Applications of Pseudofinite Model Theory

Lecture 8 (11 May 2020)

Setting: G saturated pseudofinite, $A \subseteq G$ def. + NIP, $\mathcal{B} = \langle \{gAh : g,h \in G\} \rangle$

$\mathbb{G}_A = G/G_A^{00}$ $\qquad\qquad$ $\pi : G \to \mathbb{G}_A$

$\mathbb{E}_A = \{C \in \mathbb{G}_A : C \cap A,\ C \setminus A \text{ wide}\}$ $\qquad$ $\sigma : S(\mathcal{B}) \to \mathbb{G}_A$.

Corollary 8.1 $\mathbb{E}_A$ is closed

Proof By Prop 7.6(a), $C \in \mathbb{E}_A$ iff $\exists\, p,q \in S^s(\mathcal{B})$ st $p \models C \cap A$ + $q \models C \setminus A$.

So $\mathbb{E}_A = \sigma(S^s(\mathcal{B}) \cap [A]) \cap \sigma(S^s(\mathcal{B}) \setminus [A])$.

So $\mathbb{E}_A$ is closed because σ is a closed map $\qquad \square$

Note: Given $X \in \mathcal{B}$ and $p \in S^s(\mathcal{B})$, let $D_X^p = \{a \in G : X \in ap\}$

Then D_X^p is a union of cosets of G_A^{00} since $G_A^{00} = \mathrm{Stab}(p)$ by Thm 7.7(a).

Def 8.2 $\mathbb{D}_X^p = \pi(D_X^p) = \{aG_A^{00} : X \in ap\}$

Lemma 8.3: If $p \in S^s(\mathcal{B})$ and $p \models G_A^{00}$, then $\mathbb{E}_A \subseteq \partial \mathbb{D}_A^p$

Proof: Fix $C \in \mathbb{E}_A$. Fix open $U \subseteq \mathbb{G}_A$ st $C \in U$. Fix $q, r \in S^s(\mathcal{B})$ st $q \models C \cap A$ and $r \models C \setminus A$. Then $q \in \sigma^{-1}(U) \cap [A]$ and $r \in \sigma^{-1}(U) \setminus [A]$.

By Prop 7.6(b), $\exists\, a,b \in G$ st $ap \in \sigma^{-1}(U) \cap [A]$ and $bp \in \sigma^{-1}(U) \setminus [A]$.

So $aG_A^{00} = \sigma(ap) \in U \cap \mathbb{D}_A^p$ and $bG_A^{00} = \sigma(bp) \in U \setminus \mathbb{D}_A^p$.

Tool: Let $\mathbb{G}$ be a second countable compact Hausdorff group, with Haar measure η.

Def: A Borel set $W \subseteq \mathbb{G}$ is pointwise large if $\eta(W \cap U) > 0$ $\forall$ open set U st $U \cap W \neq \emptyset$.

Theorem 8.5 (Simon) Suppose $W \subseteq \mathbb{G}$ is NIP, and W and $\mathbb{G} \setminus W$ are both

F_σ and pointwise large. Then $\eta(\partial W) = 0$.

<u>Goal</u> $\mathbb{D}^p_A$ satisfies these assumptions.

<u>Lemma 8.6</u>: If $p \in S^s(\mathcal{B})$ then $\mathbb{D}^p_A$ is NIP.

<u>Proof</u>: Let $\mathcal{A}_1 = \{C \cdot \mathbb{D}^p_A : C \in \mathcal{G}_A\}$ and $\mathcal{A}_2 = \{gA : g \in G\}$

We show $VC(\mathcal{A}_1) \leq VC(\mathcal{A}_2)$ ($< \infty$ since A is NIP).

Suppose $\mathcal{A}_2$ shatters $\{a_1 G^{00}_A, \ldots, a_n G^{00}_A\} \subseteq \mathcal{G}_A$.

Then $\forall\, s \subseteq [n]\ \exists\, g_s \in G$ st $a_i G^{00}_A \in g_s G^{00}_A \cdot \mathbb{D}^p_A$ iff $i \in s$.

So $g_s^{-1} a_i G^{00}_A \in \mathbb{D}^p_A$ iff $i \in s$, i.e., $A \in g_s^{-1} a_i p$ iff $i \in s$,

i.e., $a_i^{-1} g_s A \in p$ iff $i \in s$.

So $\exists\, b \in G$ st $b \in a_i^{-1} g_s A$ iff $i \in s$, i.e. $a_i b \in g_s A$ iff $i \in s$.

So $\mathcal{A}_2$ shatters $\{a_1 b, \ldots, a_n b\}$. $\square$.

<u>Lemma 8.7</u> If $p \in S^s(\mathcal{B})$ and $X \in \mathcal{B}$ then $\mathbb{D}^p_X$, $\mathcal{G}_A \setminus \mathbb{D}^p_X$ are F_σ in $\mathcal{G}_A$.

<u>Proof</u> Note $\mathcal{G}_A \setminus \mathbb{D}^p_X = \mathbb{D}^p_{G \setminus X}$. So it suffices to consider $\mathbb{D}^p_X$.

Let $\varphi(x;y)$ be $x \in y^{-1}X$ (this is NIP). Let $p_0 \in S_\varphi(G)$ be st

$\varphi(x,a) \in p_0$ iff $a^{-1}X \in p$. So $D^p_X = \{a \in G : X \in ap\} = \{a \in G : \varphi(x,a) \in p_0\}$.

Recall $\mathbb{D}^p_X = \pi(D^p_X)$. WLOG: Assume our language is countable.

<u>Simon</u> If p_0 is M-invariant for some ctble $M \prec G$, then $D^p_X = \bigcup_{n=0}^{\infty} Y_n$ where each Y_n is type-definable. (So $\mathbb{D}^p_X = \bigcup_{n=0}^{\infty} \pi(Y_n)$ is F_σ; see Exc 7c)

By Corollary 5.5, $\exists$ ctble $B \subseteq G$ st $\forall\, a,b \in G$ if $\mu(a^{-1}X \setminus b^{-1}X) > 0$ then $B \cap (a^{-1}X \setminus b^{-1}X) \neq \emptyset$. Let $M \prec G$ be ctble st $B \subseteq M$ and st X is M-definable.

Claim: p_0 is M-invariant (ie., if $a \equiv_M b$ then $\varphi(x,a) \in p_0$ iff $\varphi(x,b) \in p_0$)

Pf: Suppose $\exists\, a, b \in G$ st $a \equiv_M b$ and $\varphi(x,a) \wedge \neg\varphi(x,b) \in p_0$.

Then $a^{-1}X \setminus b^{-1}X \in p$. So $\mu(a^{-1}X \setminus b^{-1}X) > 0$. So $\exists\, m \in M$ st $m \in a^{-1}X \setminus b^{-1}X$. So $G \models \varphi(m,a) \wedge \neg\varphi(m,b)$, $\not\models$ to $a \equiv_M b$. // $\square$.

Def 8.8 Given $p \in S^{g}(B)$ and $X \in \mathcal{B}$, set $\eta_p(X) = \eta(\mathbb{D}^p_X)$ where η is Haar measure on G_A.

Proposition 8.9 If $p \in S^{g}(\mathcal{B})$ then η_p is a left-inv. fin. add. prob. measure on $\mathcal{B}$.

Proof Inherited from η (exercise).

Lemma 8.10: Let ν be a left inv. fin. add. prob measure on $\mathcal{B}$. Fix $X \in \mathcal{B}$. Then $\nu(X) > 0$ iff $\mu(X) > 0$.

Theorem 8.11 (Matoušek)

Let X be a finite set + fix $\mathcal{S} \subseteq \mathcal{P}(X)$ with $VC(\mathcal{S}) = d$. Fix $p \geq q \geq 2^{d+1}$ and suppose that among any p sets in $\mathcal{S}$ there are q with nontrivial intersection (i.e. $\mathcal{S}$ has the $\underline{(p,q)\text{-property}}$).

Then $\exists\, F \subseteq X$, $|F| \leq O_{p,q}(1)$, st $F \cap S \neq \emptyset$ $\forall\, S \in \mathcal{S}$.

Ex: Suppose $|S| \geq \varepsilon |X|$ $\forall\, S \in \mathcal{S}$. Then $\forall\, q$ $\exists\, p = O_{q,\varepsilon}(1)$ st $\mathcal{S}$ has the (p,q)-property (Exercise 22).

Exercise 23: Transfer 8.11 to a pseudofinite structure.