Part III - Model Theory
Examples Sheet 4

Problems 2 and 7 should be submitted through the course Moodle by 11:59 pm on Wednesday, 13 January.

1. Let T be an $\mathcal{L}$-theory. Prove that T has quantifier elimination if and only if for any $(|\mathcal{L}|+\aleph_0)^+$-saturated models $\mathcal{M}, \mathcal{N} \models T$, any finitely generated common substructure $\mathcal{A}$ of $\mathcal{M}$ and $\mathcal{N}$, and any quantifier-free $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n, y)$, if $\bar{a} \in A^n$ and $\mathcal{M} \models \exists y \varphi(\bar{a}, y)$ then $\mathcal{N} \models \exists y \varphi(\bar{a}, y)$.

2. Let $T = \mathrm{Th}(\mathbb{Z}, <)$. Describe the prime model and the countable saturated model of T.

3. Let $\mathcal{L} = \{E\}$ where E is a binary relation symbol. Let T be an $\mathcal{L}$-theory asserting that E is an equivalence relation with infinitely many infinite classes. Prove that T is complete and κ-stable for all $\kappa \geq \aleph_0$.
Bonus problem: Given an ordinal α, compute $I(T, \aleph_\alpha)$.

4. Let T be a complete theory. Show that T is stable if and only if no $\mathcal{L}$-formula of the form $\varphi(x, \bar{y})$, with x a singleton variable, has the order property with respect to T.

5. For each theory below, find a formula with the order property.

   (a) $T = \mathrm{Th}(\mathbb{N}, +)$

   (b) $T = \mathrm{Th}(\mathbb{N}, \cdot)$

   (c) T is the theory RG of Rado graphs

   (d) $T = \mathrm{Th}(\mathbb{Z}, +, A)$ where $A = \{n^2 : n \in \mathbb{N}\}$ (view A as a unary relation).

6. Let $T = \mathrm{Th}(F, +, \cdot, 0, 1)$ where F is a field. Prove that no quantifier-free formula has the order property with respect to T.

7. Suppose G is an expansion of a group and $\mathrm{Th}(G)$ is stable. Let q be a collection of bi-generic definable subsets of G, and suppose q is closed under finite intersections. Show that there is some bi-generic type $p \in S_1(G)$ extending q.

8. Suppose G is an expansion of a group and $p \in S_1(G)$. Show that if $g \in G$ then $gp \in S_1(G)$ and, moreover, if p is bi-generic then so is gp.

9. Suppose G is an expansion of a group and $\mathrm{Th}(G)$ is stable.

   (a) Assume G^0 has finite index in G. Prove that G^0 is definable.

   (b) Assume G is $(|\mathcal{L}| + \aleph_0)$-saturated and G^0 is definable. Prove that G^0 has finite index in G.