B. MISCELLANEOUS NOTES

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These notes provide details on several topics that will arise in the course, but which may be unfamiliar or have varying definitions in the literature. So the purpose of these notes is to keep everyone up to speed and on the same page. I will continue to update this document throughout the term. Feel free to send me comments, corrections, and suggestions/requests for new sections.

Contents

B.1. Probability measures on algebras

Let X be a set.

Definition B.1. A **Boolean algebra** (on X) is a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ such that:

(*i*) $X \in \mathcal{B}$,
(*ii*) if $A, B \in \mathcal{B}$ then $A \cup B \in \mathcal{B}$, and
(*iii*) if $A \in \mathcal{B}$ then $X \backslash A \in \mathcal{B}$.

Definition B.2. Suppose $\mathcal{B}$ is a Boolean algebra on X. A **finitely-additive probability measure on** $\mathcal{B}$ is a function $\mu\colon \mathcal{B} \to [0,1]$ such that $\mu(X) = 1$ and, if $A, B \in \mathcal{B}$ are disjoint, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Definition B.3. A **σ-algebra** (on X) is a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ such that:

(*i*) $X \in \mathcal{A}$,
(*ii*) if $A_0, A_1, A_2, \ldots \in \mathcal{A}$ then $\bigcup_{n=0}^{\infty} A_n \in \mathcal{A}$, and
(*iii*) if $A \in \mathcal{A}$ then $X \backslash A \in \mathcal{A}$.

Definition B.4. Suppose $\mathcal{A}$ is a σ-algebra on X. A **probability measure on** $\mathcal{A}$ is a function $\mu\colon \mathcal{A} \to [0,1]$ such that $\mu(X) = 1$ and, if $\{A_n : n \in \mathbb{N}\}$ is a collection of pairwise disjoint sets in $\mathcal{A}$, then $\mu(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} \mu(A_n)$.

Example B.5. Let X be a topological space. Then the **Borel σ-algebra** is the σ-algebra $\mathcal{A}$ of subsets of X generated by the collection of open sets. A subset of X is **Borel** if it is in $\mathcal{A}$. A **Borel probability measure on** X is a probability measure on $\mathcal{A}$.

Definition B.6. Let X be a topological space and suppose μ is a Borel probability measure on X. Then μ is **regular** if for any Borel set $W \subseteq X$,

$$\mu(W) = \sup\{\mu(K) : K \subseteq W,\ K \text{ is compact}\} = \inf\{\mu(U) : W \subseteq U,\ U \text{ is open}\}.$$

B.2. Stone spaces

Let X be a set, and let $\mathcal{B}$ be a fixed Boolean algebra on X. Then the notion of an ultrafilter relativizes nicely to $\mathcal{B}$ (see the definition below). These ultrafilters will play a significant role in the course, but in a manner very different from the construction of ultraproducts. For example, when using an ultrafilter over an index set to construct an ultraproduct, we only need to work with a single ultrafilter, which will usually need to be nonprincipal in order to achieve anything interesting. On the other hand, when working with ultrafilters over Boolean algebras (e.g., the Boolean algebra of definable subsets of some structure), we will often be concerned with the collection of *all* such ultrafilters, and how different ultrafilters interact with each other. So for pedagogical reasons, we will refer to ultrafilters over $\mathcal{B}$ as *$\mathcal{B}$-types*, and denote them with letters like p and q.

Definition B.7. A **$\mathcal{B}$-type** is a collection $p \subseteq \mathcal{B}$ such that:

(i) $\emptyset \notin p$ and $X \in p$,
(ii) if $A \in p$ and $B \in \mathcal{B}$, with $A \subseteq B$, then $B \in p$,
(iii) if $A, B \in p$ then $A \cap B \in p$, and
(iv) for any $A \in \mathcal{B}$, either $A \in p$ or $X \backslash A \in p$.

We let $S(\mathcal{B})$ denote the set of $\mathcal{B}$-types.

The letter S in the previous definition is for *Stone*, and $S(\mathcal{B})$ is referred to as the **Stone space over** $\mathcal{B}$. The use of the word "space" is not careless, as $S(\mathcal{B})$ admits a natural topological structure.

Definition B.8. The **Stone topology** on $S(\mathcal{B})$ is the topology whose basic open sets are of the form $[A] = \{p \in S(\mathcal{B}) : A \in p\}$ for $A \in \mathcal{B}$.

Exercise B.9. $S(\mathcal{B})$ is a totally disconnected compact Hausdorff space under the Stone topology.

One can view the Stone topology as a subspace topology in the following way. View $S(\mathcal{B})$ as a subset of $\{0,1\}^{\mathcal{B}}$ by identifying a $\mathcal{B}$-type with its indicator function. If we put the discrete topology on $\{0,1\}$ and the product topology on $\{0,1\}^{\mathcal{B}}$, then it is easy to show that $S(\mathcal{B})$ is closed. The Stone topology on $S(\mathcal{B})$ is precisely the induced subspace topology.

The following classical theorem provides a connection between finitely-additive probability measures on Boolean algebras and regular Borel probability measures on totally disconnected compact Hausdorff spaces. We will use this result at a crucial moment later in the course.

Theorem B.10. *For any finitely-additive probability measure μ on $\mathcal{B}$ there is a unique regular Borel probability measure $\widehat{\mu}$ on $S(\mathcal{B})$ such that $\widehat{\mu}([A]) = \mu(A)$ for any $A \in \mathcal{B}$. Moreover, the map $\mu \mapsto \widehat{\mu}$ is a bijection between finitely-additive probability measures on $\mathcal{B}$ and regular Borel probability measures on $S(\mathcal{B})$.*

This theorem will eventually be given as an official exercise. Details can also be found in [4, Section 7.1] (in a slightly more specific setting) or [1, Proposition 416Q].

B.3. Compact groups

In this section, we list some results about compact topological groups that will be used in the course. *Throughout this section, "compact" means "compact and Hausdorff".*

B.3.1. **Connected components.** Recall that a subset C of a topological space X is *connected* if it cannot be written as the union of two nonempty disjoint sets which are open in the subspace topology on C. Given $x \in X$, the *connected component of* x is the maximal (with respect to inclusion) connected subset of X containing x.

Exercise B.11. Suppose X is a compact space, and fix $x \in X$. Then the connected component of x coincides with the intersection of all clopen subsets of X containing x.

The identity component of a topological group G is denoted G^0.

Exercise B.12.

(*a*) The identity component of a topological group is a closed normal subgroup.
(*b*) Suppose K and L are compact groups and $\pi\colon K \to L$ is a continuous surjective homomorphism. Then $\pi(K^0) = L^0$.

Remark B.13. We will use the second part of the previous exercise in the following way. Suppose K is a compact group, which is obtained as an inverse limit $K = \varprojlim L_i$ of compact groups $(L_i)_{i\in I}$ such that the projection maps $K \to L_i$ are surjective. Using the second part of the previous exercise (and other basic properties of inverse/projective limits), it is not hard to show that $K^0 = \varprojlim L_i^0$. For further details on inverse limits of topological groups and spaces, see Chapter 1 of [3] (especially Corollary 1.1.8 for this particular conclusion).

Next we note an easy sufficient condition for the identity component of a compact group to be abelian. (This will be an important situation for certain parts of the course.)

Proposition B.14. *Suppose K is a compact group which has an abelian subgroup of finite index. Then K^0 is abelian.*

Proof. Suppose $H \leq K$ is abelian and finite index. Then the closure $\bar{H}$ of H is still an abelian finite-index subgroup. In particular, $\bar{H}$ is clopen, and thus contains K^0 by Exercise B.11. So K^0 is abelian. □

B.3.2. **Measures and Metrics.**

Theorem B.15. [2, Theorem 2.8] *Any compact group admits a unique left-invariant regular Borel probability measure, called the **normalized Haar measure**. Moreover, this measure is also the unique right-invariant regular Borel probability measure.*

Theorem B.16. [2, Corollary A4.19] *Any compact second countable group admits a bi-invariant metric compatible with the topology.*

B.3.3. **Compact Lie groups.** In these notes (and in the course), by a *Lie group*, we mean a topological group which is also a finite-dimensional real smooth manifold such that the group operation and the inversion map are smooth. Any finite group is a (0-dimensional) compact Lie group under the discrete topology. Another example of a compact Lie group, which we will use frequently, is the n-dimensional torus $\mathbb{T}^n$ (for any $n \geq 0$). Specifically, $\mathbb{T}^1$ is the circle group $S^1 = \mathbb{R}/\mathbb{Z}$ with the usual topology, and then we endow $\mathbb{T}^n$ with the product topology.

Extensive familiarity with Lie groups will not be necessary for the course. For the most part, one should just be aware of the previous examples and the following facts.

Exercise B.17. Let K be a compact Lie group.

(*a*) K is locally connected, and thus K^0 is clopen of finite index.
(*b*) K is second countable (and thus admits a compatible bi-invariant metric).

Theorem B.18. [2, Proposition 2.42] *A compact Lie group is connected and abelian if and only if it is topologically isomorphic to* $\mathbb{T}^n$ *for some* $n \geq 1$.

Theorem B.19 (Peter-Weyl)**.** *Any compact group* K *is (topologically isomorphic to) an inverse limit* $\varprojlim L_i$ *of some inverse system* $(L_i)_{i \in I}$ *of compact Lie groups, in which the projection maps* $K \to L_i$ *are surjective. Moreover, if* K *is second countable then one may assume* I *is* $\mathbb{N}$ *with the usual ordering.*

For the first part of the the previous theorem, see [2, Corollary 2.43]. The moreover statement can be obtained by combining [2, Corollary 2.36] and [2, Exercise E9.1].

References

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