Applications of Pseudofinite Model Theory

Lecture 1 (24 April 2020)

Ultraproduct Construction

Let I be an infinite set. An ultrafilter on I is a collection $\mathcal{U} \in \mathcal{P}(I)$ st

- $\emptyset \notin \mathcal{U}$ and $I \in \mathcal{U}$
- If $X \in \mathcal{U}$ and $X \subseteq Y$ then $Y \in \mathcal{U}$
- If $X, Y \in \mathcal{U}$ then $X \cap Y \in \mathcal{U}$
- $\forall\, X \subseteq I$, either $X \in \mathcal{U}$ or $I \setminus \mathcal{U} \in \mathcal{U}$.

E.g., Fix $i \in I$, let $\mathcal{U} = \{X \subseteq I : i \in X\}$

$\mathcal{U}$ is nonprincipal if $\mathcal{U}$ does not contain a finite subset of I.

Now fix a language L and let $\{\mathcal{M}_i : i \in I\}$ be a collection of L-structures.

Fix an ultrafilter $\mathcal{U}$ on I. We define a new L-structure $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ as follows. Let $\hat{M} = \prod_{i \in I} M_i$. Define eq. rel. $\sim$ on $\hat{M}$ st

$$(a_i)_{i \in I} \sim (b_i)_{i \in I} \text{ iff } \{i \in I : a_i = b_i\} \in \mathcal{U}$$

Let $M = \hat{M}/\sim$. This is the universe of $\mathcal{M}$.

Interpretation of L:

- If f is an n-ary function symbol and $a^1, \ldots, a^n \in M$ then

$$f^{\mathcal{M}}(a^1, \ldots, a^n) = \left[\left(f^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\right)_{i \in I}\right]_\sim$$

where $a^t = \left[(a_i^t)_{i \in I}\right]_\sim$

- If R is an n-ary relation symbol and $a^1, \dots, a^n \in \mathcal{M}$

$$R^{\mathcal{M}}(a^1, \dots, a^n) \text{ iff } \{i \in I : R^{\mathcal{M}_i}(a^1_i, \dots, a^n_i)\} \in \mathcal{U}.$$

- If c is a constant symbol, then $c^{\mathcal{M}} = [(c^{\mathcal{M}_i})_{i \in I}]_\sim$

We have a well-defined L-structure $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$

<u>Theorem 1.1</u> (Łoś's Theorem)

For any L-formula $\varphi(x_1, \dots, x_n)$ and $a^1, \dots, a^n \in \mathcal{M}$

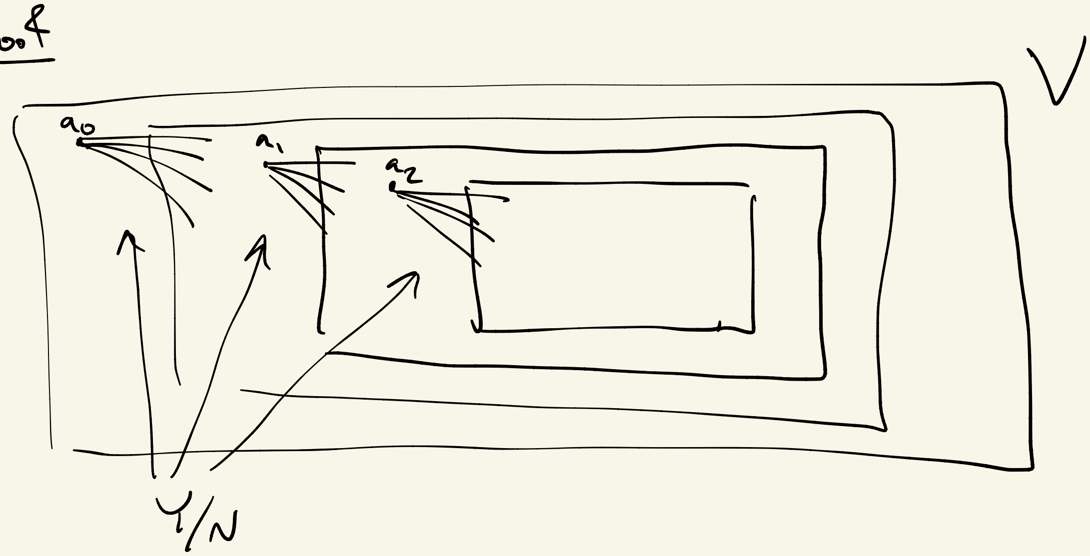
$$\mathcal{M} \models \varphi(a^1, \dots, a^n) \text{ iff } \{i \in I : \mathcal{M}_i \models \varphi(a^1_i, \dots, a^n_i)\} \in \mathcal{U}.$$

<u>Ex</u>: Ramsey's Theorem via ultraproducts.

<u>Lemma 1.2</u>

Any infinite graph contains an infinite complete set or infinite independent set.

<u>Proof</u>



<u>Theorem 1.3</u> (Ramsey's Theorem)

For any $n \geq 1$ $\exists$ $R \geq 1$ st any finite graph with R vertices contains a complete or independent set of size n.

Proof

Suppose not. There is some $n \geq 1$ st $\forall\ i \geq 1\ \exists$ a graph (V_i, E_i) st $|V_i| = i$ and (V_i, E_i) contains no complete or ind. set of size n.

Let $\mathcal{U}$ be a n.p. ultrafilter on $I = \mathbb{Z}^+$. Let $\mathcal{M}_i = (V_i, E_i)$ (L is the language of graphs) and set $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i = (V, E)$

Claim (V, E) is an infinite graph

Pf: Łoś's Thm

V is infinite: Given $m \geq 1$, let φ_m be the L-sentence saying "there are at least m elements." Then $\{i \in I : \mathcal{M}_i \models \varphi_m\} = \{i \in I : i \geq m\} \in \mathcal{U}$ since $\mathcal{U}$ is non-principal. //

By Lemma 1.2, (V, E) has a infinite complete or independent set.

Let ψ be the sentence

$$\exists v_1 \ldots \exists v_n \left(\bigwedge_{i \neq j} v_i \neq v_j \wedge \left(\bigwedge_{i \neq j} E(v_i, v_j) \vee \bigwedge_{i \neq j} \neg E(v_i, v_j) \right) \right)$$

$\mathcal{M} \models \psi$. So By Łoś's Thm, $\{i \in I : \mathcal{M}_i \models \psi\} \in \mathcal{U}$

So $\exists\ i \in I$ st $\mathcal{M}_i \models \psi$. This is a contradiction $\square$.

The Pseudofinite Counting Measure

Exercise 1 Let (C, d) be a compact metric space. Let $\mathcal{U}$ be an ultrafilter on a set I. Then for any sequence $(r_i)_{i \in I}$ from C, $\exists!\ s \in C$ st $\forall\ \varepsilon > 0$, $\{i \in I : d(r_i, s) < \varepsilon\} \in \mathcal{U}$. Notation: $s = \lim_{\mathcal{U}} r_i$.

Suppose $\{\mathcal{M}_i : i \in I\}$ is a collection of finite L-structures.

Let $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ ($\mathcal{U}$ is an ultrafilter on I).

Call a set $X \subseteq \mathcal{M}$ <u>internal</u> if $X = \prod_{\mathcal{U}} X_i \left(= \prod_I X_i /_{\sim}\right)$ for some $X_i \subseteq \mathcal{M}_i$.

<u>Def 1.4</u> The <u>normalized pseudofinite counting measure</u> of an internal set $X = \prod_{\mathcal{U}} X_i$ is $\mu(X) = \lim_{\mathcal{U}} |X_i| / |M_i|$

<u>Remark</u>: Any definable subset of $\mathcal{M}$ is internal. (Exc)

<u>Exercise 2</u>: μ is a finitely-additive prob. measure on the Boolean algebra of internal subsets of $\mathcal{M}$.