Applications of Pseudofinite Model Theory

Lecture 5 (4 May 2020)

Theorem 5.1 (Vapnik–Chervonenkis)
Suppose X is a finite set, and $\mathcal{S} \subseteq \mathcal{P}(X)$ with $VC(\mathcal{S}) = d$. Then $\forall\, \varepsilon > 0,\ n \geq 1$

$$\left| \left\{ \bar{a} \in X^n : \left| Av_{\bar{a}}(S) - |S|/|X| \right| \geq \varepsilon \text{ for some } S \in \mathcal{S} \right\} \right| \leq \frac{O(n^d)}{e^{\varepsilon^2 n/32}} |X|^n$$

Idea: For $\bar{a} \in X^n$, consider sets $\{a_1, \ldots, a_n\} \cap S$ for $S \in \mathcal{S}$.

Sauer–Shelah: If $\bar{a} \in X^n$ then $|\{\bar{a} \cap S : S \in \mathcal{S}\}| \leq O(n^d)$

Corollary 5.2 Suppose X is finite and $\mathcal{S} \subseteq \mathcal{P}(X)$ with $VC(\mathcal{S}) = d$. Then $\forall\, \varepsilon > 0\ \exists\, \bar{a} \in X^n$, with $n \leq O_{\varepsilon,d}(1)$, st $\forall\, S \in \mathcal{S}$

$$\left| Av_{\bar{a}}(S) - |S|/|X| \right| \leq \varepsilon.$$

NIP Formulas in Pseudofinite Structures

Let $\mathcal{M}$ be an $\mathcal{L}$-structure.

Def 5.3 A formula $\varphi(\bar{x};\bar{y})$ is *k-NIP* if $\nexists\, \bar{a}_1, \ldots, \bar{a}_k \in \mathcal{M}^{\bar{x}}$ and $(\bar{b}_S)_{S \subseteq [k]}$ in $\mathcal{M}^{\bar{y}}$ st $\mathcal{M} \models \varphi(\bar{a}_i, \bar{b}_S)$ iff $i \in S$

i.e., $(\mathcal{M}^{\bar{x}}, \mathcal{M}^{\bar{y}}; \varphi)$ omits $([k], \mathcal{P}([k]); \in)$

i.e., $VC(\{\varphi(\mathcal{M}; \bar{b}) : \bar{b} \in \mathcal{M}^{\bar{y}}\}) < k$. (see Exc 14(d))

$\varphi(\bar{x};\bar{y})$ is *NIP* if it is k-NIP for some $k \geq 1$.

Convention: From now on, "$\mathcal{M}$ is pseudofinite" means $\mathcal{M} \succeq \prod_{\mathcal{U}} \mathcal{M}_i$ where each $\mathcal{M}_i$ is a finite $\mathcal{L}$-structure and we work in the expanded language for μ.

In this case, μ denotes the normalized pseudofinite counting measure on all definable subsets of $\mathcal{M}$. Given a formula $\varphi(x;\bar{y})$ over $\emptyset$ and $\bar{b} \in \mathcal{M}^{\bar{y}}$

$$\mu(\varphi(x;\bar{b})) = st\left(f_{\varphi}^{\mathcal{M}}(\bar{b})\right) \quad (\text{Exc. } 4).$$

<u>Proposition 5.4</u> Suppose $\mathcal{M}$ is pseudofinite and $\varphi(x;\bar{y})$ is NIP. Then $\forall\ \varepsilon > 0\ \exists\ n \geq 1 + \bar{a} \in \mathcal{M}^n$ st $\bar{b} \in \mathcal{M}^{\bar{y}}$,

$$|Av_{\bar{a}}(\varphi(x,\bar{b})) - \mu(\varphi(x,\bar{b}))| < \varepsilon.$$

<u>Proof</u>: Let $\varphi(x,\bar{y})$ be $\psi(x;\bar{y},\bar{c})$ for some $\psi(x;\bar{y},\bar{z})$ over $\emptyset$ and $\bar{c} \in \mathcal{M}^{\bar{z}}$.
So $\mu(\varphi(x,\bar{b})) = st\left(f_{\psi}^{\mathcal{M}}(b,\bar{c})\right)$. Assume $\varphi(x,\bar{y})$ is k-NIP.

Let $\chi(\bar{z})$ express "$\psi(x;\bar{y};\bar{z})$ is k-NIP." So $\mathcal{M} \models \chi(\bar{c})$.

Fix $\varepsilon > 0$ and let $n = O_{k-1,\varepsilon}(1)$ be as in Corollary 5.2. Then $\forall\ \mathcal{M}_i$ $(\mathcal{M} \cong \prod_{\mathcal{U}} \mathcal{M}_i)$

$$\mathcal{M}_i \models \forall \bar{z}\left(\chi(\bar{z}) \to \exists v_1 \ldots \exists v_n \forall \bar{y} \,|Av_{\bar{v}}(\psi(x;\bar{y},\bar{z}) - f_{\psi}(\bar{y},\bar{z})| < \varepsilon\right).$$

So $\mathcal{M}$ satisfies this by Łoś's Theorem.

<u>Corollary 5.5</u> Suppose $\mathcal{M}$ is pseudofinite and $\varphi(x,\bar{y})$ is NIP. Then $\forall\ \varepsilon > 0$, $\exists$ finite $F \subseteq \mathcal{M}$ st $\forall\ \bar{b} \in \mathcal{M}^{\bar{y}}$ if $\mu(\varphi(x,\bar{b})) \geq \varepsilon$ then $F \cap \varphi(\mathcal{M},\bar{b}) \neq \emptyset$.

<u>Def. 5.6</u>: Given a group G and $A \subseteq G$, call A <u>left</u> <u>(right)</u> <u>generic</u> if $G = FA$ (resp. $G = AF$) for some finite $F \subseteq G$.

<u>Theorem 5.7</u> Let G be a pseudofinite expansion of a group and suppose $A \subseteq G$ is definable and NIP (ie. $\varphi(x;y) := x \in yA$ is NIP).

TFAE i) A is left generic

ii) A is right generic

iii) $\mu(A) > 0$.

<u>Proof</u> i) $\Rightarrow$ iii), ii) $\Rightarrow$ iii) by finite additivity + invariance of μ.

(iii) $\Rightarrow$ (ii). Assume $\mu(A) = \varepsilon > 0$. By Cor 5.5 (applied to "$x \in yA$") $\exists$ finite $F \subseteq G$ st $\forall g \in G$, $F \cap gA \neq \emptyset$. So $G = AF^{-1}$ $\left(ga = f \quad g^{-1} = af^{-1} \right)$

(iii) $\Rightarrow$ (i). Similar. Use the fact that "$x \in Ay$" is NIP (Exc. 18).

<u>Exercise 17</u> NIP is necessary in Thm 5.7

<u>Def 5.8</u> Let $\mathcal{M}$ be a structure, and $\varphi(\bar{x}; \bar{y})$ a formula.

1) A <u>φ-formula</u> is a Boolean combination of "instances" $\varphi(\bar{x}; \bar{b})$ for $\bar{b} \in \mathcal{M}^{\bar{y}}$

2) A subset of $\mathcal{M}^{\bar{x}}$ is <u>φ-definable</u> if its defined by a φ-formula.

3) Let $\varphi^*(\bar{y}; \bar{x})$ be the same formula $\varphi(\bar{x}; \bar{y})$ but with the roles of object variables and parameter variables exchanged.

<u>Exercise 18</u>: If $\varphi(\bar{x}; \bar{y})$ is k-NIP then $\varphi^*(\bar{y}, \bar{x})$ is 2^k-NIP.

<u>Theorem 5.9</u> Suppose $\mathcal{M}$ is pseudofinite and $\varphi(\bar{x}; \bar{y})$ is NIP. Then $\forall\, \varepsilon > 0$, the set $D_\varepsilon = \{ \bar{b} \in \mathcal{M}^{\bar{y}} : \mu(\varphi(x, \bar{b})) \leq \varepsilon \}$ is an intersection of countably many φ^*-definable sets.