A. NOTES ON MODEL THEORY

The following are some extremely brief notes on a few important topics for the course. I assume familiarity with first-order languages, structures, and the compactness theorem (essentially the material in Chapters 1 and 2 of Marker's book [1], although the first section below repeats some notions). The notation follows [1] as closely as possible. The goal of these notes is to briefly go through type spaces and saturated models (covered in Chapter 4 of [1]). I end with some details on what is meant by a "sufficiently saturated" model (sometimes called a "monster model"). The wiki article at: `https://modeltheory.fandom.com/wiki/Monster_model` is well-written and goes into more detail on this topic.

A.1. **Elementary extensions.** Let L be a first-order language (with equality) and suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures, with universes M and N, respectively. Recall that $\mathcal{M}$ and $\mathcal{N}$ are **elementarily equivalent**, written $\mathcal{M} \equiv \mathcal{N}$, if they satisfy the same L-sentences.

The following are a few ways to obtain elementarily equivalent structures.

Definition A.1.

(1) A function $\sigma\colon M \to N$ is an **L-embedding** if:

(i) for any n-ary function symbol f in L and $a_1, \ldots, a_n \in M$, we have

$$\sigma(f^{\mathcal{M}}(a_1, \ldots, a_n)) = f^{\mathcal{N}}(\sigma(a_1), \ldots, \sigma(a_n));$$

(ii) for any n-ary relation symbol R in L and $a_1, \ldots, a_n \in M$, we have

$$R^{\mathcal{M}}(a_1, \ldots, a_n) \;\Leftrightarrow\; R^{\mathcal{N}}(\sigma(a_1), \ldots, \sigma(a_n));$$

(iii) for any constant symbol c in L, we have $\sigma(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

We write $\sigma\colon \mathcal{M} \to \mathcal{N}$ to denote that σ is an L-embedding. Note that an L-embedding is automatically injective.

(2) An L-embedding $\sigma\colon \mathcal{M} \to \mathcal{N}$ is an **L-isomorphism** if it is surjective. $\mathcal{M}$ and $\mathcal{N}$ are **isomorphic**, written $\mathcal{M} \cong \mathcal{N}$, if there is an L-isomorphism $\sigma\colon \mathcal{M} \to \mathcal{N}$.

Exercise A.2.

(a) If $\sigma\colon \mathcal{M} \to \mathcal{N}$ is an L-embedding then, for any quantifier-free L-formula $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$, we have

$$\mathcal{M} \models \phi(a_1, \ldots, a_n) \;\Leftrightarrow\; \mathcal{N} \models \phi(\sigma(a_1), \ldots, \sigma(a_n)).$$

(b) If $\mathcal{M} \cong \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$.

Definition A.3.

(1) Given $A \subseteq M$, a function $f\colon A \to N$ is a **partial elementary map** if, for any L-formula $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in A$, we have

$$\mathcal{M} \models \phi(a_1, \ldots, a_n) \;\Leftrightarrow\; \mathcal{N} \models \phi(\sigma(a_1), \ldots, \sigma(a_n)).$$

A partial elementary map $\sigma\colon M \to N$ is called an **elementary L-embedding**. (Note that an elementary L-embedding is an L-embedding.)

(2) $\mathcal{M}$ is an **elementary substructure** of $\mathcal{N}$, written $\mathcal{M} \preceq \mathcal{N}$, if $M \subseteq N$ and the inclusion map from M to N is an elementary L-embedding. In this case, we also say that $\mathcal{N}$ is an **elementary extension** of $\mathcal{M}$.

Exercise A.4.

(1) If $\sigma \colon \mathcal{M} \to \mathcal{N}$ is an L-isomorphism, then it is an elementary L-embedding.
(2) If $\mathcal{M} \preceq \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$.
(3) If $\mathcal{M}$ is infinite then for any cardinal $\kappa \geq |M|$, there is some $\mathcal{N} \succeq \mathcal{M}$ such that $|N| \geq \kappa$.

Theorem A.5 (Downward Löwenheim-Skolem)**.** [1, Theorem 2.3.7] *If $\mathcal{M}$ is an L-structure and $C \subseteq M$ then there is $\mathcal{N} \preceq \mathcal{M}$ such that $C \subseteq N$ and $|N| \leq |C| + |L| + \aleph_0$.*

A.2. **Type Spaces.** Let T be a complete first-order L-theory with infinite models. (For example, fix an infinite L-structure $\mathcal{M}$ and let $T = \mathrm{Th}(\mathcal{M})$.) We will no longer use different notation for structures and their underlying universes.

Definition A.6. Fix $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$.

(1) A formula is **over** A if all of its parameters are in A.
(2) A **type over** A is a finitely consistent collection of L-formulas over A.
(3) Given a tuple of variables $\bar{x}$ (possibly infinite), we say that p is a **complete type over** A **in the variables** $\bar{x}$ if p is a type over A, the free variables of any formula in p come from $\bar{x}$, and for any formula ϕ over A, with free variables from $\bar{x}$, either $\phi \in p$ or $\neg\phi \in p$. Let $S^{\mathcal{M}}_{\bar{x}}(A)$ denote the set of complete types over A in the variables X. If $\bar{x}$ has finite length n, then we may also write $S^{\mathcal{M}}_n(A)$.

In general, we write a formula as $\phi(\bar{x})$ to indicate that the free variables in the formula all come from the tuple $\bar{x}$.

Given a tuple of variables $\bar{x}$, we let $\mathcal{M}^{\bar{x}}$ denote $\mathcal{M}^{|\bar{x}|}$, where $|\bar{x}|$ is the length of the tuple. A subset $X \subseteq \mathcal{M}^{\bar{x}}$ is **definable (over** A**)** if there is a formula $\phi(\bar{x})$ (over A) such that $X = \{\bar{a} \in M^{\bar{x}} : \mathcal{M} \models \phi(\bar{a})\}$.

Remark A.7. The space $S^{\mathcal{M}}_{\bar{x}}(A)$ above can be identified with the set of ultrafilters over the Boolean algebra of subsets of $\mathcal{M}^{\bar{x}}$ definable over A.

Given the previous remark, we can think of type spaces as special cases of Stone spaces over Boolean algebras. In particular, given a set X and a Boolean algebra $\mathcal{B}$ of subsets of X, a **complete $\mathcal{B}$-type** is a maximal finitely consistent collection of subsets of $\mathcal{B}$ (i.e., an ultrafilter over $\mathcal{B}$). In this context, we will often use the notation $S(\mathcal{B})$ for the space of complete $\mathcal{B}$-types. (These spaces are topological spaces under the Stone topology, which we will discuss during the course.)

We also give special notation for type spaces "generated" by a single formula. In general, we often "partition" the free variables in a formula as $\phi(\bar{x}; \bar{y})$. The (lefthand) $\bar{x}$ variables are called **object variables** and the (righthand) $\bar{y}$ variables are called **parameter variables**.

Definition A.8. Fix $\mathcal{M} \models T$ and let $\phi(\bar{x}; \bar{y})$ be a formula over $\emptyset$.

(1) A **instance** of $\phi(\bar{x}; \bar{y})$ is a formula of the form $\phi(\bar{x}; \bar{a})$ for some $\bar{a} \in \mathcal{M}^{\bar{y}}$.
(2) A **ϕ-formula** is a finite Boolean combination of instances of $\phi(\bar{x}; \bar{y})$.
(3) A subset of $\mathcal{M}^{\bar{x}}$ is **ϕ-definable (over** A**)** if it is defined by a ϕ-formula (over A).

(4) Given $A \subseteq \mathcal{M}$, a **ϕ-type** over A is a finitely consistent collection of ϕ-formulas over A. A ϕ-type p over A is **complete** if for any ϕ-formula $\psi(\bar{x})$, either $\psi(\bar{x}) \in p$ or $\neg\psi(\bar{x}) \in p$.
(5) Let $S_\phi^{\mathcal{M}}(A)$ denote the set of complete ϕ-types over A.

Note that $S_\phi^{\mathcal{M}}(A)$ can be identified with $S(\mathcal{B})$ where $\mathcal{B}$ is the Boolean algebra of subsets of $\mathcal{M}^{\bar{x}}$ that are ϕ-definable over A.

Remark A.9. In the same way that any filter (or even any collection of sets with the finite intersection property) extends to an ultrafilter, one can show that any type over A extends to a complete type over A (and similarly for ϕ-types).

A.3. **Saturation.** Let T be a complete first-order L-theory with infinite models.

Definition A.10. Suppose p is a type in the variables $\bar{x}$. A **realization** of p, in some model $\mathcal{M} \models T$, is a tuple $\bar{a} \in \mathcal{M}^{\bar{x}}$ such that $\mathcal{M} \models \phi(\bar{a})$ for all $\phi(\bar{x}) \in p$. We may also write $\mathcal{M} \models p(\bar{a})$, or $\bar{a} \models p$, or just say "$p(\bar{a})$ holds".

Exercise A.11. If $\mathcal{M} \models T$, $A \subseteq \mathcal{M}$, and $p \in S_{\bar{x}}^{\mathcal{M}}(A)$, then p is realized in some $\mathcal{N} \succeq \mathcal{M}$.

Definition A.12. Given an infinite cardinal κ, we say that $\mathcal{M}$ is **κ-saturated** if for any $A \subseteq \mathcal{M}$, with $|A| < \kappa$, any $p \in S_1^{\mathcal{M}}(A)$ is realized in $\mathcal{M}$. We say that $\mathcal{M}$ is **saturated** if it is $|\mathcal{M}|$-saturated.

Exercise A.13. Given $\mathcal{M} \models T$ and an infinite cardinal κ, the following are equivalent.

(i) $\mathcal{M}$ is κ-saturated.
(ii) Any type in one variable, over a set $A \subseteq \mathcal{M}$ with $|A| < \kappa$, is realized in $\mathcal{M}$.
(iii) Any type in κ variables, over a set $A \subseteq \mathcal{M}$ with $|A| < \kappa$, is realized in $\mathcal{M}$.

Note that if $\mathcal{M}$ is κ-saturated then $\kappa \leq |\mathcal{M}|$ since the type $\{x \neq a : a \in \mathcal{M}\}$ (which *is a type*) is not realized in $\mathcal{M}$. On the other hand:

Theorem A.14. [1, Theorem 4.3.12] *Given any $\mathcal{M} \models T$ and any infinite cardinal κ, there is a κ^+-saturated elementary extension $\mathcal{N} \succeq \mathcal{M}$ with $|\mathcal{N}| \leq |\mathcal{M}|^\kappa$.*

In addition to saturation, we will also want structures to have "strong homogeneity".

Definition A.15. Given an infinite cardinal κ, we say that $\mathcal{M}$ is **strongly κ-homogeneous** if for any $A \subseteq \mathcal{M}$ with $|A| < \kappa$, any partial elementary map $f \colon A \to \mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

If the cardinal is chosen carefully, then strong homogeneity follows from saturation:

Exercise A.16. If $\mathcal{M}$ is saturated then it is strongly $|\mathcal{M}|$-homogeneous.

However, given a cardinal κ, there may not be a saturated model of T of cardinality κ. The simplest way to address this is via certain set-theoretic assumptions, which we dicuss below. But first, let us describe the goal more accurately.

Throughout the course, we will say that $\mathcal{M} \models T$ is **sufficiently saturated** if it is κ-saturated and strongly κ-homogenous for some uncountable cardinal κ, which is large enough to accomplish some prescribed list of tasks. It is usually annoying to write down precisely what these tasks are beforehand, but one can always go back through a proof and figure out exactly which properties of κ were used. For example:

We will often be in the situation where $\mathcal{M}$ is an expansion of a group G, and we will want to know that if a subgroup $\Gamma \leq G$ has index less than κ then so does its "normal core" $\bigcap_{g\in G} g\Gamma g^{-1}$. For this, it suffices to have that $2^\lambda < \kappa$ whenever $\lambda < \kappa$, i.e., that κ is a **strong limit**. An example of an *uncountable* strong limit is the cardinal $\beth_\omega = \sup\{\beth_n : n \geq 0\}$, where $\beth_0 = \aleph_0$ and $\beth_{n+1} = 2^{\beth_n}$.

On the other hand, $\beth_\omega$ may not be suitable for other tasks since it has countable cofinality. (The **cofinality** of a cardinal κ is the smallest cardinal λ such that there is an unbounded function $f\colon \lambda \to \kappa$.) So instead one might work with $\kappa = \beth_{\omega_1}$, where ω_1 denotes the first uncountable ordinal. But this still might not be good enough for some other tasks, e.g., applications of the Erdös-Rado Theorem.

Often, the easiest thing to do is assume that κ is **strongly inaccessible**, i.e., it is an uncountable strong limit and is equal to its own cofinality. However, ZFC cannot prove that such cardinals exist. So one usually makes a set-theoretic assumption like this and then, if it is important enough, goes back and isolates precisely what properties of κ are needed and checks that they are provable in ZFC.

Altogether, throughout the course, when we say that $\mathcal{M} \models T$ is "sufficiently saturated", we can take this to mean that $\mathcal{M}$ is κ-saturated and strongly κ-homogeneous for some strongly inaccessible cardinal κ. Assuming we have such a κ, the existence of a "sufficiently saturated" model is a consequence of Exercise A.16 and the following theorem.

Theorem A.17. [1, Theorem 4.3.14] *If $\mathcal{M} \models T$ and $\kappa > |\mathcal{M}|$ is strongly inacessible, then there is a saturated elementary extension $\mathcal{N} \succeq \mathcal{M}$ of cardinality κ.*

Going back to the previous discussion, if one really did want to remove the "large cardinal assumption" then the first thing to do would be to write down all properties of κ required for the proof (e.g., strong limit, uncountable cofinality, etc.). The second thing to do is check that such a cardinal κ can be proved to exist in ZFC. If one successfully does this, then the following (ZFC) theorem can be used to obtain a "sufficiently saturated" model.

Theorem A.18. *Fix $\mathcal{M} \models T$ and let κ be an infinite cardinal. Then there is a κ^+-saturated and strongly κ^+-homogeneous elementary extension $\mathcal{N} \succeq \mathcal{M}$, with $|\mathcal{N}| \leq \beth_{\kappa^+}(|\mathcal{M}|)$.*

As stated, this theorem doesn't appear in [1], but the techniques are similar to Theorem A.14 (see also Proposition 4.11 of Pillay's course notes [2]).

References

[1] David Marker, *Model theory*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002.
[2] Anand Pillay, *Lecture notes - model theory*, Math 411 notes, `https://www3.nd.edu/~apillay/pdf/lecturenotes_modeltheory.pdf`, 2002.