC}(\mathcal{S}) \leq 2^{d+1}$.

Exercise 14.

(a) ($\sharp$) Let $\mathcal{S}$ be the collection of open intervals in $\mathbb{R}$. Show that $\mathrm{VC}(\mathcal{S}) = 2$.

(b) ($\sharp$) Let $\mathcal{S}$ be the collection of axis-parallel rectangles in $\mathbb{R}^2$. Show that $\mathrm{VC}(\mathcal{S}) = 4$.

(c) ($\sharp$) Let $\mathcal{S}$ be the collection of convex sets in $\mathbb{R}^2$. Show that $\mathrm{VC}(\mathcal{S}) = \infty$.

(d) Let $\Gamma = (V, W; E)$ be a bipartite graph, and let $\mathcal{S}$ be the collection of neighborhoods $E_w := \{v \in V : E(v, w)\}$ for all $w \in W$. Show that Γ is k-NIP if and only if $\mathrm{VC}(\mathcal{S}) < k$. (In particular, if G is a group then $A \subseteq G$ is k-NIP if and only if $\mathrm{VC}(\{gA : g \in G\}) < k$.)

Exercise 15. Given $n \geq 1$ and $0 < \delta \leq 1/2$, let $\mathcal{S}_{n,\delta}$ be the collection of "axis-parallel cubes of side length 2δ" in $\mathbb{T}^n$, i.e., subsets of $\mathbb{T}^n$ of the form $I_1 \times \ldots \times I_n$, where each I_t is an open interval in S^1 of length 2δ (using the normalized arclength metric on S^1).

(a) ($\dagger, \sharp$) Show that $\mathrm{VC}(\mathcal{S}_{n,\delta}) \leq (2 + o(1)) n \log_2 n$.

Hint: Compute $\mathrm{VC}(\mathcal{P}_{1,\delta})$ and then use an argument similar to Exercise 13(b).

(b) ($\sharp$) Let B be a (δ, n)-Bohr set in a group G and set $\mathcal{S} = \{gB : g \in G\}$. Show that $\mathrm{VC}(\mathcal{S}) \leq \mathrm{VC}(\mathcal{S}_{n,\delta})$.

Remark: One can improve the bound in (a) for $\delta \leq 1/4$ by using the fact that, in this case, periodic cubes in $\mathcal{S}_{n,\delta}$ behave a bit more like axis-parallel cubes in $\mathbb{R}^n$. Specifically, if $\mathcal{T}_{n,\delta}$ denotes sets in $\mathbb{R}^n$ of the form $I_1 \times \ldots \times I_n$, where each $I_t \subseteq \mathbb{R}$ is an open interval of length 2δ, then $\mathrm{VC}(\mathcal{S}_{n,\delta}) \leq \mathrm{VC}(\mathcal{T}_{n,\delta})$ when $\delta \leq 1/4$. Despres (arXiv 1412.6612) showed that the collection of *all* cubes in $\mathbb{R}^n$ has VC-dimension exactly $\lfloor (3n+1)/2 \rfloor$. So this improves the bound in (a) for $\delta \leq 1/4$. But the precise value of $\mathrm{VC}(\mathcal{S}_{n,\delta})$ is unknown in general, as far as I know. A very recent result in this area is on $\mathrm{VC}(\mathcal{S}_n)$, where $\mathcal{S}_n = \bigcup_{\delta} \mathcal{S}_{n,\delta}$ is the collection of all axis-parallel cubes in $\mathbb{T}^n$. Gillibert, Lachmann, and Müllner (arXiv 2004.13861) show that $\mathrm{VC}(\mathcal{S}_n) \sim n \log_2 n$.

Exercises from Lecture 5 (4 May)

Exercise 16. Let $\mathcal{M}$ be a structure, and suppose $\phi(\bar{x};\bar{y})$ and $\psi(\bar{x};\bar{z})$ are NIP formulas. Show that $\neg\phi(\bar{x};\bar{y})$ and $\theta(x;\bar{y},\bar{z}) := \phi(\bar{x};\bar{y}) \wedge \psi(\bar{x};\bar{z})$ are NIP.

Hint: Use Exercise 13.

Exercise 17. $(\dagger, \sharp)$

(a) Find an example of a pseudofinite group G and an internal set $A \subseteq G$ such that A is left generic in G but not right generic in G.

(b) Find an example of a pseudofinite group G and an internal set $A \subseteq G$ such that $\mu(A) > 0$ but A is not left or right generic in G.

Exercise 18. Let $\mathcal{M}$ be a structure, and suppose $\phi(\bar{x};\bar{y})$ is a NIP formula. Show that $\phi^*(\bar{y};\bar{x})$ is NIP.

Exercises from Lecture 6 (6 May)

Exercise 19. $(\dagger, \sharp)$ Find an example of a group G and a set $A \subseteq G$ such that A is NIP but the "formula" $\phi(x;y,z)$ given by $x \in yAz$ is not NIP. (As an extra challenge, do this with a pseudofinite group G and an internal set A.)

Exercise 20. Let G be a group. Show that the collection of NIP subsets of G is a bi-invariant Boolean algebra.

Exercises from Lecture 7 (8 May)

Exercise 21. Let X be a set and let $\mathcal{B}$ be a Boolean algebra of subsets of X. Let $S(\mathcal{B})$ be the Stone space of ultrafilters over $\mathcal{B}$. (See Section B.2 in *Miscellaneous Notes.*)

(a) Show that $S(\mathcal{B})$ is a totally disconnected compact Hausdorff space.

(b) Show that for any $A, B \in \mathcal{B}$, $[A \cup B] = [A] \cup [B]$ and $S(\mathcal{B})\backslash[A] = [X\backslash A]$.

(c) $(*)$ Show that a subset $K \subseteq S(\mathcal{B})$ is clopen if and only if it is of the form $[A]$ for some $A \in \mathcal{B}$.

(d) $(*)$ Show that if $K \subseteq S(\mathcal{B})$ is closed, $U \subseteq S(\mathcal{B})$ is open, and $K \subseteq U$ then there is some $A \in \mathcal{B}$ such that $K \subseteq [A] \subseteq U$.

Exercises from Lecture 8 (11 May)

Exercise 22. Show that for any real number $\epsilon > 0$ and integer $q \geq 1$, there is a real number $\delta > 0$ and an integer $p \geq 1$ such that the following holds. Let B be a Boolean algebra and suppose ν is a finitely additive probability measure on B. Suppose $x_1, \ldots, x_p \in B$ are such that $\nu(x_i) \geq \epsilon$ for all $i \leq p$. Show that there is a q-element set $I \subseteq [p]$ such that $\nu(\bigwedge_{i \in I} x_i) \geq \delta$.

Exercise 23. Transfer Theorem 8.11 to the pseudofinite setting as follows. Suppose $\mathcal{M}$ is a pseudofinite structure and $\phi(x;\bar{y})$ is a k-NIP formula. Fix $p \geq q \geq 2^k$ and assume $\{\phi(\mathcal{M};\bar{b}) : \bar{b} \in \mathcal{M}^{\bar{y}}\}$ has the (p,q)-property. Show that there is a finite set $F \subseteq \mathcal{M}$ such that $\phi(\mathcal{M};\bar{b}) \cap F \neq \emptyset$ for all $\bar{b} \in \mathcal{M}^{\bar{b}}$.

Exercises from Lecture 9 (13 May)

Exercise 24. Let G be a sufficiently saturated group, and let $\mathcal{B}$ be a left-invariant Boolean algebra of definable subsets of G. Suppose $\Gamma \leq G$ is a $\mathcal{B}$-type-definable bounded-index subgroup of G. Let $\pi\colon G \to G/\Gamma$ be the quotient map, and let $\sigma\colon S(\mathcal{B}) \to G/\Gamma$ be such that, for $p \in S(\mathcal{B})$, $\sigma(p)$ is the unique left coset of Γ on which p concentrates.

(a) Show that σ is surjective.
(b) ($*$) Show if $K \subseteq G/\Gamma$ is closed and $X \in \mathcal{B}$, then $\pi^{-1}(K) \subseteq X$ if and only if $\sigma^{-1}(K) \subseteq [X]$.
(c) ($\sharp$) Show that the logic topology on G/Γ is the finest topology for which σ is continuous.
(d) ($\sharp$) Show that $K \subseteq G/\Gamma$ is closed if and only if $\pi^{-1}(K)$ is $\mathcal{B}$-type-definable.
Hint: Show that if $X \subseteq G/\Gamma$ is type-definable then $\pi^{-1}(\pi(X))$ is $\mathcal{B}$-type-definable.

Exercise 25. ($*$) Let X be a set and let $\mathcal{B}$ be a Boolean algebra of subsets of X.

(a) Show that if ν is a regular Borel probability measure on $S(\mathcal{B})$ then, for any closed $K \subseteq S(\mathcal{B})$, $\nu(K) = \inf\{\nu(U) : U \text{ is clopen and } K \subseteq U\}$.
(b) Show that for any finitely additive probability measure μ on $\mathcal{B}$, there is a unique regular Borel probability measure $\widetilde{\mu}$ on $S(\mathcal{B})$ such that $\widetilde{\mu}([X]) = \mu(X)$ for any $X \in \mathcal{B}$.
(c) Show that the map $\mu \mapsto \widetilde{\mu}$ is a bijection between finitely additive probability measures on $\mathcal{B}$ and regular Borel probability measures on $S(\mathcal{B})$. Show that if X is a group G and $\mathcal{B}$ is left-invariant then this map preserves left-invariance.

Exercises from Lecture 10 (15 May)

Exercise 26. ($\sharp$) Suppose G is a finite group of exponent r, and $A \subseteq G$ is k-NIP. Show that for any $\epsilon > 0$, there is a normal subgroup $H \leq G$ of index $O_{k,r,\epsilon}(1)$, and a set $Z \subseteq G$ with $|Z| < \epsilon|G|$, such that if $g \in G \backslash Z$ then either $|gH \cap A| < \epsilon|H|$ or $|gH \backslash A| < \epsilon|H|$.

Hint: Use the fact that a compact Hausdorff torsion group is profinite (Iltis 1968).