Applications of Pseudofinite Model Theory

Lecture 2 (27 April 2020)

$$\underset{\text{finite structures}}{(\mathcal{M}_i)_{i\in I}} \xrightarrow{\text{ultraproduct}} \prod_{\mathcal{U}} \mathcal{M}_i \xrightarrow{\text{elementary extension}} \mathcal{M}^*$$

$$\text{normalized counting measures} \xrightarrow{\text{ultralimits}} \mu \longrightarrow ?$$

Let $\mathcal{R} = (\mathbb{R}, +, <)$ with constants for all elements of $\mathbb{R}$

Exercise 3 Suppose $\mathcal{R}^* \succeq \mathcal{R}$. Call $x \in \mathcal{R}^*$ finite if $|x| < r$ for some $r \in \mathbb{R}$, and call x infinitesimal if $|x| < r$ $\forall r \in \mathbb{R}^+$.

Then $\forall$ finite $x \in \mathcal{R}^*$ $\exists !$ $r \in \mathbb{R}$ st $x - r$ is infinitesimal.

In this case, r is the standard part of x.

Let $\mathcal{U}$ be ultrafilter on I and $\{\mathcal{M}_i : i \in I\}$ finite L-structures.

Expand each $\mathcal{M}_i$ to a 2-sorted structure $\tilde{\mathcal{M}}_i = (\mathcal{M}_i, \mathcal{R})$

For any L-formula $\varphi(x; y_1, \dots, y_n)$, add an n-ary function symbol f_φ st $f_\varphi^{\tilde{\mathcal{M}}_i}(b_1, \dots, b_n) = |\varphi(\mathcal{M}_i; b_1, \dots, b_n)| / |\mathcal{M}_i|$

$$\text{Let } \tilde{\mathcal{M}} = \prod_{\mathcal{U}} \tilde{\mathcal{M}}_i = \Big(\underbrace{\prod_{\mathcal{U}} \mathcal{M}_i}_{=\mathcal{M}}, \underbrace{\prod_{\mathcal{U}} \mathcal{R}}_{=\mathcal{R}^* \succeq \mathcal{R}}\Big)$$

Exercise 4 For any L-formula $\varphi(x; \bar{y})$ and $\bar{b} \in \mathcal{M}^{\bar{y}}$

$$\mu(\varphi(\mathcal{M}; \bar{b})) = st\left(f_\varphi^{\tilde{\mathcal{M}}}(\bar{b})\right)$$

Type-Definable Subgroups & the Logic Topology

Setting: $\mathcal{M}$ is a <u>sufficiently saturated</u> L-structure, i.e. $\mathcal{M}$ is κ-saturated and strongly κ-homogeneous for some strongly <u>inaccessible</u> κ (i.e., uncountable, $\kappa = \mathrm{cf}(\kappa)$, $\lambda < \kappa \Rightarrow 2^\lambda < \kappa$).

<u>Key point</u>: If $\{X_i : i \in I\}$ is a collection of definable subsets of $\mathcal{M}$ st $|I| < \kappa$ and $\bigcap_{i \in I_0} X_i \neq \emptyset$ $\forall$ finite $I_0 \subseteq I$, then $\bigcap_{i \in I} X_i \neq \emptyset$.

We say that a set is <u>bounded</u> or <u>small</u> if its cardinality is $< \kappa$.

<u>Def 2.1</u>: A subset $X \subseteq \mathcal{M}$ is <u>type-definable</u> if $X = \bigcap_{i \in I} X_i$ where I is bounded and each X_i is definable. If I is countable then X is <u>countably type-definable</u>

<u>Exercise 5</u>: Basic properties of type-definable sets.

Now let $\mathcal{M} = G$ be an expansion of a group.

<u>Exercise 6</u>: Basic prop. of type-def. subgroups.

<u>Def 2.2</u> Suppose $\Gamma \leq G$ is a type-def., normal, bounded index subgroup of G. Then $K \subseteq G/\Gamma$ is <u>closed</u> if $\pi^{-1}(K)$ is type-definable, where $\pi : G \to G/\Gamma$ is the quotient map.

<u>Exercise 7</u> Let $\Gamma \leq G$ be as above.

1) G/Γ is a compact Hausdorff group.

2) $K \subseteq G/\Gamma$ is clopen iff $\pi^{-1}(K)$ is definable.

3) If $X \subseteq G$ is type-def then $\pi(X)$ is closed.

4) If $X \subseteq G$ is definable, and $U = \{C \in G/\Gamma : C \subseteq X\}$ then $\pi^{-1}(U) \subseteq X$ and U is open.

5) G/Γ is second countable iff Γ is countably type-definable

6) G/Γ is profinite iff Γ is an intersection of definable subgroups of G.

Example 2.3

1) $G \models Th(\mathbb{Z}, +, 0)$ and $\Gamma = \bigcap_{n \geq 1} nG$. Γ is countable type-definable and $[G : \Gamma] = 2^{\aleph_0}$. Fact: $G/\Gamma \cong \hat{\mathbb{Z}}$ (independent of the choice of G)

2) $G \models Th(S^1, \cdot, cyc(x,y,z))$. Let $\Gamma = \bigcap_{n=1}^{\infty} X_n$ where $X_n = \{x \in G : cyc(e^{-i\pi/n}, x, e^{i\pi/n})\}$. Γ is ctbly type-def, index $2^{\aleph_0}$.

Fact: $G/\Gamma \cong S^1$.

Compactifications & Pseudofinite Groups

Def 2.4: An L-structure $\mathcal{M}$ is pseudofinite if any L-sentence true in $\mathcal{M}$ is true in some finite L-structure.

Fact: $\mathcal{M}$ is pseudofinite iff $\mathcal{M} \equiv \prod_{\mathcal{U}} \mathcal{M}_i$ where each $\mathcal{M}_i$ is finite.

Def 2.5 Let C be a compact Hausdorff space. Then a function $f : \mathcal{M} \to C$ is definable if $\forall$ closed $K \subseteq C$ and $U \subseteq C$, if $K \subseteq U$, then $\exists$ definable set $X \subseteq \mathcal{M}$ st $f^{-1}(K) \subseteq X \subseteq f^{-1}(U)$.

Exercise 8 : If $\mathcal{M}$ is sufficiently saturated and C is small then $f : \mathcal{M} \to C$ is definable iff $f^{-1}(K)$ is type-def. $\forall$ closed $K \subseteq C$.

Def 2.6: Let G be an expansion of a group. Then a definable compactification of G is a definable homomorphism $\tau : G \to C$, where C is a compact

Hausdorff group and $\tau(G)$ is dense in C

Example 2.7 If G is sufficiently saturated, and $\Gamma \leq G$ is type-def., normal, bounded index, then $\pi : G \to G/\Gamma$ is a definable comp. of G.