The Asymptotic Behavior of Degenerate Oscillatory Integrals in Two Dimensions

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1. Introduction

In this paper we are interested in the following type of oscillatory integral. Suppose $S(x,y)$ is a smooth real-valued function defined in a neighborhood of the origin in $\mathbf{R}^2$, and $\phi(x,y) \in C_c^\infty(\mathbf{R}^2)$ is real-valued and supported in a small neighborhood of the origin. We define

$$J_{S,\phi}(\lambda) = \int_{\mathbf{R}^2} e^{i\lambda S(x,y)}\phi(x,y)\,dx\,dy \tag{1.1}$$

Here λ is a real parameter and we want to understand the behavior of $J_{S,\phi}(\lambda)$ as $\lambda \to +\infty$. Oscillatory integrals of the form (1.1) and their higher-dimensional analogues come up frequently in several areas of analysis, including PDE's, mathematical physics, and harmonic analysis. For example, such oscillatory integrals arise when analyzing the decay of Fourier transforms of surface-supported measures such as in [IKM]. We refer to chapter 8 of [S] for an overview of such issues. The stability of oscillatory integrals (1.1) under perturbations of the phase function $S(x,y)$ is connected to various issues in complex geometry and has been considered for example in [PSSt] and [V].

Since one can always factor out an $e^{i\lambda S(0,0)}$, it does no harm to assume that $S(0,0) = 0$. If $\nabla S(0,0) \neq 0$, in a small enough neighborhood of the origin one can integrate by parts arbitrarily many times in (1.1) and get that $J_{S,\phi}(\lambda)$ decays faster than $C_N\lambda^{-N}$ for any N. Hence in this paper we always assume that the origin is a critical point for S; that is, we assume that

$$S(0,0) = 0, \quad \nabla S(0,0) = 0$$

In this case, if $S(x,y)$ is real-analytic, using resolution of singularities (see [G1] for an elementary proof) one always has an asymptotic expansion

$$J_{S,\phi}(\lambda) \sim \sum_{j=0}^{\infty}(d_j(\phi)\lambda^{-s_j} + d_j'(\phi)\ln(\lambda)\lambda^{-s_j}) \tag{1.2}$$

Here $\{s_j\}$ is an increasing arithmetic progressions of positive rational numbers independent of ϕ deriving from the resolution of singularities of S. We always assume s_0 is chosen to

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be minimal such that in any sufficiently small neighborhood U of the origin $d_0(\phi)$ or $d_0'(\phi)$ is nonzero for some ϕ supported in U. In this paper, we will give explicit formulas for the leading term of (1.2) once one is in certain coordinate systems which we call "superadapted", in analogy with the adapted coordinate systems of [V]. In the smooth case, we will find appropriate weaker analogues.

Definition 1.1. The *oscillatory index* of S is defined to be s_0. If in any small neighborhood of the origin there is some ϕ for which $d_0'(\phi)$ is nonzero, then we say s_0 has multiplicity 1. Otherwise, we say it has multiplicity zero.

In the case where S is a smooth function whose Hessian determinant at the origin is nonvanishing, one can do a smooth coordinate change such that $S(x,y)$ becomes $\pm x^2 \pm y^2$ in the new coordinates. Then one can use the well-known one dimensional theory (see Chapter 8 of [S]) and explicitly obtain an asymptotic expansion for $J_{S,\phi}(\lambda)$; the leading term will be given by $\frac{2\pi i}{D^{\frac{1}{2}}}\phi(0,0)\lambda^{-1}$ where D denotes the Hessian determinant of S at the origin. Hence our concern in this paper will be when $D = 0$; that is, when S has a degenerate critical point at the origin.

In the real-analytic situation, there is a close relationship between $J_{S,\phi}(\lambda)$ and the function $I_{S,\phi}(\epsilon)$ defined by

$$I_{S,\phi}(\epsilon) = \int_{\{(x,y):0<S(x,y)<\epsilon\}} \phi(x,y)\,dx\,dy \tag{1.3}$$

Analogous to (1.2), when $S(0,0) = 0$ the functions $I_{S,\phi}(\epsilon)$ and $I_{-S,\phi}(\epsilon)$ have asymptotic expansions which we may write as

$$I_{S,\phi}(\epsilon) \sim \sum_{j=0}^{\infty}(c_j(\phi)\epsilon^{r_j} + c_j'(\phi)\ln(\epsilon)\epsilon^{r_j}) \tag{1.4a}$$

$$I_{-S,\phi}(\epsilon) \sim \sum_{j=0}^{\infty}(C_j(\phi)\epsilon^{r_j} + C_j'(\phi)\ln(\epsilon)\epsilon^{r_j}) \tag{1.4b}$$

Analogous to before, $\{r_j\}$ is an increasing arithmetic progression of positive rational numbers independent of ϕ deriving from the resolution of singularities of S, and r_0 is chosen to be minimal such that in any sufficiently small neighborhood U of the origin there is some ϕ supported in U for which at least one of $c_0(\phi)$, $c_0'(\phi)$, $C_0(\phi)$, or $C_0'(\phi)$ is nonzero.

Using well-known methods (see Ch 7 of [AGV]), when the oscillatory index is less than 1, corresponding to the degenerate case, $r_0 = s_0$ and the coefficient of the leading term of (1.2) can always be expressed in terms of those of the corresponding terms of (1.4a) and (1.4b). Often $I_{S,\phi}$ and $I_{-S,\phi}$ are easier to deal with than $J_{S,\phi}$ due to the absence of cancellations which can make it difficult to find lower bounds for $J_{S,\phi}$ directly.

The $I_{S,\phi}$ are closely related to the $H_{S,U}$ defined by

$$H_{S,U}(\epsilon) = |\{x \in U : 0 < S(x,y) < \epsilon\}| \tag{1.5}$$

Here U is a small open set containing the origin, and the goal is to understand how $H_{S,U}(\epsilon)$ behaves as $\epsilon \to 0$. Many of our results concerning the $I_{S,\phi}(\epsilon)$ will immediately imply corresponding results about the $H_{S,U}(\epsilon)$. Specifically, one chooses ϕ_1 supported in U and equal to 1 outside some δ-neighborhood of the boundary of U, then chooses ϕ_2 equal to 1 on U and supported on a δ-neighborhood of U. One compares the theorems for $I_{S,\phi_1}(\epsilon)$ and $I_{S,\phi_2}(\epsilon)$ and then lets δ go to zero if necessary. In the case where S has an isolated zero at the origin, for ϵ small enough the set $\{x \in U : 0 < S(x,y) < \epsilon\}$ will be a subset of a set where the theorems hold, so one can simply take $\phi = 1$ and then $H_{S,U}(\epsilon) = I_{S,\phi}(\epsilon)$ for such ϵ.

In [V], Varchenko developed some ideas that went a long way towards understanding the case of degenerate real-analytic phase. To describe his work, we need some pertinent definitions.

Definition 1.2. Let $S(x,y) = \sum_{a,b} s_{ab}x^a y^b$ denote the Taylor expansion of $S(x,y)$ at the origin. Assume there is at least one (a,b) for which s_{ab} is nonzero. For any (a,b) for which $s_{ab} \neq 0$, let Q_{ab} be the quadrant $\{(x,y) \in \mathbf{R}^2 : x \geq a, y \geq b\}$. Then the *Newton polygon* $N(S)$ of $S(x,y)$ is defined to be the convex hull of the union of all Q_{ab}.

In general, a Newton polygon consists of finitely many (possibly zero) bounded edges of negative slope as well as an unbounded vertical ray and an unbounded horizontal ray.

Definition 1.3. The *Newton distance* $d(S)$ of $S(x,y)$ is defined to be $\inf\{t : (t,t) \in N(S)\}$.

Throughout this paper, we will use the (t_1, t_2) coordinates to write equations of lines relating to Newton polygons, so as to distinguish from the x-y variables of the domain of $S(x,y)$. The line in the t_1-t_2 plane with equation $t_1 = t_2$ comes up so frequently it has its own name:

Definition 1.4: The *bisectrix* is the line in the t_1-t_2 plane with equation $t_1 = t_2$.

A key role in the above theorems as well as our theorems to follow is played by the following polynomials.

Definition 1.5. Suppose e is a compact edge of $N(S)$. Define $S_e(x,y)$ by $S_e(x,y) = \sum_{(a,b)\in e} s_{ab}x^a y^b$. In other words $S_e(x,y)$ is the sum of the terms of the Taylor expansion of S corresponding to $(a,b) \in e$. If $S(x,y)$ is real-analytic, we use the same terminology when e is the vertical or horizontal ray of $N(S)$.

In [V], Varchenko showed that when S is real-analytic the oscillatory index is always at most $\frac{1}{d(S)}$, and that there is necessarily a coordinate system in which it is actually equal to $\frac{1}{d(S)}$. He also showed that the coordinate change to such coordinates can always be made of the form $(x,y) \rightarrow (x, y - f(x))$ or $(x,y) \rightarrow (x - f(y), y)$ for real analytic f. Coordinate systems where $d(S)$ achieves the maximum possible value are referred to as "adapted coordinates". He also showed that the multiplicity of the oscillatory index is equal to 1 if and only if there are adapted coordinates where the bisectrix intersects $N(S)$ at a vertex. Otherwise the multiplicity is 0; the leading term of (1.2) will not have the $\ln(\lambda)$ factor in it. The issue of finding an expression for the leading coefficient $d_0(\phi)$ or $d_0'(\phi)$ is not treated in [V]. However, in the case where the $S(x,y)$ has a critical point of finite Milnor number at the origin, it is shown in [V] that the leading coefficient $d_0(\phi)$ or $d_0'(\phi)$ is some fixed multiple of $\phi(0,0)$ depending on the phase; precisely which multiple is not determined. These results were later extended in [Sh] to convex finite-type functions.

Smooth analogues are proven in [IM] and [IKM]. In [IM] it is shown that adapted coordinates exist in the smooth case. In [IKM] it is shown that in smooth adapted coordinates one has the estimate $|J_{S,\phi}(\lambda)| < C \ln|\lambda||\lambda|^{-\frac{1}{d(S)}}$ for large $|\lambda|$. There are also operator versions of such results. For example, in [PS] it is proven that $||\int e^{i\lambda S(x,y)}\phi(x,y)f(y)\,dy||_{L^2}$ $< C|\lambda|^{-\frac{1}{2d(\tilde{S})}}||f||_{L^2}$ for real analytic phase, where $\tilde{S}(x,y) = S(x,y) - S(0,y) - S(x,0)$. The exponent $\frac{1}{2d(\tilde{S})}$ is sharp. Generalizations to smooth phase were proven in [R] and [G2]. These results use subdivisions into curved regions as will be done here. However, there are significant differences since one gets stronger results for operators; in particular, adapted coordinates are not needed.

Our theorems below will require us to be in certain adapted coordinate systems which we call "superadapted" coordinate systems:

Definition 1.6. One is in *superadapted coordinates* if whenever e is a compact edge of $N(S)$ intersecting the bisectrix, both of the functions $S_e(1,y)$ and $S_e(-1,y)$ have no real zero of order $d(S)$ or greater other than possibly $y = 0$.

It can be shown that an equivalent definition is obtained by stipulating the same condition on $S_e(x,1)$ and $S_e(x,-1)$ instead of $S_e(1,y)$ and $S_e(-1,y)$; we choose the y-variable for definiteness. In section 7 we will prove any phase function can be put in superadapted coordinates using some ideas from two-dimensional resolution of singularities.

Lemma 1.0. In a superadapted coordinate system, a critical point of $S(x,y)$ at the origin is nondegenerate if and only if $d(S) = 1$.

Proof. Write $S(x,y) = ax^2 + bxy + cy^2 + O(|x|^3 + |y|^3)$. The only way $d(S)$ could be greater than 1 is for either a and b to both be zero, or for c and b to both be zero. In either case, the Hessian at the origin is zero and the phase is degenerate. So we assume that $d(S) = 1$, and we will show that in a superadapted coordinate system the phase is nondegenerate.

First consider the case where $N(S)$ has a vertex at $(1,1)$. Then $b \neq 0$ and either a or c is zero. Suppose $a = 0$ but $c \neq 0$; we claim that this implies the coordinate system is not superadapted. For in this case there is an edge e connecting $(1,1)$ and $(0,2)$. Then $S_e(x,y) = bxy + cy^2$ and thus $S_e(1,y) = by + cy^2$ has a zero at $-\frac{b}{c} \neq 0$, inconsistent with the definition of superadapted. Thus in a superadapted coordinate system, if $N(S)$ has a vertex at $(1,1)$ and $a = 0$, then $c = 0$ and thus the Hessian is nonzero at the origin; the phase is nondegenerate. The case where $c = 0$ but $a \neq 0$ leads to a similar contradiction. We conclude that if $N(S)$ has a vertex at $(1,1)$ and $d(S) = 1$ then the phase is nondegenerate.

Next, consider the case where $(1,1)$ is in the interior of an edge e of $N(S)$. In this case, the endpoints of e are $(2,0)$ and $(0,2)$. Hence a and c are nonzero. In a superadapted coordinate system, one must have that $S(1,y) = a + by + cy^2$ has no real zeroes other than $y = 0$. Since $a \neq 0$, this is equivalent to $a + by + cy^2$ having no real zeroes at all, which happens exactly when $b^2 < 4ac$. This is equivalent to the Hessian determinant at the origin being nonzero, and thus the phase is nondegenerate in this situation too. This completes the proof of Lemma 1.0.

We now come to our theorems. We will use the shorthand d to denote the Newton distance $d(S)$. If $N(S)$ intersects the bisectrix in the interior of an edge, bounded or unbounded, we denote this edge by e_0 and its slope by $-\frac{1}{m}$, where $0 \leq m \leq \infty$. We use the shorthand $S_0(x,y)$ to denote $S_{e_0}(x,y)$.

Theorem 1.1 is our main result. It gives explicit expressions for the leading term of $(1.4a)$ in superadapted coordinates. Applying the theorem to $-S$ gives analogous formulas for the expansion $(1.4b)$. As indicated above, $(1.4a)$ – $(1.4b)$ directly imply formulas for the leading term of the asymptotic expansion (1.2) in the degenerate case; these are given in Theorem 1.2.

Theorem 1.1. Suppose $S(x,y)$ is a smooth phase function in superadapted coordinates with $d > 1$. If the function $\phi(x,y)$ is supported in a sufficiently small neighborhood of the origin, then $r_0 = \frac{1}{d}$ and the following hold.

a) Suppose the bisectrix intersects $N(S)$ in the interior of a compact edge. Define the function $S_0^+(x,y)^{-\frac{1}{d}}$ to be $S_0(x,y)^{-\frac{1}{d}}$ when $S_0(x,y) > 0$ and zero otherwise. Then we have

$$\lim_{\epsilon \to 0} \frac{I_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}} = (m+1)^{-1}\phi(0,0) \int_{-\infty}^{\infty} (S_0^+(1,y)^{-\frac{1}{d}} + S_0^+(-1,y)^{-\frac{1}{d}})\, dy \tag{1.6}$$

In particular, if $S(x,y)$ is real-analytic then the coefficient $c_0'(\phi)$ in $(1.4a)$ is always zero and $c_0(\phi)$ is given by (1.6).

b) Suppose the bisectrix intersects $N(S)$ at a vertex (d,d). Let $s_{dd}x^dy^d$ denote the corresponding term of the Taylor expansion of S; hence $s_{dd} \neq 0$. Denote the slopes of the two edges of $N(S)$ meeting at (d,d) by s_1 and s_2, where $-\infty \leq s_2 < s_1 \leq 0$. Then

$$\lim_{\epsilon \rightarrow 0} \frac{I_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}} \ln(\epsilon)} = \eta(S)|s_{dd}|^{-\frac{1}{d}} \phi(0,0)(\frac{1}{s_1 - 1} - \frac{1}{s_2 - 1}) \tag{1.7}$$

Here $\eta(S) = 4$ if $s_{dd} > 0$ and d is even, $\eta(S) = 2$ if d is odd, and $\eta(S) = 0$ if $s_{dd} < 0$ and d is even.

c) Suppose $S(x,y)$ is real-analytic and the bisectrix intersects $N(S)$ in the interior of the horizontal ray. Write $S_0(x,y) = a(x)y^d$ where $a(x)$ is real-analytic. Let $\alpha(x)$ denote the one-dimensional measure of $\{y : 0 \leq a(x)y^d \leq 1\}$. In particular, $\alpha(x) = |a(x)|^{-\frac{1}{d}}$ when d is odd. Then $c_0'(\phi) = 0$, and $c_0(\phi)$ is given by

$$c_0(\phi) = \int_{-\infty}^{\infty} \alpha(x)\phi(x,0)\,dx \tag{1.8}$$

The case where the bisectrix intersects the interior of the vertical ray has the analogous formula.

As can be seen, parts a), b) and c) give quite different formulas. Correspondingly, in our subsequent theorems we break up into three cases. Case 1 is when the bisectrix intersects $N(S)$ in the interior of a compact edge, case 2 is when the bisectrix intersects $N(S)$ at a vertex (d,d), and case 3 is when the bisectrix intersects $N(S)$ in the interior of one of the unbounded rays. Notice that in cases 1 and 3, for a given $\phi(x,y)$ the expressions of Theorem 1.1 depend only on $S_0(x,y)$, and that in case 2 it depends on $s_{dd}x^dy^d$ as well as the slopes of the edges of $N(S)$ meeting at (d,d).

Our next theorem gives the oscillatory integral version of Theorem 1.1 for the real-analytic case.

Theorem 1.2. Assume $S(x,y)$ is real-analytic and is in superadapted coordinates with $d > 1$. Then $s_0 = \frac{1}{d}$. In case 1 and 3, the coefficient $d_0'(\phi)$ of (1.2) is always zero and $d_0(\phi)$ is given by

$$d_0(\phi) = \frac{\Gamma(\frac{1}{d})}{d}(e^{i\frac{\pi}{2d}}c_0(\phi) + e^{-i\frac{\pi}{2d}}C_0(\phi)) \tag{1.9a}$$

In case 2, one has

$$d_0'(\phi) = -\frac{\Gamma(\frac{1}{d})}{d}(e^{i\frac{\pi}{2d}}c_0'(\phi) + e^{-i\frac{\pi}{2d}}C_0'(\phi)) \tag{1.9b}$$

Proof. We will be sketchy here since the method for proving Theorem 1.2 from Theorem 1.1 is well-known; we refer to chapter 7 of [AGV] for more details. Note that

$$J_{S,\phi}(\lambda) = \int_0^{\infty} (\partial_\epsilon I_{S,\phi}(\epsilon))e^{i\lambda\epsilon}\,d\epsilon + \int_0^{\infty} (\partial_\epsilon I_{-S,\phi}(\epsilon))e^{-i\lambda\epsilon}\,d\epsilon \tag{1.10a}$$

By [F], if $\gamma \in C_c(\mathbf{R})$ with $\gamma(t) = 1$ near 0 and if $\alpha > -1$, for any l we have

$$\int_0^\infty e^{i\lambda t} t^\alpha \ln(t)^m \gamma(t)\, dt = \frac{\partial^m}{\partial \alpha^m} \frac{\Gamma(\alpha+1)}{(-i\lambda)^{\alpha+1}} + O(\lambda^{-l}) \tag{1.10b}$$

Inserting (1.4a) and (1.4b) into (1.10a) and using Theorem 1.1 and (1.10b) gives the theorem.

Comment 1. One does need that $d > 1$ for Theorem 1.2 to hold. When $S(x,y)$ has a nondegenerate saddle critical point, there are coordinates where $S(x,y) = xy$. This falls under case 2, and one has that $c_0'(\phi) = C_0'(\phi)$. This means that the two terms of (1.9b) will cancel. And in fact when $\phi(0,0) \neq 0$, $|I_{S,\phi}(\epsilon)|$, $|I_{-S,\phi}(\epsilon)| \sim |\epsilon \ln(\epsilon)|$, while $|J_{S,\phi}(\lambda)| \sim \lambda^{-1}$.

Comment 2. In cases 1 and 2, the expressions (1.9a) and (1.9b) for $d_0(\phi)$ and $d_0'(\phi)$ will always be nonzero when $d > 1$ and $\phi(0,0) \neq 0$. This is because the expressions given by Theorem 1.1 for $c_0(\phi)$, $C_0(\phi)$, $c_0'(\phi)$, and $C_0'(\phi)$ are real multiples of $\phi(0,0)$, while the ratio of the $e^{i\frac{\pi}{2d}}$ and $e^{-i\frac{\pi}{2d}}$ factors is never real when $d > 1$.

Comment 3. In any dimension, when the phase satisfies an appropriate nondegeneracy condition there are reasonably explicit formulas for the leading coefficient of the leading term of the asymptotic expansion of oscillatory integrals such as (1.1). Such formulas are proven in [DS] and [DNS].

Next, we give some less precise C^∞ analogues for the $I_{S,\phi}(\epsilon)$. The lower bounds involve $I_{|S|,\phi}(\epsilon) = I_{S,\phi}(\epsilon) + I_{-S,\phi}(\epsilon) = \int_{\{(x,y): |S(x,y)| < \epsilon\}} \phi(x,y)\, dx\, dy$.

Theorem 1.3a. Suppose now that $S(x,y)$ is a smooth phase function in superadapted coordinates with $d > 1$. If ϕ is supported in a sufficiently small neighborhood of the origin, then there is a positive $B_{S,\phi}$ such that:

In cases 1 and 3 one has

$$I_{S,\phi}(\epsilon) < B_{S,\phi} \epsilon^{\frac{1}{d}} \tag{1.11}$$

In case 2 one has

$$I_{S,\phi}(\epsilon) < B_{S,\phi} |\ln(\epsilon)| \epsilon^{\frac{1}{d}} \tag{1.12}$$

One has some analogous lower bounds for $I_{|S|,\phi}(\epsilon)$. They are sharp in cases 1 and 2, and almost sharp in case 3. (Sharp lower bounds do not hold in general in case 3, as explicit examples show).

Theorem 1.3b. Suppose we are in the setting of Theorem 1.3a). Suppose also that $\phi(0,0) \neq 0$.

In case 1 there exists a $A_{S,\phi} > 0$ such that for ϵ sufficiently small we have

$$I_{|S|,\phi}(\epsilon) > A_{S,\phi} \epsilon^{\frac{1}{d}} \tag{1.13a}$$

In case 2, one similarly has

$$I_{|S|,\phi}(\epsilon) > A_{S,\phi}|\ln(\epsilon)|\epsilon^{\frac{1}{d}} \tag{1.13b}$$

In case 3, one has analogous almost-sharp lower bounds, at least if $\phi(x,y)$ is nonnegative. Namely, for any $\delta > 0$ one has

$$I_{|S|,\phi}(\epsilon) > A_{S,\phi,\delta}\epsilon^{\frac{1}{d}+\delta} \tag{1.13c}$$

The next lemma will be used to show that the three cases of superadapted coordinates are mutually exclusive.

Lemma 1.4. Assume $S(x,y)$ is smooth and is in case 3 of superadapted coordinates with $d > 1$. Then for any M one can find a smooth function $S_M(x,y)$ such that $S_M - S$ has a zero of order at least M at the origin, but such that in a small enough neighborhood U of the origin one has

$$\int_U |S_M(x,y)|^{-\frac{1}{d}}\,dx\,dy < \infty \tag{1.14}$$

Proof. Suppose for example that the bisectrix intersects $N(S)$ in the interior of the horizontal ray. Then $S_M(x,y) = S(x,y) + x^M$ agrees with $S(x,y)$ to order M at the origin and has Newton distance less than that of $S(x,y)$. The Newton distance is also greater than 1 for M large and the relevant polynomials $(S_M)_e(1,y)$ and $(S_M)_e(-1,y)$ have zeroes of order at most 1. Hence $S_M(x,y)$ is in case 1 superadapted coordinates and one can apply Theorem 1.1a to conclude that (1.14) is finite for a small enough neighborhood U of the origin.

Lemma 1.5. Any smooth degenerate phase $S(x,y)$ can be put in superadapted coordinates in exactly one of cases 1, 2, or 3.

Proof: In section 7 we will show that one can always put $S(x,y)$ into some superadapted coodinate system. Equations (1.11) (for $S(x,y)$ and $-S(x,y)$) and (1.13b) cannot simultaneously hold, so a case 2 coordinate system cannot be put in a case 1 or 3 coordinate system. Suppose $S(x,y)$ has a case 3 coordinate system as well as a case 1 coordinate system; we will derive a contradiction. Since it has a case 3 coordinate system, we may adjust $S(x,y)$ to arbitrarily high order and cause (1.14) to hold. In its case 1 coordinates, an adjustment of high enough order will not affect the fact that it is in case 1 and thus equation (1.11) will still hold. Using the well-known relationship between L^p norms and distribution functions (on the function $(I_{|S|,\phi}(\epsilon))^{-1}$), one then gets that the integral (1.14) in the new coordinates is infinite, a contradiction. Thus the three cases are mutually exclusive.

For oscillatory integrals with smooth phase, one has some analogues of Theorem 1.3. It should be noted that Theorem 1.6a can be proved using the results of [IKM].

Theorem 1.6a. Suppose $S(x,y)$ is smooth and is in superadapted coordinates with $d > 1$ In cases 1 and 3 as $\lambda \rightarrow \infty$ one has

$$|J_{S,\phi}(\lambda)| < C\lambda^{-\frac{1}{d}} \tag{1.15a}$$

In case 2 one has

$$|J_{S,\phi}(\lambda)| < C\lambda^{-\frac{1}{d}}\ln(\lambda) \tag{1.15b}$$

Theorem 1.6b. Suppose $\phi(x,y)$ is nonnegative with $\phi(0,0) > 0$.

In case 1, one has

$$\limsup_{\lambda\rightarrow\infty}\left|\frac{J_{S,\phi}(\lambda)}{\lambda^{-\frac{1}{d}}}\right| > 0 \tag{1.16a}$$

In case 2, one has

$$\limsup_{\lambda\rightarrow\infty}\left|\frac{J_{S,\phi}(\lambda)}{\lambda^{-\frac{1}{d}}\ln(\lambda)}\right| > 0 \tag{1.16b}$$

In case 3, for any $\delta > 0$ one has

$$\limsup_{\lambda\rightarrow\infty}\left|\frac{J_{S,\phi}(\lambda)}{\lambda^{-\frac{1}{d}-\delta}}\right| = \infty \tag{1.16c}$$

Although we will not prove it here, it can be shown with some additional argument that the conditions on $\phi(x,y)$ in (1.16a) – (1.16b) can be weakened to just that $\phi(0,0) \neq 0$.

In all of cases 1, 2, and 3 we will divide the domain of integration of the expressions (1.1) and (1.3) for $J_{S,\phi}$ and $I_{S,\phi}$ into 4 parts, depending on whether or not x and y are positive or negative. Adding the resulting formulas and estimates will give the theorems. Without loss of generality we will always focus on the $x, y > 0$ as the other quadrants are always dealt with the same way. Hence our goal is to understand $I_{S,\phi}^+$ and $J_{S,\phi}^+$, where

$$I_{S,\phi}^+(\epsilon) = \int_{\{(x,y):x>0,y>0,\ 0<S(x,y)<\epsilon\}} \phi(x,y)\,dx\,dy \tag{1.17a}$$

$$J_{S,\phi}^+(\lambda) = \int_{\{(x,y):x>0,y>0\}} e^{i\lambda S(x,y)}\phi(x,y)\,dx\,dy \tag{1.17b}$$

In turn, the domains of (1.17a) – (1.17b) will be written as the union of various "curved triangles" (such as those of Lemma 2.0 below). On a given curved triangle, one typically Taylor expands $S(x,y)$ or one of its derivatives and then uses Van der Corput-type lemmas in the x or y direction to get a desired estimate. For the oscillatory integrals, the traditional van der Corput (see Ch 8 of [S]) is used, while for sublevel integrals the version of [C] is used. Van der Corput-type lemmas have been considered in some detail, such as in [ArKaCu] and [CaCW], as well as the early work of Vinogradov [Vi]. We refer to [CaCW] for further results and references.

Throughout this paper, we will often have a constant C appearing on the right-hand side of an inequality. This always denotes a constant depending on S and ϕ. Occasionally we will need further constants C', C'', etc which also depend on S and ϕ.

2. Some useful lemmas for Cases 1 and 2.

Suppose G is an open subset of $\mathbf{R}^2$. Then throughout the course of this paper we will make frequent use of $I_{S,\phi}^G(\epsilon)$ and $J_{S,\phi}^G(\lambda)$ defined by

$$I_{S,\phi}^G(\epsilon) = \int_{\{(x,y)\in G: 0<S(x,y)<\epsilon\}} \phi(x,y)\,dx\,dy \tag{2.1a}$$

$$J_{S,\phi}^G(\lambda) = \int_G e^{i\lambda S(x,y)}\phi(x,y)\,dx\,dy \tag{2.1b}$$

A certain type of G comes up in several contexts in this paper, and relevant estimates we need for $I_{S,\phi}^G(\epsilon)$ and $J_{S,\phi}^G(\lambda)$ are given by the following lemma.

Lemma 2.0. Suppose for some $A, m > 0$, $0 < \delta < 1$, we let $G = \{(x,y) \in [0,\delta]\times[0,\delta] : 0 < y < Ax^m\}$, and suppose there are nonnegative integers a and b with $a > b$ and $a \geq 2$ such that for some constant C_0 the following holds on G.

$$\partial_y^b S(x,y) > C_0 x^a \tag{2.2a}$$

If $b = 1$, assume also that

$$\partial_y^2 S(x,y) < C_0 x^{a-m} \tag{2.2b}$$

If $b = 0$, instead of (2.2b) assume also that $a > m+1$ and that for some constant C_1 we have

$$\partial_x S(x,y) > C_1 x^{a-1}, \quad \partial_x^2 S(x,y) < C_0 x^{a-2} \tag{2.2c}$$

Then for some $\zeta_{ab} > 0$ and C depending on S, ϕ, and C_0 (and C_1 if $b=0$), if the support of ϕ is contained in $[-\delta,\delta]\times[-\delta,\delta]$ one has

$$|I_{S,\phi}^G(\epsilon)| < C\epsilon^{\frac{m+1}{a+mb}} A^{\zeta_{ab}} \tag{2.3a}$$

$$|J_{S,\phi}^G(\lambda)| < C|\lambda|^{-\frac{m+1}{a+mb}} A^{\zeta_{ab}} \tag{2.3b}$$

Proof. We first consider the case where $b > 0$; we will do the $b = 0$ argument afterwards. By (2.2a) and the Van der Corput lemma in the y direction (see [C] for example) one has that for a given x we have

$$\Big|\{y : |S(x,y)| < \epsilon\}\Big| < C\epsilon^{\frac{1}{b}} x^{-\frac{a}{b}} \tag{2.4}$$

As a result, if G^x denotes the vertical cross section of G at x, of length Ax^m, then one has

$$\Big|\{y \in G^x : |S(x,y)| < \epsilon\}\Big| < C\min(Ax^m, \epsilon^{\frac{1}{b}} x^{-\frac{a}{b}}) \tag{2.5}$$

Consequently, we have

$$|I^G_{S,\phi}| < C\int_0^\delta \min(Ax^m, \epsilon^{\frac{1}{b}}x^{-\frac{a}{b}})\,dx \tag{2.6}$$

It is natural to break the integral (2.6) into two parts, depending on whether or not $Ax^m > \epsilon^{\frac{1}{b}}x^{-\frac{a}{b}}$. The two quantities are equal at $x_0 = A^{-\frac{b}{a+mb}}\epsilon^{\frac{1}{a+mb}}$. The left integral becomes

$$\int_0^{x_0} Ax^m\,dx = \frac{A}{m+1}x_0^{m+1} = \frac{1}{m+1}A^{\frac{a-b}{a+mb}}\epsilon^{\frac{m+1}{a+mb}} \tag{2.7}$$

The right integral is computed to be

$$\epsilon^{\frac{1}{b}}\int_{x_0}^\delta x^{-\frac{a}{b}}\,dx < \epsilon^{\frac{1}{b}}\int_{x_0}^\infty x^{-\frac{a}{b}}\,dx$$

$$= \frac{b}{a-b}\epsilon^{\frac{1}{b}}x_0^{\frac{a-b}{b}} = \frac{b}{a-b}A^{\frac{a-b}{a+mb}}\epsilon^{\frac{m+1}{a+mb}} \tag{2.8}$$

Adding together, we obtain that $|I^G_{S,\phi}| < CA^{\frac{a-b}{a+mb}}\epsilon^{\frac{m+1}{a+mb}}$ as needed.

The estimates for $J^G_{S,\phi}(\lambda)$ for $b \geq 2$ are done in a similar fashion. First suppose $b \geq 2$. Then one can use the usual Van der Corput lemma (see [S] ch 8) in the y direction to obtain

$$\left|\int_{G^x} e^{i\lambda S(x,y)}\phi(x,y)\,dy\right| < C|\lambda|^{-\frac{1}{b}}x^{-\frac{a}{b}} \tag{2.9}$$

This is the analogue to (2.4) with ϵ replaced by $|\lambda|^{-1}$. As a result, similar to (2.6) we get

$$|J^G_{S,\phi}| < C\int_0^\delta \min(Ax^m, |\lambda|^{-\frac{1}{b}}x^{-\frac{a}{b}})\,dx \tag{2.10}$$

The result is

$$|J^G_{S,\phi}| \leq CA^{\frac{a-b}{a+mb}}|\lambda|^{-\frac{m+1}{a+mb}} \tag{2.11}$$

This gives (2.3b). We next prove (2.3b) when $b = 1$. If one integrates by parts in the y variable one gets several terms each of which can be bounded using (2.2a) and (2.2b). If one works it out, one gets that these terms are bounded by $C|\lambda|^{-1}x^{-(a+m)}$. It is thus natural to split the integral into two parts at the x_0 satisfying $|\lambda|^{-1}{x_0}^{a+m} = 1$, in other words, at $x_0 = |\lambda|^{-\frac{1}{a+m}}$. We accordingly write $G = G_1 \cup G_2$, with G_1 the portion where $x_0 < |\lambda|^{-\frac{1}{a+m}}$. We then have

$$J^{G_1}_{S,\phi}(\lambda) < C|G_1| < C'A|\lambda|^{-\frac{m+1}{m+a}} \tag{2.12}$$

As for $J^{G_2}_{S,\phi}(\lambda)$, we integrate by parts in y, obtaining the $C|\lambda|^{-1}x^{-(a+m)}$ factor, and one gets that

$$|J^{G_2}_{S,\phi}(\lambda)| < C\int_{G_2} |\lambda|^{-1}x^{-(m+a)}$$

$$\leq C\int_{|\lambda|^{-\frac{1}{a}}}^{1}\int_{0}^{Ax^m}|\lambda|^{-1}x^{-(m+a)}\,dy\,dx$$

$$= CA\int_{|\lambda|^{-\frac{1}{a+m}}}^{1}|\lambda|^{-1}x^{-a}\,dx$$

$$< CA|\lambda|^{-\frac{m+1}{m+a}} \tag{2.13}$$

Adding (2.12) to (2.13) give the oscillatory integral estimates for $b = 1$.

We now consider the oscillatory integral when $b = 0$. Here we do the integrations by parts in the x direction. This time, by (2.2a) and (2.2c) an integration by parts incurs a factor of $|\lambda|^{-1}x^{-a}$. Hence we subdivide $G = G_1 \cup G_2$, where $G_1 = \{(x, y) \in G : 0 < x < |\lambda|^{-\frac{1}{a}}\}$. Note that the measure of G_1 is $A|\lambda|^{-\frac{m+1}{a}}$, so that

$$|J_{S,\phi}^{G_1}(\lambda)| < CA|\lambda|^{-\frac{m+1}{a}} \tag{2.14}$$

For the G_2 piece one obtains

$$|J_{S,\phi}^{G_2}(\lambda)| < C\int_G |\lambda|^{-1}x^{-a} = \int_{|\lambda|^{-\frac{1}{a}}}^{1}\int_{0}^{Ax^m}|\lambda|^{-1}x^{-a}\,dy\,dx$$

$$= C|\lambda|^{-1}\int_{|\lambda|^{-\frac{1}{a}}}^{1}Ax^{m-a}\,dx$$

$$= CA|\lambda|^{-\frac{m+1}{a}} \tag{2.15}$$

(For the last equality we use the hypothesis that $a > m+1$). Adding (2.14) to (2.15) gives the estimate we seek, (2.3b). Lastly, we prove the bounds for $I_{S,\phi}^{G}(\epsilon)$ when $b = 0$. In this case, since $|S(x, y)| > C_0x^a$, we have

$$|I_{S,\phi}^{G}| < C|\{(x, y) \in G : C_0x^a < \epsilon\}| < C'\{(x, y) \in G : x < \epsilon^{\frac{1}{a}}\} = C''A\epsilon^{\frac{m+1}{a}} \tag{2.16}$$

This concludes the proof of Lemma 2.0.

In Lemmas 2.1 and 2.2 below, $S(x, y)$ is a smooth phase function in Case 1 or 2 of superadapted coordinates with $d > 1$. If there is a compact edge E of $N(S)$ such that the bisectrix contains either the upper vertex of E or an interior point of E, then we denote the equation of this edge by $t_1 + mt_2 = \alpha$, and for some large but fixed number N we let $A_1 = \{(x, y) \in [0, 1] \times [0, 1] : y < \frac{1}{N}x^m\}$. Similarly, if there is some edge E' with equation $t_1 + m't_2 = \alpha'$ such that the bisectrix contains either the lower vertex of E' or an interior point of E', we let $A_2 = \{(x, y) \in [0, 1] \times [0, 1] : x < \frac{1}{N}y^{\frac{1}{m'}}\}$. Note that in case 1 both A_1 and A_2 exist. We focus our attention on $I_{S,\phi}^{A_i}(\epsilon)$ and $J_{S,\phi}^{A_i}(\lambda)$. The relevant information about them (if they exist) is provided by the following lemma.

Lemma 2.1. There exists an $\eta > 0$ such that if the support of $\phi(x,y)$ is sufficiently small, then for $i = 1, 2$ we have

$$|I^{A_i}_{S,\phi}(\epsilon)| < C\epsilon^{\frac{1}{d}} N^{-\eta} \tag{2.17}$$

$$|J^{A_i}_{S,\phi}(\lambda)| < C|\lambda|^{-\frac{1}{d}} N^{-\eta} \tag{2.18}$$

Proof. By symmetry, it suffices to prove the bounds for A_1. Let (a,b) denote the lowest vertex of E. Thus $b < a$. Since (a,b) and (d,d) are both on E, we have $a + mb = (1+m)d$ or $\frac{1}{d} = \frac{m+1}{a+mb}$. We will show that the hypotheses of Lemma 2.0 hold for these values of a, b, and m, setting $A = N^{-1}$. Since $\frac{1}{d} = \frac{m+1}{a+mb}$, Lemma 2.1 will follow. For a large but fixed M, we write $S(x,y) = \sum_{p<M,\ q<M} s_{pq}x^p y^q + E_M(x,y)$. By standard estimates, for $0 \leq \alpha, \beta \leq M$ we have

$$|\partial_x^\alpha \partial_y^\beta E_M(x,y)| < C(|x|^{M-\alpha} + |y|^{M-\beta}) \tag{2.19}$$

We can write

$$\partial_y^b S(x,y) = \sum_{p<M,\ q<M-b} s'_{pq}x^p y^q + \partial_y^b E_M(x,y) \tag{2.20}$$

Here $s'_{a0} \neq 0$. We next show that the sum in (2.20) is dominated by the term $s'_{a0}x^a$ in a sufficiently small neighborhood of the origin. To this end, note that $(a,0)$ is a vertex of the Newton polygon of $\partial_y^b S(x,y)$, and that a horizontal ray and an edge of this Newton polygon with equation $t_1 + mt_2 = a$ intersect at $(a,0)$. As a result, for any term $s'_{pq}x^p y^q$ in the sum of (2.20) other than $s'_{a0}x^a$, either $p \geq a$, or $p < a$, $q > 0$, and $p + mq \geq a$. Correspondingly, we let $T_1 = \{(p,q) : p < a,\ 0 < q < M-b,\ p+mq \geq a\}$ and $T_2 = \{(p,q) : a \leq p \leq M,\ 0 \leq q \leq b-a,\ (p,q) \neq (a,0)\}$ and we rewrite (2.20) as

$$\partial_y^b S(x,y) = s'_{a0}x^a + \sum_{T_1} s'_{pq}x^p y^q + \sum_{T_2} s'_{pq}x^p y^q + \partial_y^b E_M(x,y) \tag{2.21}$$

We examine a given term $s'_{pq}x^p y^q$ in the T_1 sum. Since (x,y) is in the domain A_1, one has $y < \frac{x^m}{N}$. As a result, $|s'_{pq}x^p y^q| < |s'_{pq}N^{-q}x^{p+mq}| \leq |s'_{pq}N^{-q}x^a|$. Thus so long as N is chosen sufficiently large (depending on M), we may assume that the absolute value of the T_1 sum is at most $\frac{1}{4}|s'_{a0}|x^a$. We next examine a term $s'_{pq}x^p y^q$ in the T_2 sum. Here we have $|s'_{pq}x^p y^q| < |s'_{pq}x^a|(|x| + |y|)$. Hence if we are in a sufficiently small neighborhood of the origin, we can assume the absolute value of the T_2 sum is also at most $\frac{1}{4}|s'_{a0}|x^a$. Lastly, by (2.19), in a sufficiently small neighborhood of the origin $|\partial_y^b E_M(x,y)|$ is also bounded by $\frac{1}{4}|s'_{a0}|x^a$. Putting these together, we conclude that on the domain A_1 we have

$$|\partial_y^b S(x,y)| > \frac{1}{4}|s'_{a0}|x^a \tag{2.22}$$

Note that when $b > 1$, (2.22) ensures hypotheses of Lemma 2.0 hold. Thus one may apply Lemma 2.0, giving Lemma 2.1. When $b = 0$, in order to apply Lemma 2.0 one needs also that $a > m+1$ and that (2.2c) holds. But taking an x derivative of $S(x,y)$ just shifts the Newton polygon to the left by 1, so exactly as in (2.22) the first and second derivative conditions of (2.2c) will hold. As for the requirement that $a > m+1$, note that (d,d) and $(a,0)$ are on the line $t_1 + mt_2 = \alpha$ and that $d > 1$. This means $a = a + m0 = d + md > 1 + m$ as needed.

Lastly, we show that the supplemental hypothesis (2.2b) holds when $b = 1$. It helps to view things in (x, y') coordinates where $(x,y) = (x, x^m y')$. The condition (2.2b) becomes that $|\partial_{y'}^2 S(x, x^m y')| \leq Cx^{a+m}$. Also, since the terms $s_{pq}x^p y^q$ of $S(x,y)$'s Taylor expansion with minimal $p + mq$ ($= \alpha$) are exactly the terms of $S_E(x,y)$, the expansion $S(x,y) = \sum_{p<M,\ q<M} s_{pq}x^p y^q + E_M(x,y)$ becomes of the following form, where T_M is a polynomial in y' and a fractional power of x.

$$S(x, x^m y') = x^\alpha S_E(1, y') + x^{\alpha+\epsilon} T_M(x, y') + E_M(x, x^m y') \tag{2.23}$$

Using (2.23) and the error estimates (2.19) we have $|\partial_{y'}^2 S(x, x^m y')| \leq Cx^\alpha$. But $(a,b) = (a,1)$ is on the line $t_1 + mt_2 = \alpha$ and therefore $a + m = \alpha$. This gives the second derivative bounds of (2.2b), and thus the hypotheses of Lemma 2.0 hold. This concludes the proof of Lemma 2.1.

Lemma 2.2. Let $S(x,y)$ be a smooth phase function in case 1 or 2 superadapted coordinates with $d > 1$. Suppose e is a compact edge of $N(S)$ intersecting the bisectrix and has equation given by $t_1 + mt_2 = \alpha$. Define $B = \{(x,y) \in [0,1] \times [0,1] : \frac{x^m}{N} < y < Nx^m\}$, where N is some large but fixed constant. Then if the support of $\phi(x,y)$ is sufficiently small, depending on N, one has the estimates $|I_{S,\phi}^B(\epsilon)| < C_N \epsilon^{\frac{1}{d}}$ and $|J_{S,\phi}^B(\lambda)| < C_N |\lambda|^{-\frac{1}{d}}$.

Proof. In the above (x, y') coordinates one has

$$I_{S,\phi}^B(\epsilon) = \int_{\{(x,y') \in [0,1] \times [N^{-1}, N] : 0 < S(x, x^m y') < \epsilon\}} x^m \phi(x, x^m y')\, dx\, dy'$$

$$J_{S,\phi}^B(\lambda) = \int_{[0,1] \times [N^{-1}, N]} e^{i\lambda S(x, x^m y')} x^m \phi(x, x^m y')\, dx\, dy'$$

In view of (2.23), the zeroes of $S_e(1, y')$ might be expected to play a significant role in the analysis. To this end, we assume that N is large enough so that any zeroes of $S_e(1, y')$ for $y' > 0$ are in $[N^{-1}, N]$, and denote these zeroes by $z_1, ..., z_k$ (if there are any). Let $v_1, ...v_k$ denote the orders of these zeroes, and let I_i denote the interval $[z_i - \frac{1}{N}, z_i + \frac{1}{N}]$. By (2.23) and the error term derivative estimates (2.19) we can assume that on the (sufficiently small, depending on N) support of $\phi(x, x^m y')$, if $(x, y') \in [0,1] \times I_i$ then

$$|\partial_{y'}^{v_i} S(x, x^m y')| \geq Cx^\alpha \tag{2.24a}$$

In the case that $v_i = 1$, we similarly have

$$|\partial_{y'}^2 S(x, x^m y')| \leq Cx^\alpha \tag{2.24b}$$

We now translate this into the orginal (x, y) coordinates. Each $[0,1] \times I_i$ becomes a set D_i of the form $\{(x, y) \in [0,1] \times [0,1] : (z_i - \frac{1}{N})x^m < y < (z_i + \frac{1}{N})x^m$, and on D_i (2.24a) becomes

$$|\partial_y^{v_i} S(x,y)| \geq C x^{\alpha - m v_i} \tag{2.25a}$$

In the case of $v_i = 1$, (2.24b) becomes

$$|\partial_y^2 S(x,y)| \leq C x^{\alpha - 2m} \tag{2.25b}$$

Since we are superadapted coordinates, $0 < v_i < d$. Thus $(m+1)v_i < (m+1)d$. Since (d,d) is on the edge e, we have $\alpha = (m+1)d$. Thus $(m+1)v_i < \alpha$ or $\alpha - mv_i > v_i$. As a result, the sets D_i have vertical cross sections of length $\frac{2}{N}x^m$ and satisfying (2.25a) – (2.25b) with $\alpha - mv_i > v_i$. Hence after doing a coordinate change of the form $(x, y) \to (x, y - f(x))$, we are in the set-up of Lemma 2.0 and for some $\eta > 0$ we get

$$|I_{S,\phi}^{D_i}| \leq C \epsilon^{\frac{m+1}{\alpha}} N^{-\eta} \tag{2.26a}$$

$$|J_{S,\phi}^{D_i}| \leq C |\lambda|^{-\frac{m+1}{\alpha}} N^{-\eta} \tag{2.26b}$$

Since $\frac{m+1}{\alpha} = \frac{1}{d}$ the above becomes

$$|I_{S,\phi}^{D_i}| \leq C \epsilon^{\frac{1}{d}} N^{-\eta} \tag{2.27a}$$

$$|J_{S,\phi}^{D_i}| \leq C |\lambda|^{-\frac{1}{d}} N^{-\eta} \tag{2.27b}$$

Next, write $[N^{-1}, N] - \cup_i I_i$ as the union of intervals J_i. Then since $S_e(1, y')$ has no zeroes on any J_i, by the expansion (2.23) and the error derivative bounds (2.19), if δ_N is sufficiently small then on $[0, \delta_N] \times J_i$ we have

$$|\partial_x S(x, x^m y')| > C_N x^{\alpha - 1} \qquad |\partial_x^2 S(x, x^m y')| < C_N' x^{\alpha - 2} \tag{2.28}$$

Separating at $x = |\lambda|^{-\frac{1}{\alpha}}$ and integrating the right portion by parts in x using (2.28) as in the proof of Lemma 2.0 gives

$$\left| \int_{[0,\delta_N] \times J_i} e^{i\lambda S(x, x^m y')} x^m \phi(x, x^m y') \right| dx\, dy' < C_N'' |\lambda|^{-\frac{m+1}{\alpha}} \tag{2.29a}$$

Converting back into (x, y) coordinates and using that $\frac{1}{d} = \frac{m+1}{\alpha}$, (2.29$a$) becomes the following, where E_i denotes the set $[0, \delta_N] \times J_i$ in the (x, y) coordinates.

$$\left| \int_{E_i} e^{i\lambda S(x,y)} \phi(x,y)\, dx\, dy \right| \leq C_N'' |\lambda|^{-\frac{1}{d}} \tag{2.29b}$$

Moving now to $I_{S,\phi}(\epsilon)$, by (2.23) one has $|S(x, x^m y')| > C_N x^\alpha$ on $[0, \delta_N] \times J_i$ for sufficiently small $\delta_N > 0$ and some C_N (not necessarily the same constant as above). As a result,

$$\left| \int_{\{(x,y') \in [0,\delta_N] \times J_i : 0 < S(x, x^m y') < \epsilon\}} x^m \phi(x, x^m y')\, dx\, dy' \right|$$

$$\leq \Big| \int_{\{(x,y')\in[0,\delta_N]\times J_i: 0<C_N x^\alpha<\epsilon\}} x^m \phi(x, x^m y')\,dx\,dy'\Big|$$

$$< C'_N \int_0^{\epsilon^{\frac{1}{\alpha}}} x^m\,dx$$

$$= C'_N \epsilon^{\frac{m+1}{\alpha}} \tag{2.30a}$$

Again going back to the (x,y) coordinates and using that $\frac{1}{d} = \frac{m+1}{\alpha}$, we conclude that

$$\Big| \int_{\{(x,y)\in E_i: 0<S(x,y)<\epsilon\}} \phi(x,y)\,dx\,dy\Big| \leq C_N \epsilon^{\frac{1}{d}} \tag{2.30b}$$

Lemma 2.2 now follows by adding $(2.29b) - (2.30b)$ to $(2.27a) - (2.27b)$.

3. Case 1 proofs.

Assume now that we are in Case 1. We start by proving the upper bounds for smooth phase.

Theorem 3.1. The right-hand sides of (1.11) and $(1.15a)$ hold.

Proof. In case 1 of superadapted coordinates, the domain of integration of $I^+_{S,\phi}(\epsilon)$ or $J^+_{S,\phi}(\lambda)$ is the union of A_1, A_2, and B, where A_1 and A_2 are as in Lemma 2.1 and where B is as in Lemma 2.2 for the edge of $N(S)$ intersecting the bisectrix. Thus the theorem follows by fixing some N and adding the inequalities of Lemmas 2.1 and 2.2 to the corresponding inequalities for the other quadrants.

The next lemma will be useful in getting the formulas for the real-analytic case.

Lemma 3.2. Suppose $S(x,y)$ is a smooth case 1 phase function like before. Then there is a natural number $D < d$ and a neighborhood U of the origin such that if $\phi(x,y)$ is supported in U and is zero on a neighborhood of the origin, then $|I_{S,\phi}(\epsilon)| < C\epsilon^{\frac{1}{D}}$.

Proof: It suffices to fix some N and show that each $I^{A_i}_{S,\phi}(\epsilon)$, $I^{D_i}_{S,\phi}(\epsilon)$, $I^{E_i}_{S,\phi}(\epsilon)$ satisfies the upper bounds, where the D_i and E_i correspond to the edge of $N(S)$ intersecting the bisectrix. We start with $I^{A_i}_{S,\phi}$. Without loss of generality we may take $i = 1$. Since $\phi(x,y)$ is zero on a neighborhood of the origin, there is some $\delta > 0$ such that $\phi(x,y)$ is zero for $(x,y) \in A_1$ with $0 < x < \delta$. (2.22) says that $\partial_y^e S(x,y)$ is bounded below on A_1, where $e < d$ denotes the y-coordinate of the lower vertex of the edge of $N(S)$ intersecting the bisectrix. If $e > 0$, by the Van der Corput lemma in the y direction, we have

$$|\{y \in A_1^x : 0 < S(x,y) < \epsilon\}| < C\epsilon^{\frac{1}{e}} \tag{3.1}$$

Thus we have

$$|I^{A_1}_{S,\phi}(\epsilon)| \leq C\int_\delta^1 |\{y \in A_1^x : 0 < S(x,y) < \epsilon\}|\,dx < C\epsilon^{\frac{1}{e}} \tag{3.2}$$

This is the desired estimate for $e > 0$. If $e = 0$, then (2.22) says that $S(x, y)$ is bounded below on the support of the integrand of $I^{A_1}_{S,\phi}(\epsilon)$ and then (3.2) holds trivially. Thus we have the desired bounds for the $I^{A_1}_{S,\phi}(\epsilon)$. The $I^{D_i}_{S,\phi}(\epsilon)$ are dealt with in a similar way. This time, one uses (2.24a) to obtain $|I^{D_i}_{S,\phi}(\epsilon)| < C\epsilon^{\frac{1}{v_i}}$.

Lastly, we look at the $|I^{E_i}_{S,\phi}(\epsilon)|$. As mentioned below (2.29b), by (2.23) one has $|S(x, y)| > Cx^\alpha$ on each E_i. Hence since $\phi(x, y)$ is zero on a neighborhood of the origin, $|S(x, y)|$ is bounded below on the support of the integrand of $I^{E_i}_{S,\phi}(\epsilon)$ and (3.2) again holds trivially. This completes the proof of Lemma 3.2.

We now move to the case 1 formulas of Theorems 1.1. We write $\phi = \phi_1 + \phi_2$, where ϕ_1 is supported in a smaller neighborhood of the origin and ϕ_2 is zero on a neighborhood of the origin. By Lemma 2.3, $\lim_{\epsilon\to 0} \frac{I_{S,\phi_2}(\epsilon)}{\epsilon^{\frac{1}{d}}} = 0$ and therefore $\lim_{\epsilon\to 0} \frac{I_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}} = \lim_{\epsilon\to 0} \frac{I_{S,\phi_1}(\epsilon)}{\epsilon^{\frac{1}{d}}}$. Hence when proving the limit (1.6) one can always replace ϕ by ϕ_1 at will, regardless of how small the support of ϕ_1 is.

Our strategy will involve fixing N and analyzing the $I^{E_i}_{S,\phi_1}$, where ϕ_1 has small support depending on N. Lemma 2.1 and (2.27a) will ensure that the contributions of the $I^{D_i}_{S,\phi_1}$ and $I^{A_i}_{S,\phi_1}$ will be $O(N^{-\eta})$ smaller than that of the $I^{E_i}_{S,\phi_1}$ as ϵ goes to zero. Letting N go to infinity will give $\lim_{\epsilon\to 0} \frac{I^{+}_{S,\phi_1}(\epsilon)}{\epsilon^{\frac{1}{d}}} = \sum_i \lim_{\epsilon\to 0} \frac{I^{E_i}_{S,\phi_1}(\epsilon)}{\epsilon^{\frac{1}{d}}}$. Adding up the latter limits along with their analogues in the other three quadrants will give (1.6).

Proof of (1.6)**.** It suffices to assume $\phi(0, 0) > 0$ as the general case can be obtained by writing $\phi = \phi' - \phi''$ where $\phi'(0, 0)$ and $\phi''(0, 0)$ are positive. We work in the (x, y') coordinates like above. Namely, we write

$$I^{E_i}_{S,\phi}(\epsilon) = \int_{\{(x,y')\in[0,1]\times J_i: 0<S(x,x^m y')<\epsilon\}} x^m \phi(x, x^m y')\, dx\, dy' \tag{3.3}$$

J_i was defined so that $S_0(1, y')$ has no zeroes on J_i. As a result, by (2.23) (and using (2.19) to deal with $E_M(x, x^m y')$), we may let $\delta > 0$ such that on $[0, \delta] \times J_i$ either $S(x, x^m y')$ is negative, or we have

$$0 < (1 - \frac{1}{N})S(x, x^m y') \leq x^\alpha S_0(1, y') \leq (1 + \frac{1}{N})S(x, x^m y') \tag{3.4}$$

Shrinking δ further if necessary, we assume δ is small enough so that on $[0, \delta] \times J_i$ we have

$$\phi(x, x^m y') < (1 + \frac{1}{N})\phi(0, 0) \tag{3.5}$$

As described above, one can multiply $\phi(x, y)$ by a cutoff function supported on $|x| < \delta$ without affecting $\lim_{\epsilon\to 0} \frac{I^{E_i}_{S,\phi_1}(\epsilon)}{\epsilon^{\frac{1}{d}}}$. Hence we assume $\phi(x, y)$ is supported on $|x| < \delta$, and that the multiplying cutoff was chosen so that for $(x, y') \in [0, \delta] \times J_i$, equation (3.5) holds. We also assume the multiplying cutoff was chosen so that for $(x, y') \in [0, \frac{\delta}{2}] \times J_i$ one has

$$(1 - \frac{1}{N})\phi(0,0) < \phi(x, x^m y') \tag{3.6}$$

We now proceed to our main estimates. If $S(x, x^m y')$ is negative on $[0, \delta] \times J_i$, $I_{S,\phi}^{E_i}(\epsilon)$ becomes zero. If on the other hand (3.4) holds, then by (3.4) and (3.5) one has

$$I_{S,\phi}^{E_i}(\epsilon) \leq (1 + \frac{1}{N})\phi(0,0) \int_{\{(x,y') \in [0,\delta] \times J_i : 0 < x^\alpha S_0(1,y') < (1+\frac{1}{N})\epsilon\}} x^m \, dx \, dy' \tag{3.7a}$$

On the other hand, by (3.4) and (3.6) we also have

$$I_{S,\phi}^{E_i}(\epsilon) \geq (1 - \frac{1}{N})\phi(0,0) \int_{\{(x,y') \in [0,\frac{1}{2}\delta] \times J_i : 0 < x^\alpha S_0(1,y') < (1-\frac{1}{N})\epsilon\}} x^m \, dx \, dy' \tag{3.7b}$$

We now change coordinates from x to $x' = x^{m+1}$ in the integrals $(3.7a) - (3.7b)$. Using the fact that $\frac{\alpha}{m+1} = d$, which follows from the fact that (d, d) is on the line $t_1 + mt_2 = \alpha$, $(3.7a) - (3.7b)$ become the following, where $\delta' = \delta^{m+1}$.

$$I_{S,\phi}^{E_i}(\epsilon) \leq (m+1)^{-1}(1 + \frac{1}{N})\phi(0,0) \int_{\{(x,y') \in [0,\delta'] \times J_i : 0 < x^d S_0(1,y') < (1+\frac{1}{N})\epsilon\}} dx \, dy' \tag{3.8a}$$

$$I_{S,\phi}^{E_i}(\epsilon) \geq (m+1)^{-1}(1 - \frac{1}{N})\phi(0,0) \int_{\{(x,y') \in [0,\frac{1}{2}\delta'] \times J_i : 0 < x^d S_0(1,y') < (1-\frac{1}{N})\epsilon\}} dx \, dy' \tag{3.8b}$$

Doing the x integrals first, $(3.8a) - (3.8b)$ become

$$I_{S,\phi}^{E_i}(\epsilon) \leq (m+1)^{-1}(1 + \frac{1}{N})\phi(0,0) \int_{J_i} \min(\delta', [(1 + \frac{1}{N})\epsilon]^{\frac{1}{d}} S_0(1,y')^{-\frac{1}{d}}) \, dy' \tag{3.9a}$$

$$I_{S,\phi}^{E_i}(\epsilon) \geq (m+1)^{-1}(1 - \frac{1}{N})\phi(0,0) \int_{J_i} \min(\frac{\delta'}{2}, [(1 - \frac{1}{N})\epsilon]^{\frac{1}{d}} S_0(1,y')^{-\frac{1}{d}}) \, dy' \tag{3.9b}$$

These can be written as

$$\frac{I_{S,\phi}^{E_i}(\epsilon)}{\epsilon^{\frac{1}{d}}} \leq (m+1)^{-1}(1 + \frac{1}{N})^{\frac{d+1}{d}} \phi(0,0) \int_{J_i} \min(\frac{\delta'}{[(1 + \frac{1}{N})\epsilon]^{\frac{1}{d}}}, S_0(1,y')^{-\frac{1}{d}}) \, dy' \tag{3.10a}$$

$$\frac{I_{S,\phi}^{E_i}(\epsilon)}{\epsilon^{\frac{1}{d}}} \geq (m+1)^{-1}(1 - \frac{1}{N})^{\frac{d+1}{d}} \phi(0,0) \int_{J_i} \min(\frac{\delta'}{2[(1 - \frac{1}{N})\epsilon]^{\frac{1}{d}}}, S_0(1,y')^{-\frac{1}{d}}) \, dy' \tag{3.10b}$$

We now take limits of $(3.10a) - (3.10b)$ as $\epsilon \to 0$. We obtain

$$\liminf_{\epsilon \to 0} \frac{I_{S,\phi}^{E_i}(\epsilon)}{\epsilon^{\frac{1}{d}}} \geq (m+1)^{-1}(1 - \frac{1}{N})^{\frac{d+1}{d}} \phi(0,0) \int_{J_i} S_0(1,y')^{-\frac{1}{d}} \, dy' \tag{3.11a}$$

$$\limsup_{\epsilon\to 0}\frac{I_{S,\phi}^{E_i}(\epsilon)}{\epsilon^{\frac{1}{d}}}\leq (m+1)^{-1}(1+\frac{1}{N})^{\frac{d+1}{d}}\phi(0,0)\int_{J_i} S_0(1,y')^{-\frac{1}{d}}\,dy' \tag{3.11b}$$

Note that the integrals here are automatically finite since we are in superadapted coordinates and therefore all zeroes of $S_0(1,y')$ are of order at most $d-1$. We now take limits as $N\to\infty$. If the endpoints of J_i are two zeroes z and z' of $S_0(1,y')$, then the interval will converge to $[z,z']$. Otherwise, the left endpoint of J_i may converge to zero and the right endpoint may go off to ∞. In any event, the J_i goes to some (possibly unbounded) interval K_i, and $(3.11a)-(3.11b)$ both converge to

$$(m+1)^{-1}\phi(0,0)\int_{K_i} S_0(1,y')^{-\frac{1}{d}}\,dy' \tag{3.12}$$

The integral in (3.12) is finite since $S_0(1,y')$ must have degree greater than d and has no zeroes of order d or greater since the coordinate system being used is superadapted.

Furthermore, as $N\to\infty$, the upper bounds of Lemma 2.1 and $(2.27a)$ for $I_{S,\phi}^{D_i}(\epsilon)$ and $I_{S,\phi}^{A_i}(\epsilon)$ go to zero. There is no issue of the constants appearing depending on N due to the cutoffs' dependence on N; the bounds of Lemma 2.1 and $(2.27a)$ depend on $||\phi||_\infty$ and no other properties of ϕ. Hence we conclude that $\lim_{\epsilon\to 0}\frac{I_{S,\phi_1}^{+}(\epsilon)}{\epsilon^{\frac{1}{d}}}$ is given by the sum of (3.12) over all i, along with their analogues in the other three quadrants. This gives exactly (1.6) (recall the terms where $S(x,x^m y')$ is negative gives no contribution) and we are done.

Lastly, we prove $(1.13a)$ of Theorem 1.3a. Either $S_0(1,y)$ is positive for some $y>0$ or $-S_0(1,y)$ is positive for some $y>0$. Since the result we are trying to prove is symmetric in S and $-S$, without loss of generality we may assume that $S_0(1,y)$ has this property. Then (1.6) says that $I_{S,\phi}(\epsilon)>A_{S,\phi}\epsilon^{\frac{1}{d}}$ for sufficiently small ϵ. This implies $(1.13a)$ since $I_{|S|,\phi}(\epsilon)\geq I_{S,\phi}(\epsilon)$ and we are done.

4. Case 2 proofs.

We now assume $S(x,y)$ is a smooth phase function in case 2 of superadapted coordinates with $d>1$. Hence the bisectrix intersects $N(S)$ at (d,d). As in the case 1 proofs, we consider $I_{S,\phi}^{+}(\epsilon)$ and $J_{S,\phi}^{+}(\lambda)$ and we will do some subdivisions of the domains of these integrals to prove the estimates and formulas. We proceed as follows. If (d,d) is the lower vertex of a compact edge e_2, denote its equation by $t_1+m_2t_2=\alpha_2$. Where N is a large natural number, fixed for now, we let $A_2=\{(x,y)\in[0,1]\times[0,1]: x<\frac{1}{N}y^{\frac{1}{m_2}}\}$ and $B_2=\{(x,y)\in[0,1]\times[0,1]:\frac{1}{N}y^{\frac{1}{m_2}}<x<Ny^{\frac{1}{m_2}}\}$. Define $C_2=A_2\cup B_2$. If (d,d) is not the lower vertex of a compact edge (i.e. (d,d) is on the vertical ray), then define $C_2=\{(x,y)\in[0,1]\times[0,1]: x<y^L\}$. Here L is large number to be determined by our future arguments. Similarly, if there is a compact edge e_1 whose upper vertex is (d,d), we write its equation as $t_1+m_1t_2=\alpha_1$. We then let $A_1=\{(x,y)\in[0,1]\times[0,1]: y<\frac{1}{N}x^{m_1}\}$ and $B_1=\{(x,y)\in[0,1]\times[0,1]:\frac{1}{N}x^{m_1}<y<Nx^{m_1}\}$. We then define $C_1=A_1\cup B_1$. If

(d,d) is not the upper vertex of a compact edge, then define $C_1 = \{(x,y) \in [0,1]\times[0,1] : y < x^L\}$.

In all cases, define $D = [0,1]\times[0,1] - (A_2 \cup B_2)$ We will see that the terms $I^D_{S,\phi}(\epsilon)$ and $J^D_{S,\phi}(\lambda)$ dominate; the contributions from C_1 and C_2 to the main term of the asymptotics can be made arbitrarily small as $L \to \infty$. In fact, for the case when C_i comes from a compact edge, Lemmas 2.1 and 2.2 give that

$$|I^{A_i}_{S,\phi}(\epsilon)| < CN^{-\eta}\epsilon^{\frac{1}{d}} \qquad\qquad |I^{B_i}_{S,\phi}(\epsilon)| < C_N\epsilon^{\frac{1}{d}} \tag{4.1a}$$

$$|J^{A_i}_{S,\phi}(\lambda)| < CN^{-\eta}|\lambda|^{-\frac{1}{d}} \qquad\qquad |J^{B_i}_{S,\phi}(\lambda)| < C_N|\lambda|^{-\frac{1}{d}} \tag{4.1b}$$

Lemma 4.1. For sufficiently small ϵ we have

$$|I^D_{S,\phi}(\epsilon)| < C|\ln(\epsilon)|\epsilon^{\frac{1}{d}} \qquad\qquad |J^D_{S,\phi}(\lambda)| < C\ln|\lambda||\lambda|^{-\frac{1}{d}} \tag{4.2a}$$

Furthermore, for $i = 1,2$ if C_i derives from the horizontal or vertical ray we have

$$|I^{C_i}_{S,\phi}(\epsilon)| < \frac{C}{L+1}|\ln(\epsilon)|\epsilon^{\frac{1}{d}} \qquad\qquad |J^{C_i}_{S,\phi}(\lambda)| < \frac{C}{L+1}\ln|\lambda||\lambda|^{-\frac{1}{d}} \tag{4.2b}$$

Proof: We start with the proof of (4.2a). We divide $D = D_1 \cup D_2$, where $D_1 = \{(x,y) \in D : y < x^m\}$. Here m is chosen such that if e_1 exists, then $m < m_1$, and if e_2 exists then $m > m_2$. The estimates for D_1 and D_2 are proven the same way, so we restrict our attention to proving the estimates for D_1. Taylor expand $S(x,y)$ arount the origin as $S(x,y) = \sum_{a<M,\ b<M} s_{ab}x^ay^b + E_M(x,y)$, where like in (2.19) for $0 \leq \alpha,\beta \leq M$ we have

$$|\partial^\alpha_x\partial^\beta_y E_M(x,y)| < C(|x|^{M-\alpha} + |y|^{M-\beta}) \tag{4.3}$$

Correspondingly, we have

$$\partial^d_y S(x,y) = \sum_{a<M,\ b<M-d} s'_{ab}x^ay^b + \partial^d_y E_M(x,y) \tag{4.4}$$

The Newton polygon of $\partial^d_y S(x,y)$ has a vertex at $(d,0)$ and thus $s'_{d0} \neq 0$. If (d,d) is the lower vertex of some compact edge e_2, then there is a compact edge e' of the Newton polygon of $\partial^d_y S(x,y)$ containing $(d,0)$ with equation $t_1 + m_2t_2 = d$. Hence every nonzero $s'_{ab}x^ay^b$ appearing in (4.4) satisfies $a + m_2b \geq d$ and we can rewrite (4.4) as

$$\partial^d_y S(x,y) = s'_{d0}x^d + \sum_{a<d,\ 0<b<M-d,\ a+m_2b\geq d} s'_{ab}x^ay^b$$

$$+ \sum_{d\leq a<M,\ 0\leq b<M-d,\ (a,b)\neq(d,0)} s'_{ab}x^ay^b + \partial^d_y E_M(x,y) \tag{4.5}$$

If (d,d) is not the lower vertex of such a compact edge, (4.5) is still valid if we take the first sum to be empty. If the first sum is not empty, then since $y < N^{-m_2}x^{m_2}$ for all $(x,y) \in D$, if (x,y) is in D then each term $s'_{ab}x^a y^b$ in the first sum of (4.5) is bounded in absolute value by $N^{-m_2 b}|s'_{ab}|x^{a+m_2 b} \leq N^{-m_2 b}|s'_{ab}|x^d$. Thus if N were chosen sufficiently large, the absolute value of the whole first sum is less than $\frac{1}{4}|s'_{d0}|x^d$.

Next, note that the absolute value of a given term $s'_{ab}x^a y^b$ in the second sum is at most $|s'_{ab}|x^d(x+y)$. As a result, if the support of $\phi(x,y)$ is sufficiently small, then for (x,y) in this support the absolute value of the second sum is also at most $\frac{1}{4}|s'_{d0}|x^d$. Similarly, if the support of $\phi(x,y)$ is sufficiently small, then by (4.2) and the fact that $y < x^m$, for (x,y) in this support $|\partial_y^d E_M(x,y)|$ can also be assumed to be at most $\frac{1}{4}|s'_{d0}|x^d$. Consequently, for such (x,y) in the support of ϕ we can assume

$$|\partial_y^d S(x,y)| > \frac{1}{4}|s'_{d0}|x^d \tag{4.6}$$

Denote the vertical cross section of D at x by D_x. By (4.6) and the measure version of Van der Corput's lemma, for each x in this range we have

$$|\{y \in D_x : 0 < S(x,y) < \epsilon\}| < C\epsilon^{\frac{1}{d}}\frac{1}{x}$$

Also, $|\{y \in D_x : 0 < S(x,y) < \epsilon\}|$ is at most $|D_x|$, which is at most x^m if the support of ϕ is sufficiently small, which we may assume. Thus we have

$$|\{y \in D_x : 0 < S(x,y) < \epsilon\}| < C\min(x^m, \epsilon^{\frac{1}{d}}\frac{1}{x})$$

Consequently, integrating with respect to y first one has

$$|I^D_{S,\phi}(\epsilon)| < C\int_0^1 \min(x^m, \epsilon^{\frac{1}{d}}\frac{1}{x})\,dx \tag{4.7}$$

For small enough ϵ, the quantities x^m and $\epsilon^{\frac{1}{d}}\frac{1}{x}$ are equal at $x_0 = \epsilon^{\frac{1}{(m+1)d}}$, with x^m smaller on the left and $\epsilon^{\frac{1}{d}}\frac{1}{x}$ smaller on the right. Doing a computation gives

$$\int_0^1 \min(x^m, \epsilon^{\frac{1}{d}}\frac{1}{x})\,dx = \frac{1}{m+1}\epsilon^{\frac{1}{d}} + \frac{1}{(m+1)d}|\ln(\epsilon)|\epsilon^{\frac{1}{d}} \tag{4.8}$$

Hence $|I^D_{S,\phi}(\epsilon)| < C|\ln(\epsilon)|\epsilon^{\frac{1}{d}}$ as desired. As for the $J^D_{S,\phi}(\lambda)$, by (4.6) the traditional Van der Corput lemma in the y direction gives

$$|\int_{D_x} e^{i\lambda S(x,y)}\phi(x,y)\,dy| \leq C|\lambda|^{-\frac{1}{d}}x^{-1} \tag{4.9}$$

Consequently, we have

$$|J^D_{S,\phi}(\lambda)| < C\int_0^1 \min(N^{-m}x^m, |\lambda|^{-\frac{1}{d}}\frac{1}{x})\,dx \tag{4.10}$$

This is exactly (4.8) with ϵ replaced by $|\lambda|^{-1}$. Thus instead of (4.8) for large $|\lambda|$ we get the estimate

$$|J_{S,\phi}^{D}(\lambda)| < C\ln|\lambda||\lambda|^{-\frac{1}{d}} \tag{4.11}$$

This completes the proof of (4.2a). Equation (4.2b) is done the same way; the only difference is that x^m is replaced by x^L. Equation (4.8) and its oscillatory integral analogue then give (4.2b) and we are done.

We now have proven the upper bounds for the smooth case:

Lemma 4.2. Equations (1.12) and (1.15b) hold.

Proof. Add (4.2a) to (4.2b) or (4.1a) – (4.1b).

Our next result is an analogue of Lemma 3.2.

Lemma 4.3. There is a neighborhood U of the origin such that if $\phi(x,y)$ is supported in U and $\phi(x,y)$ is zero in a neighborhood of the origin, then

$$|I_{S,\phi}(\epsilon)| < C\epsilon^{\frac{1}{d}} \qquad\qquad |J_{S,\phi}(\lambda)| < C|\lambda|^{-\frac{1}{d}} \tag{4.12}$$

Proof. Fix some N. By (4.1a) – (4.1b), the upper bounds of (4.12) hold for $I_{S,\phi}^{A_i}(\epsilon)$, $J_{S,\phi}^{A_i}(\lambda)$, $I_{S,\phi}^{B_i}(\epsilon)$, and $J_{S,\phi}^{B_i}(\lambda)$. Thus it suffices to prove these upper bounds for $I_{S,\phi}^{D}(\epsilon)$ and $J_{S,\phi}^{D}(\epsilon)$, as well as $I_{S,\phi}^{C_i}(\lambda)$ and $J_{S,\phi}^{C_i}(\lambda)$ if they derive from the vertical or horizontal ray. These are all done basically the same way, so we restrict our attention to $I_{S,\phi}^{D}(\epsilon)$ and $J_{S,\phi}^{D}(\epsilon)$. As in the proof of Lemma 4.1, we divide $D = D_1 \cup D_2$ along the curve $y = x^m$. The two pieces are done similarly, so we will only consider D_1, the part where $y < x^m$.

Each vertical cross section $(D_1)_x$ of the set D_1 is the a subset of the interval $[0, x^m]$. Hence there is some $\delta > 0$ such that on D, $\phi(x,y) = 0$ for $x < \delta$. Doing the y integration first we have

$$|I_{S,\phi}^{D_1}(\epsilon)| < C\int_{\delta}^{1} |\{y \in (D_1)_x : 0 < S(x,y) < \epsilon\}|\, dx \tag{4.13}$$

By (4.6), $|\partial_y^d S(x,y)|$ is bounded below on $x > \delta$. Hence by the Van der Corput lemma in the y direction, we have $|\{y \in (D_1)_x : 0 < S(x,y) < \epsilon\}| < C\epsilon^{\frac{1}{d}}$ uniformly in $x > \delta$. Inserting this back into (4.13) gives the desired bounds. For the oscillatory integral, one similarly uses the Van der Corput lemma in the y direction to get $|\int_{(D_1)_x} e^{i\lambda S(x,y)}\phi(x,y)\,dy| < C|\lambda|^{-\frac{1}{d}}$ uniformly in $x > \delta$. Thus

$$|J_{S,\phi}^{D_1}(\lambda)| = |\int_{\delta}^{1}(\int_{(D_1)_x} e^{i\lambda S(x,y)}\phi(x,y)\,dy)\,dx| < C|\lambda|^{-\frac{1}{d}} \tag{4.14}$$

These are the sought-after bounds for $J^D_{S,\phi}(\lambda)$ and we are done.

We now proceed to the proof of the explicit formula (1.7). The general methodology is similar to that of the case 1 arguments of section 3. If one writes $\phi = \phi_1 + \phi_2$, where ϕ_1 is supported in a smaller neighborhood of the origin and ϕ_2 is zero on a neighborhood of the origin then by Lemma 4.3, $\limsup_{\epsilon \to 0} \frac{I^+_{S,\phi}(\epsilon)}{\ln(\epsilon)\epsilon^{\frac{1}{d}}} = \limsup_{\epsilon \to 0} \frac{I^+_{S,\phi_1}(\epsilon)}{\ln(\epsilon)\epsilon^{\frac{1}{d}}}$ and $\liminf_{\epsilon \to 0} \frac{I^+_{S,\phi}(\epsilon)}{\ln(\epsilon)\epsilon^{\frac{1}{d}}} = \liminf_{\epsilon \to 0} \frac{I^+_{S,\phi_1}(\epsilon)}{\ln(\epsilon)\epsilon^{\frac{1}{d}}}$. So when proving (1.7) one can always replace ϕ by ϕ_1 at will, regardless of how small the support of ϕ_1 is.

We will show if one first chooses the parameter L of (4.2b) sufficiently large, and then chooses ϕ_1 to be supported in a sufficiently small neighborhood of the origin, then the above limsup and liminf, added to their analogues from the other three quadrants, can both be made as arbitrarily close to the limit given in (1.7). To do this, in view of (4.2b), it suffices to show that if the support of ϕ_1 is sufficiently small, the quantities $\limsup_{\epsilon \to 0} \frac{I^D_{S,\phi_1}}{\ln(\epsilon)\epsilon^{\frac{1}{d}}}$ and $\liminf_{\epsilon \to 0} \frac{I^D_{S,\phi_1}}{\ln(\epsilon)\epsilon^{\frac{1}{d}}}$, can be made arbitrarily close to the appropriate expression. For a fixed L we will find lower bounds for the limsup and upper bounds for the liminf. In doing so, we will choose the parameter M of the Taylor expansions in terms of L, and then the parameter N of (4.1a) – (4.1b) in terms of L and M. Analogous to in section 3 taking limits as L goes to infinity, both expressions will converge to the same limit. Adding this limit to its analogues in the other 3 quadrants will give (1.7).

We start with the following lemma. As before, $s_{dd}x^dy^d$ denotes the (d,d) term of the Taylor expansion $S(x,y) = \sum_{a,b} s_{ab}x^ay^b$ of S at the origin.

Lemma 4.4. There are constants β and C depending on $S(x,y)$, and a neighborhood U of the origin depending on $S(x,y)$ and L, such that $|S(x,y) - s_{dd}x^dy^d| < |CL^{-\beta}x^dy^d|$ on $D \cap U$.

Proof. Analogous to (4.5), we may Taylor expand

$$S(x,y) - s_{dd}x^dy^d = \sum_{M>a\geq d,\ M>b\geq d,\ (a,b)\neq(d,d)} s'_{ab}x^ay^b$$

$$+ \sum_{a<d,\ M>b>d,\ a+m_2b\geq\alpha_2} s_{ab}x^ay^b + \sum_{M>a>d,\ b<d,\ a+m_1b\geq\alpha_1} s_{ab}x^ay^b + E_M(x,y) \qquad (4.15)$$

Here the second term is nonempty only if (d,d) is the lower vertex of a compact edge, and the third term is nonempty only if (d,d) is the upper vertex of a compact edge.

The first sum can be made less than $\frac{1}{L}x^dy^d$ in absolute value by making the radius of U sufficiently small depending on M and $S(x,y)$. If the second sum is nonempty, then the domain D is a subset of $\{(x,y) \in [0,1] \times [0,1] : 0 < y < N^{-m_2}x^{m_2}\}$. As a result if one changes coordinates from (x,y) to (x,y'), where $y' = x^{m_2}y$, D becomes a subset of $D' = \{(x,y) \in [0,1] \times [0,1] : 0 < y < N^{-m_2}\}$. Observe that a given term $s_{ab}x^a y^b$ of the second sum becomes $s_{ab}x^{a+m_2 b}(y')^b$. Since $a + m_2 b \geq \alpha_2$ and $b > d$ in each term in the second sum, the entire sum can be written as $y'(x^{\alpha_2}(y')^d)f(x,y')$ for some $f(x,y')$ which is a polynomial in y and a fractional power of x. Thus the sum is of absolute value at most $C_M N^{-m_2} x^{\alpha_2}(y')^d$ in a small enough neighborhood of the origin. Note that $s_{dd}x^d y^d = s_{dd}x^{d+dm_2}(y')^d$, and this is equal to $s_{dd}x^{\alpha_2}(y')^d$ since (d,d) is on the edge with equation $t_1 + m_2 t_2 = \alpha_2$. As a result, in the original (x,y) coordinates, the sum is of absolute value at most $C_M N^{-m_2} x^d y^d$. Thus if one chooses N sufficiently large for fixed L and M, one has the desired bounds.

The third sum is dealt with in exactly the same way, reversing the roles of the x and y axes. Since D necessarily lies in the range $x^{\frac{1}{L}} > y > x^L$, the error term $E_M(x,y)$ can be made less than $\frac{1}{L}x^d y^d$ by making the radius of U sufficiently small. This completes the proof of Lemma 4.4.

Proof of (1.7)**.** As before it suffices to assume $\phi(0,0) > 0$ as the general case can be obtained by writing $\phi = \phi' - \phi''$ where $\phi'(0,0)$ and $\phi''(0,0) > 0$ are positive. Let $\delta_L > 0$ be such that on the ball $B(0,\delta_L)$ one has

$$(1 - L^{-\beta})\phi(0,0) < \phi(x,y) < (1 + L^{-\beta})\phi(0,0) \tag{4.16}$$

Further assume that δ_L is small enough that $B(0,\delta_L) \subset U$, where U is as in the previous lemma. Let $\psi(x,y)$ be a nonnegative cutoff function such that $0 \leq \psi \leq 1$, $\psi(x,y)$ is supported on $B(0,\delta_L)$, and $\psi(x,y) = 1$ on $[0,\frac{\delta_L}{2}] \times [0,\frac{\delta_L}{2}]$. Then by the discussion following (4.14) we may replace $\phi(x,y)$ by $\phi(x,y)\psi(x,y)$ without affecting the liminf or limsup. We have

$$I^D_{S,\phi}(\epsilon) < (1 + L^{-\beta})\phi(0,0)\big|\{(x,y) \in D : 0 < S(x,y) < \epsilon\}\big|$$

By Lemma 4.4 this is bounded by

$$(1 + L^{-\beta})\phi(0,0)\big|\{(x,y) \in D : 0 < (s_{dd} - CL^{-\beta})x^d y^d < \epsilon\}\big| \tag{4.17a}$$

In addition,

$$I^D_{S,\phi}(\epsilon) > (1 - L^{-\beta})\phi(0,0)\big|\{(x,y) \in D \cap ([0,\frac{\delta_L}{2}] \times [0,\frac{\delta_L}{2}]) : 0 < S(x,y) < \epsilon\}\big|$$

$$> (1 - L^{-\beta})\phi(0,0)\big|\{(x,y) \in D \cap [0,(\frac{\delta_L}{2}] \times [0,\frac{\delta_L}{2}]) : 0 < (s_{dd} + CL^{-\beta})x^d y^d < \epsilon\}\big| \tag{4.17b}$$

If $s_{dd} < 0$, then by (4.17a) $I^D_{S,\phi}(\epsilon) = 0$ for large enough L. For $s_{dd} > 0$, we need the following lemma, whose proof is routine.

Lemma 4.5. Suppose $0 < m_2 < m_1$ and $\delta_0 > 0$. Then as $t \to 0$,

$$\big|\{(x,y) \in (0,\delta_0] \times (0,\delta_0] : x^{m_1} < y < x^{m_2},\ y < \frac{t}{x}\}\big| = (\frac{1}{m_1+1} - \frac{1}{m_2+1})t\ln(t) + O(t)$$

$$\left|\{(x,y)\in(0,\delta_0]\times(0,\delta_0]:0<y<x^{m_2},\ y<\frac{t}{x}\}\right| = -\frac{1}{m_2+1}t\ln(t)+O(t)$$

We now apply Lemma 4.5 to $(4.17a)$ – $(4.17b)$. We get that

$$I^D_{S,\phi}(\epsilon) < (1+L^{-\beta})(s_{dd}-CL^{-\beta})^{-\frac{1}{d}}\phi(0,0)\epsilon^{\frac{1}{d}}\ln(\epsilon)(\frac{1}{m_1+1}-\frac{1}{m_2+1})+C\epsilon^{\frac{1}{d}} \quad (4.18a)$$

$$I^D_{S,\phi}(\epsilon) > (1-L^{-\beta})(s_{dd}+CL^{-\beta})^{-\frac{1}{d}}\phi(0,0)\epsilon^{\frac{1}{d}}\ln(\epsilon)(\frac{1}{m_1+1}-\frac{1}{m_2+1})-C\epsilon^{\frac{1}{d}} \quad (4.18b)$$

(When (d,d) is on the horizontal ray one substitutes $m_1=L$, and when it is on the vertical ray one substitutes $m_2=\frac{1}{L}$). Hence we have

$$\limsup_{\epsilon\to 0}\frac{I^D_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}\ln(\epsilon)} \leq (1+L^{-\beta})(s_{dd}-CL^{-\beta})^{-\frac{1}{d}}\phi(0,0)(\frac{1}{m_1+1}-\frac{1}{m_2+1}) \quad (4.19a)$$

$$\liminf_{\epsilon\to 0}\frac{I^D_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}\ln(\epsilon)} \geq (1-L^{-\beta})(s_{dd}+CL^{-\beta})^{-\frac{1}{d}}\phi(0,0)(\frac{1}{m_1+1}-\frac{1}{m_2+1}) \quad (4.19b)$$

We take limits as $L\to\infty$. Both expressions converge to $\phi(0,0)(\frac{1}{m_1+1}-\frac{1}{m_2+1})$, where now m_1 is taken as ∞ when (d,d) is on the horizontal ray and m_2 is taken as 0 when it is on the vertical ray. Hence by the discussion following (4.14) we conclude that

$$\lim_{\epsilon\to 0}\frac{I^+_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}\ln(\epsilon)} = s_{dd}^{-\frac{1}{d}}\phi(0,0)(\frac{1}{m_1+1}-\frac{1}{m_2+1})$$

Letting s_i be the slope $-\frac{1}{m_i}$ of the edge $t_1+m_it_2=\alpha_i$, this becomes

$$\lim_{\epsilon\to 0}\frac{I^+_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}\ln(\epsilon)} = s_{dd}^{-\frac{1}{d}}\phi(0,0)(\frac{1}{s_1-1}-\frac{1}{s_2-1}) \quad (4.20)$$

In summary, if $s_{dd}>0$ then (4.20) gives the contribution to (1.7) from the upper right-hand quadrant. If $s_{dd}<0$ then the contribution is zero as mentioned above Lemma 4.5. Adding this to its analogues over the other three quadrants gives exactly the formula of (1.7) and we are done.

Our final task is to prove $(1.13b)$:

Proof of (1.13b). Since the result is symmetric in S and $-S$, we may replace S by $-S$ if necessary and assume that $s_{dd}>0$. As in the proof of $(1.13a)$, we write $\phi=\phi_1+\phi_2$, where ϕ_1 is nonnegative and ϕ_2 is zero on a neighborhood of the origin. By Lemma 4.3, $\lim_{\epsilon\to 0}\frac{I_{|S|,\phi_2}(\epsilon)}{\ln(\epsilon)\epsilon^{\frac{1}{d}}}=0$. So to prove $(1.13b)$ it suffices to show $I_{|S|,\phi_1}(\epsilon)>A_{S,\phi_1}|\ln(\epsilon)|\epsilon^{\frac{1}{d}}$ for some $A_{S,\phi_1}>0$. Since $I_{|S|,\phi_1}(\epsilon)\geq I_{S,\phi_1}(\epsilon)$, it further suffices to show that $I_{S,\phi_1}(\epsilon)>$

$A_{S,\phi}|\ln(\epsilon)|\epsilon^{\frac{1}{d}}$ for some A_{S,ϕ_1}. For this we use Lemma 4.4, which implies that there is a $\delta > 0$ such that on $([0,\delta]\times[0,\delta])\cap D$ we have

$$S(x,y) < \frac{3}{2}s_{dd}x^d y^d \tag{4.21}$$

As a result, shrinking δ to ensure that $\phi(x,y) > \frac{1}{2}\phi(0,0)$ on $[0,\delta]\times[0,\delta]$ if necessary, we have

$$I_{S,\phi_1}(\epsilon) > \frac{1}{2}\phi(0,0)|\{(x,y)\in D\cap([0,\delta]\times[0,\delta]) : \frac{3}{2}s_{dd}x^d y^d < \epsilon\}| \tag{4.22}$$

Using Lemma 4.5, we conclude that there is some A_{S,ϕ_1} with $|I_{S,\phi_1}(x,y)| > A_{S,\phi_1}|\ln(\epsilon)|\epsilon^{\frac{1}{d}}$ as needed. This gives (1.13b) and we are done.

5. Case 3 proofs.

In this section, $S(x,y)$ is a smooth phase function in case 3 of superadapted coordinates with $d > 1$. We restrict ourselves to the situation where the bisectrix intersects the horizontal ray in its interior, as the case of a vertical ray is entirely analogous. Thus the lowest vertex of $N(S)$ is of the form (c,d), where $c < d$ and d is also the Newton distance of S. As in sections 3 and 4 we will focus our attention on the analysis of $I^+_{S,\phi}(\epsilon)$ and $J^+_{S,\phi}(\lambda)$. We divide $[0,1]\times[0,1]$ into two parts. For a sufficiently large positive integer k (to be determined by our arguments), we let $D_1 = \{(x,y)\in[0,1]\times[0,1] : y < x^k\}$ and $D_2 = \{(x,y)\in[0,1]\times[0,1] : y > x^k\}$. Our first lemma is the following.

Lemma 5.1. $|I^{D_1}_{S,\phi}(\epsilon)| < C\epsilon^{\frac{1}{d}}$ and $|J^{D_1}_{S,\phi}(\lambda)| < C|\lambda|^{-\frac{1}{d}}$.

Proof. As in cases 1 and 2, we write the Taylor expansion of S at the origin as $S(x,y) = \sum_{a<M,\ b<M} s_{ab}x^a y^b + E_M(x,y)$, where for $0 \leq \alpha,\beta \leq M$ the function $E_M(x,y)$ satisfies the error estimates

$$|\partial_x^\alpha\partial_y^\beta E_M(x,y)| < C(|x|^{M-\alpha} + |y|^{M-\beta}) \tag{5.1}$$

The dth y-derivative can be written as

$$\partial_y^d S(x,y) = \sum_{a<M,\ b<M-d} s'_{ab}x^a y^b + \partial_y^d E_M(x,y) \tag{5.2}$$

Furthermore the Newton polygon of $\partial_y^d S(x,y)$ has a vertex at $(c,0)$, contained either in the vertical ray of $N(\partial_y^d S)$ or an edge of $N(\partial_y^d S)$ with equation $t_1 + mt_2 = c$ with $m > 0$. Hence each (a,b) in the sum of (5.2) satisfies $a + mb \geq c$. Analogous to (4.5) we rewrite (5.2) as

$$\partial_y^d S(x,y) = s'_{c0}x^c + \sum_{a<c,\ 0<b<M-d,\ a+mb\geq c} s'_{ab}x^a y^b$$

$$+ \sum_{c\leq a<M,\ 0\leq b<M-d,\ (a,b)\neq(c,0)} s'_{ab}x^a y^b + \partial_y^d E_M(x,y) \tag{5.3}$$

In the case where $(c,0)$ is on a vertical ray of $N(\partial_y^d S)$, the first sum of (5.3) is empty. We now argue like after (4.5). Since $y < x^k$ for all $(x,y) \in D_1$, if (x,y) is in D_1 each term $s'_{ab}x^a y^b$ in the first sum of (4.5) is bounded in absolute value by $|s'_{ab}|x^{a+kb} \leq |s'_{ab}|x^{k-m}(x^{a+mb}) \leq |s'_{ab}|x^{k-m+c}$. Thus as long as k were chosen greater than m, which we may assume, then if the support of $\phi(x,y)$ is sufficiently small, then for (x,y) in this support, the absolute value of the whole first sum is less than $\frac{1}{4}|s'_{c0}|x^c$. Also, the absolute value of a given term $s'_{ab}x^a y^b$ in the second sum is at most $|s'_{ab}|x^c(x+y)$. As a result, for such (x,y) the absolute value of the second sum is also at most $\frac{1}{4}|s'_{c0}|x^c$. Similarly, using (5.1) and the fact that $0 < y < x^k$, for such (x,y) the quantity $|\partial_y^d E_M(x,y)|$ can also be assumed to be at most $\frac{1}{4}|s'_{c0}|x^c$. Consequently, for these (x,y) we have

$$|\partial_y^d S(x,y)| > \frac{1}{4}|s'_{c0}|x^c \tag{5.4}$$

As a result, by the measure version of Van der Corput's lemma of [C], for each x in this range we have

$$|\{y : 0 < S(x,y) < \epsilon\}| < C\epsilon^{\frac{1}{d}}x^{-\frac{c}{d}} \tag{5.5}$$

Consequently, integrating with respect to y first one has

$$|I_{S,\phi}^{D_1}(\epsilon)| < C\int_0^1 \epsilon^{\frac{1}{d}}x^{-\frac{c}{d}}\,dx = C'\epsilon^{\frac{1}{d}} \tag{5.6}$$

This is the desired upper bound for $I_{S,\phi}^{D_1}(\epsilon)$. As for the oscillatory integral analogue, by (5.4) the normal Van der Corput lemma in the y direction gives

$$|\int e^{i\lambda S(x,y)}\phi(x,y)\,dy| \leq C|\lambda|^{-\frac{1}{d}}x^{-\frac{c}{d}} \tag{5.7}$$

Hence by integrating first with respect to y one has

$$|J_{S,\phi}^{D_1}(\lambda)| < C\int_0^1 |\lambda|^{-\frac{1}{d}}x^{-\frac{c}{d}}\,dx = C'|\lambda|^{-\frac{1}{d}} \tag{5.8}$$

This completes the proof of Lemma 5.1.

It turns out that one gets stronger estimates for the $I_{S,\phi}^{D_2}(\epsilon)$ and $J_{S,\phi}^{D_2}(\lambda)$. Observe that since $c < d$, the quantity $\frac{1+k}{c+kd}$ is greater than $\frac{1}{d}$. We have the following.

Lemma 5.2. $|I_{S,\phi}^{D_2}(\epsilon)| < C\epsilon^{\frac{1+k}{c+kd}}$ and $|J_{S,\phi}^{D_2}(\lambda)| < C|\lambda|^{-\frac{1+k}{c+kd}}$.

Proof. We will verify the hypotheses of Lemma 2.0, with the roles of the x and the y variables reversed. Because (c,d) is the rightmost vertex of $N(S)$, the Newton polygon of $\partial_x^c S(x,y)$ has a single vertex at $(0,d)$. The Taylor expansion of $\partial_x^c S(x,y)$ can be written in the form

$$\partial_x^c S(x,y) = r_{0d}\,y^d + \sum_{0\leq a<M,\ d\leq b<M\ \ (a,b)\neq(0,d)} r_{ab}x^a y^b + E_M(x,y) \tag{5.9}$$

Similar to elsewhere in this paper, bounding the error term using the fact that $x < y^{\frac{1}{k}}$ on D_2, in a small enough neighborhood of the origin on D_2 one has

$$|\partial_x^c S(x,y)| > \frac{1}{2}|r_{0d}|y^d$$

Thus if $c \geq 2$, one can apply Lemma 2.0 and immediately get this lemma. If $c = 1$, to apply Lemma 2.0 one also needs that $|\partial_x^2 S(x,y)| < Cy^{d-\frac{1}{k}}$. But in fact since the Newton polygon of $\partial_x^2 S(x,y)$ is a subset of $\{(x,y) : y \geq d\}$, by expanding as in (5.9) one even has the stronger estimate $|\partial_x^2 S(x,y)| < Cy^d$. Thus Lemma 2.0 applies here. If $c = 0$, to apply Lemma 2.0 one needs (2.2c) to hold (with the x and y variables reversed) which here means one needs $|\partial_y S(x,y)| > Cy^{d-1}$ and $|\partial_y^2 S(x,y)| < C'y^{d-2}$. Since the Newton polygon of $\partial_y^i S(x,y)$ has a single vertex at $(0, d-i)$ this holds as in the $c = 1$ case. Lastly, to apply Lemma 2.0 for $c = 0$ one also needs that $d > \frac{1}{k} + 1$. We can make this true simply choosing k large enough since d is at least 2. This completes the proof of Lemma 5.2.

Corollary 5.3. (1.11) and (1.15a) hold in Case 3.

Proof. Add the estimates from Lemmas 5.1 and 5.2 and their analogues from the other three quadrants.

Next, we prove the lower bounds of (1.13c). Since we are not trying to prove sharp estimates, the arguments are not that intricate. Assume $\phi(0,0) \neq 0$, and let M be some large positive integer. We examine the behavior of $S(x,y)$ on the set $Z_N = \{(x,y) : x > 0,\ x^N < y < 2x^N\}$ for N sufficiently large. We Taylor expand $S(x,y) = \sum_{a<M,\ b<M} s_{ab}x^a y^b + E_M(x,y)$ as above. If N is large enough, the term $s_{cd}x^c y^d$ dominates this Taylor expansion much the way $r_{0d}y^d$ dominates (5.9) or $s'_{c0}x^c$ dominates (5.3). Hence in a small enough neighborhood U of the origin, on Z_N we have

$$|S(x,y)| < 2|s_{cd}|x^c y^d \tag{5.10}$$

Shrinking U, we may assume that $|\phi(x,y)| > \frac{1}{2}|\phi(0,0)|$. Hence for $c_\phi = \frac{1}{2}|\phi(0,0)|$ we have

$$I_{|S|,\phi}(\epsilon) > c_\phi|\{(x,y) \in U \cap Z_N : 2|s_{cd}|x^c y^d < \epsilon\}| \tag{5.11}$$

It is easy to compute that the curve $y = 2x^N$ intersects the curve $2|s_{cd}|x^c y^d = \epsilon$ at $x = a\epsilon^{\frac{1}{Nd+c}}$ for some a depending on $S(x,y)$. Hence for ϵ sufficiently small the measure of the set in the right-hand side of (5.11) is at least the measure of the portion of Z_N between $x = \frac{a}{2}\epsilon^{\frac{1}{Nd+c}}$ and $x = a\epsilon^{\frac{1}{Nd+c}}$, given by $a'\epsilon^{\frac{N+1}{Nd+c}}$ where now a' also depends on N. Thus we can write

$$I_{|S|,\phi}(\epsilon) > c'_{S,\phi,N}\epsilon^{\frac{N+1}{Nd+c}} \tag{5.12}$$

Since $d > c$, the exponent in (5.12) is larger than $\frac{1}{d}$ but as $N \rightarrow \infty$ it tends to $\frac{1}{d}$. This gives us (1.13c).

We now move to the case of real-analytic phase. Our goal here is to prove Theorem 1.1 c). So assume $S(x,y)$ is real-analytic. It suffices to show that $\lim_{\epsilon\rightarrow 0}\frac{I_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}}$ exists and is given by (1.8). As in cases 1 and 2, we will give an expression for $\lim_{\epsilon\rightarrow 0}\frac{I^+_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}}$ and the full limit will follow by adding this and the analogues from the other quadrants. Also, by Lemma 5.2, $\lim_{\epsilon\rightarrow 0}\frac{I^{D_2}_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}}=0$ so it suffices to show $\lim_{\epsilon\rightarrow 0}\frac{I^{D_1}_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}}$ exists and has the desired value. Because the bisectrix intersects $N(S)$ in the interior of its horizontal ray and (c,d) is the lowest vertex of $N(S)$, the real-analytic $S(x,y)$ can be written as

$$S(x,y)=s_{cd}x^cy^d+x^{c+1}y^dg(x)+\sum_{b\geq d+1}s_{ab}x^ay^b \tag{5.13a}$$

Here $g(x)$ is real-analytic. Changing coordinates from (x,y) to (x,y') where $y=x^ky'$, we have

$$S(x,x^ky')=s_{cd}x^{c+kd}(y')^d+x^{c+kd+1}(y')^dg(x)+\sum_{b\geq d+1}s_{ab}x^{a+kb}(y')^b \tag{5.13b}$$

Since the line $t_1+mt_2=\alpha$ is an edge of $N(S)$ containing (c,d), each (a,b) in the sum (5.13a) satisfies $a+mb\geq c+md$. Furthermore, $b>d$, and therefore

$$a+kb=a+mb+(k-m)b\geq c+md+(k-m)b>c+md+(k-m)d=c+kd$$

Hence we can rewrite $\sum_{b\geq d+1}s_{ab}x^{a+kb}(y')^b$ as $x^{c+kd+1}(y')^{d+1}f(x,y)$ where $f(x,y)$ is real-analytic. Thus we have

$$S(x,x^ky')=s_{cd}x^{c+kd}(y')^d+x^{c+kd+1}(y')^dg(x)+x^{c+kd+1}(y')^{d+1}f(x,y') \tag{5.14}$$

In the (x,y') coordinates $I^{D_1}_{S,\phi}(\epsilon)$ becomes

$$I^{D_1}_{S,\phi}(\epsilon)=\int_{\{(x,y')\in[0,1]\times[0,1]:0<S(x,x^ky')<\epsilon\}}x^k\phi(x,x^ky')\,dx\,dy' \tag{5.15}$$

Let d' be between c and d. The exact value of d' will be dictated by our arguments. Then the portion of (5.15) over $x<\epsilon^{\frac{1}{d'(k+1)}}$ has absolute value at most $C\int_0^{\epsilon^{\frac{1}{d'(k+1)}}}x^k\,dx=C\epsilon^{\frac{1}{d'}}$. Since this is $o(\epsilon^{\frac{1}{d}})$, this portion of the integral will can be removed without affecting $\lim_{\epsilon\rightarrow 0}\frac{I^{D_1}_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}}$. In other words, we may replace $I^{D_1}_{S,\phi}(\epsilon)$ by $I'_{S,\phi}(\epsilon)$ where

$$I'_{S,\phi}(\epsilon)=\int_{\{(x,y')\in[\epsilon^{\frac{1}{d'(k+1)}},1]\times[0,1]:0<S(x,x^ky')<\epsilon\}}x^k\phi(x,x^ky')\,dx\,dy' \tag{5.16}$$

We now fix $x>\epsilon^{\frac{1}{d'(k+1)}}$ and look at the set $E_x=\{y\in[0,1]:0<S(x,x^ky')<\epsilon\}$. We may assume the support of $\phi(x,y)$ is small enough so that if $\phi(x,y)\neq 0$ then x is small enough so that

$$|x^{c+kd+1}(y')^dg(x)|+|x^{c+kd+1}(y')^{d+1}f(x,y)|<\frac{1}{2}|s_{cd}|x^{c+kd}y^d \tag{5.17}$$

Note that if s_{cd} is negative, by (5.14) and (5.17) $S(x, x^k y')$ is always negative and thus $I'_{S,\phi}(\epsilon) = 0$. So assume that $s_{cd} > 0$. Then by (5.14) and (5.17) we have

$$E_x \subset \{y \in [0,1] : 0 < s_{cd} x^{c+kd} y^d < 2\epsilon\} \subset [0, C\epsilon^{\frac{1}{d}} x^{-\frac{c+kd}{d}}] \tag{5.18}$$

Since $x \geq \epsilon^{\frac{1}{d'(k+1)}}$, we have

$$\epsilon^{\frac{1}{d}} x^{-\frac{c+kd}{d}} \leq \epsilon^{\frac{1}{d}} \epsilon^{-\frac{c+kd}{d(d')(k+1)}} = \epsilon^{\frac{kd'+d'}{d(d')(k+1)}} \epsilon^{-\frac{c+kd}{d(d')(k+1)}} = \epsilon^{\frac{k(d'-d)+(d'-c)}{dd'(k+1)}}$$

Hence if d' were chosen close enough to d, there is some $\eta' > 0$ such that for $x \geq \epsilon^{\frac{1}{d'(k+1)}}$ one has $\epsilon^{\frac{1}{d}} x^{-\frac{c+kd}{d}} < \epsilon^{\eta'}$ and thus

$$E_x \subset [0, C\epsilon^{\eta'}] \tag{5.19}$$

Next, we write $I'_{S,\phi}(\epsilon) = I''_{S,\phi}(\epsilon) + I'''_{S,\phi}(\epsilon)$, where

$$I''_{S,\phi}(\epsilon) = \int_{\{(x,y') \in [\epsilon^{\frac{1}{d'(k+1)}}, 1] \times [0,1] : 0 < S(x, x^k y') < \epsilon\}} x^k \phi(x, 0)\, dx\, dy' \tag{5.20a}$$

$$I'''_{S,\phi}(\epsilon) = \int_{\{(x,y') \in [\epsilon^{\frac{1}{d'(k+1)}}, 1] \times [0,1] : 0 < S(x, x^k y') < \epsilon\}} x^k (\phi(x, x^k y') - \phi(x, 0))\, dx\, dy' \tag{5.20b}$$

Note that due to (5.19), the factor $(\phi(x, x^k y') - \phi(x, 0))$ in (5.20b) is bounded in absolute value by $C'\epsilon^{\eta'}$, so we have

$$I'''_{S,\phi}(\epsilon) \leq C'\epsilon^{\eta'} \int_{\{(x,y') \in [\epsilon^{\frac{1}{d'(k+1)}}, 1] \times [0,1] : 0 < S(x, x^k y') < \epsilon\}} x^k\, dx\, dy' \tag{5.21}$$

In (5.21), we perform the y integration by inserting the second inclusion of (5.18). We then have

$$I'''_{S,\phi}(\epsilon) \leq C''\epsilon^{\eta'} \int_{\epsilon^{\frac{1}{d'(k+1)}}}^{1} x^k \epsilon^{\frac{1}{d}} x^{-\frac{c+kd}{d}}\, dx\, dy' \leq C'''\epsilon^{\frac{1}{d}+\eta'} \int_0^1 x^{-\frac{c}{d}}\, dx = C''''\epsilon^{\frac{1}{d}+\eta'} \tag{5.22}$$

Thus $\lim_{\epsilon \to 0} \frac{I'''_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}} = 0$. Hence $\lim_{\epsilon \to 0} \frac{I'_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}} = \lim_{\epsilon \to 0} \frac{I''_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}}$, and our goal now becomes to prove the latter limit gives the portion of (1.8) coming from the upper right quadrant. Next, we rewrite $I''_{S,\phi}(\epsilon)$ as

$$I''_{S,\phi}(\epsilon) = \int_{\epsilon^{\frac{1}{d'(k+1)}}}^{1} x^k \phi(x, 0) |\{y' : 0 < S(x, x^k y') < \epsilon\}|\, dx\, dy' \tag{5.23}$$

To analyze (5.23), first note that $\frac{x^{c+kd+1}}{s_{cd} x^{c+kd} + x^{c+kd+1} g(x)}$ is a real-analytic function in a neighborhood of the origin, which we denote by $h(x)$. Then by (5.14) we have

$$S(x, x^k y') = (s_{cd} x^{c+kd} + x^{c+kd+1} g(x))((y')^d + x h(x) (y')^{d+1} f(x, y')) \tag{5.24}$$

Since s_{cd} is being assumed to be positive, in (5.24) $(s_{cd}x^{c+kd} + x^{c+kd+1}g(x))^{\frac{1}{d}}$ is positive and there is $j(x,y')$ with $\partial_y j(x,0) = 1$ and $\partial_x j(x,0) = 0$ such that (5.24) can be rewritten as
$$S(x, x^k y')^{\frac{1}{d}} = (s_{cd}x^{c+kd} + x^{c+kd+1}g(x))^{\frac{1}{d}} j(x,y') \tag{5.25}$$
Consequently, in (5.23), one has
$$|\{y' : 0 < S(x, x^k y') < \epsilon\}| = \left|\{y' : 0 < j(x,y') < \epsilon^{\frac{1}{d}}(s_{cd}x^{c+kd} + x^{c+kd+1}g(x))^{-\frac{1}{d}}\}\right| \tag{5.26}$$
By the inverse function theorem, $(x, j(x,y'))$ has an inverse function which can be written as $(x, k(x,y'))$ for some $k(x,y')$ which satisfies $\partial_y k(x,0) = 1$ and $\partial_x k(x,0) = 0$ By (5.19) the interval of (5.26) has length at most $C\epsilon^{\eta'}$. As a result, as long as ϵ is small enough we can use a linear approximation to $k(x,y')$ and get that
$$\left|\{y' : 0 < j(x,y') < \epsilon^{\frac{1}{d}}(s_{cd}x^{c+kd} + x^{c+kd+1}g(x))^{-\frac{1}{d}}\}\right|$$
$$= \epsilon^{\frac{1}{d}}(s_{cd}x^{c+kd} + x^{c+kd+1}g(x))^{-\frac{1}{d}} + \epsilon^{\eta'} O(\epsilon^{\frac{1}{d}}(s_{cd}x^{c+kd} + x^{c+kd+1}g(x))^{-\frac{1}{d}}) \tag{5.27}$$
Thus we have
$$I''_{S,\phi}(\epsilon) = \int_{\epsilon^{\frac{1}{d'(k+1)}}}^{1} x^k \phi(x,0)\epsilon^{\frac{1}{d}}(s_{cd}x^{c+kd} + x^{c+kd+1}g(x))^{-\frac{1}{d}}\,dx$$
$$+O(\epsilon^{\eta'}\int_{\epsilon^{\frac{1}{d'(k+1)}}}^{1} x^k |\phi(x,0)|\epsilon^{\frac{1}{d}}(s_{cd}x^{c+kd} + x^{c+kd+1}g(x))^{-\frac{1}{d}}\,dx) \tag{5.28}$$
Because $\eta' > 0$, the ratio of the second term to $\epsilon^{\frac{1}{d}}$ goes to zero as $\epsilon > 0$ (assuming the integral is finite, which we will see shortly). Thus the second term does not contribute to $\lim_{\epsilon\to 0} \frac{I''_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}}$ and we have
$$\lim_{\epsilon\to 0} \frac{I''_{S,\phi}(\epsilon)}{\epsilon^{\frac{1}{d}}} = \int_0^1 x^k (s_{cd}x^{c+kd} + x^{c+kd+1}g(x))^{-\frac{1}{d}}\phi(x,0)\,dx$$
$$= \int_0^1 (s_{cd}x^c + x^{c+1}g(x))^{-\frac{1}{d}}\phi(x,0)\,dx \tag{5.29}$$
Since $|s_{cd}x^{c+kd} + x^{c+kd+1}g(x)| > \frac{1}{2}x^{c+kd}$ and $c < d$ the integral (5.29) is finite as needed. Going back to the definition of $g(x)$, $S(x,y) = (s_{cd}x^c + x^{c+1}g(x))y^d + O(y^{d+1})$. As a result, (5.29) translates into the part of equation (1.8) coming from the upper right-hand quadrant. (Recall that (5.29) is for $s_{cd} > 0$ and that the limit is zero when $s_{cd} < 0$). Adding the analogous expressions from the remaining three quadrants gives (1.8) and we are done.

6. Proof of Theorem 1.6b.

Suppose $S(x,y)$ is a smooth phase function in case 1 superadapted coordinates, and $\phi(x,y)$ is nonnegative with $\phi(0,0)>0$. Let $\psi(t)$ be a nonnegative function in $C_c(\mathbf{R})$ such that $\psi(t)>1$ on $[-1,1]$. Then by (1.13a), if j is sufficiently large we have

$$|\int \psi(2^j S(x,y))\phi(x,y)\,dx\,dy| > |\int_{\{(x,y):|S(x,y)|<2^{-j}\}} \phi(x,y)\,dx\,dy| > A_{S,\phi}2^{-\frac{j}{d}} \tag{6.1}$$

We also have

$$\int \psi(2^j S(x,y))\phi(x,y)\,dx\,dy = 2^{-j}\int (\int \hat{\psi}(2^{-j}\lambda)e^{i\lambda S(x,y)}d\lambda)dx\,dy$$

$$= 2^{-j}\int \hat{\psi}(2^{-j}\lambda)(\int e^{i\lambda S(x,y)}\phi(x,y)\,dx\,dy)d\lambda = 2^{-j}\int \hat{\psi}(2^{-j}\lambda)J_{S,\phi}(\lambda)\,d\lambda \tag{6.2}$$

In order to prove (1.16a), we argue by contradiction. Suppose that we were in the setup of (1.16a) but $\limsup_{\lambda\to\infty}|\frac{J_{S,\phi}(\lambda)}{\lambda^{-\frac{1}{d}}}|=0$. Then for any $\delta>0$, we may let M_δ be such that for $|\lambda|>M_\delta$ we have $|J_{S,\phi}(\lambda)|<\delta|\lambda|^{-\frac{1}{d}}$. We then have

$$|2^{-j}\int \hat{\psi}(2^{-j}\lambda)J_{S,\phi}(\lambda)\,d\lambda| =$$

$$|2^{-j}\int_{|\lambda|<M_\delta} \hat{\psi}(2^{-j}\lambda)J_{S,\phi}(\lambda)\,d\lambda + 2^{-j}\int_{|\lambda|>M_\delta} \hat{\psi}(2^{-j}\lambda)J_{S,\phi}(\lambda)\,d\lambda|$$

$$< C2^{-j}M_\delta + 2^{-j}\delta\int_{|\lambda|>M_\delta} |\hat{\psi}(2^{-j}\lambda)||\lambda|^{-\frac{1}{d}}\,d\lambda \tag{6.3}$$

In turn, equation (6.3) is bounded by

$$C2^{-j}M_\delta + 2^{-j}\delta\int_{\mathbf{R}} |\hat{\psi}(2^{-j}\lambda)||\lambda|^{-\frac{1}{d}}\,d\lambda$$

$$= C2^{-j}M_\delta + 2^{-\frac{j}{d}}\delta\int_{\mathbf{R}} |\hat{\psi}(\lambda)||\lambda|^{-\frac{1}{d}}d\lambda$$

$$< C2^{-j}M_\delta + C'\delta 2^{-\frac{j}{d}} \tag{6.4}$$

When j is sufficiently large, (6.4) is at most $2C'\delta 2^{-\frac{j}{d}}$. On the other hand, by (6.1), it must also be at least $A_{S,\phi}2^{-\frac{j}{d}}$. This gives a contradiction if δ were chosen less than $\frac{A_{S,\phi}}{2C'}$. This contradiction implies that $\limsup_{\lambda\to\infty}|\frac{J_{S,\phi}(\lambda)}{\lambda^{-\frac{1}{d}}}|$ is in fact positive, giving (1.16a).

That the lim sup of (1.16c) is positive is proven from (1.13c) exactly as (1.16a) is proven from (1.13a), so we do not include a proof here. Since it holds for all $\delta>0$ the lim sup is automatically infinite. Equation (1.16b) is proved similarly to (1.16a), using (1.13b) in place of (1.13a). Namely, suppose $S(x,y)$ is a smooth phase function in case 2 superadapted coordinates, and $\phi(x,y)$ is a nonnegative function with $\phi(0,0) > 0$. Using (1.13b) we have

$$|\int \psi(2^j S(x,y))\phi(x,y)\,dx\,dy| > |\int_{\{(x,y):|S(x,y)|<2^{-j}\}} \phi(x,y)\,dx\,dy| > A_{S,\phi}\frac{j}{d}2^{-\frac{j}{d}} \tag{6.5}$$

Exactly as above we also have

$$\int \psi(2^j S(x,y))\phi(x,y)\,dx\,dy = 2^{-j}\int \hat{\psi}(2^{-j}\lambda)J_{S,\phi}(\lambda)\,d\lambda \tag{6.6}$$

Proceeding by contradiction again, suppose (1.16b) does not hold. Therefore for every $\delta > 0$ there is some L_δ such that for $|\lambda| > L_\delta$ we have $|J_{S,\phi}(\lambda)| < \delta|\lambda|^{-\frac{1}{d}}\ln|\lambda|$. Analogous to (6.3) we have

$$\begin{aligned} 2^{-j}|\int \hat{\psi}(2^{-j}\lambda)J_{S,\phi}(\lambda)\,d\lambda| \;&<\; C2^{-j}L_\delta + 2^{-j}\delta\int_{|\lambda|>L_\delta} |\hat{\psi}(2^{-j}\lambda)||\lambda|^{-\frac{1}{d}}\ln|\lambda|\,d\lambda \\ &\leq C2^{-j}L_\delta + 2^{-j}\delta\int_{\mathbf{R}} |\hat{\psi}(2^{-j}\lambda)||\lambda|^{-\frac{1}{d}}\ln|\lambda|\,d\lambda \end{aligned} \tag{6.7}$$

Changing variables, this in turn is equal to

$$\begin{aligned} &C2^{-j}L_\delta + 2^{-\frac{j}{d}}\delta\int_{\mathbf{R}} |\hat{\psi}(\lambda)||\lambda|^{-\frac{1}{d}}\ln(2^j|\lambda|)d\lambda \\ &= C2^{-j}L_\delta + 2^{-\frac{j}{d}}\delta\int_{\mathbf{R}} |\hat{\psi}(\lambda)||\lambda|^{-\frac{1}{d}}\ln|\lambda|\,d\lambda + j2^{-\frac{j}{d}}\ln(2)\delta\int_{\mathbf{R}} |\hat{\psi}(\lambda)||\lambda|^{-\frac{1}{d}}\,d\lambda \\ &< C2^{-j}L_\delta + C''\delta j2^{-\frac{j}{d}} \end{aligned} \tag{6.8}$$

If j is sufficiently large, (6.4) is at most $2C''\delta j2^{-\frac{j}{d}}$, while by (6.5) it is at least $A_{S,\phi}\frac{j}{d}2^{-\frac{j}{d}}$. This is a contradiction if $\delta < \frac{A_{S,\phi}}{2dC''}$. Therefore $\limsup_{\lambda\to\infty}|\frac{J_{S,\phi}(\lambda)}{\ln(\lambda)\lambda^{-\frac{1}{d}}}|$ must in fact be positive. Hence we have (1.16b) and we are done.

7. Superadapted coordinates.

In this section we prove the existence of superadapted coordinates for smooth phase functions. Here we always assume $S(x,y)$ is a smooth phase function defined on a neighborhood of the origin such that $S(0,0) = 0$ and $S(x,y)$ has nonvanishing Taylor expansion at the origin. The three cases of superadapted coordinates can be written as follows:

Case 1. The bisectrix intersects $N(S)$ in the interior of a bounded edge e and any real zero $r \neq 0$ of $S_e(1,y)$ or $S_e(-1,y)$ has order less than $d(S)$.

Case 2. The bisectrix intersects $N(S)$ at a vertex (d,d) and if e is a compact edge of $N(S)$ containing (d,d) then any real zero $r \neq 0$ of $S_e(1,y)$ or $S_e(-1,y)$ has order less than $d(S)$.

Case 3. The bisectrix intersects $N(S)$ in the interior of one of the unbounded edges.

Lemma 7.0. Any superadapted coordinate system is adapted.

Proof. By the main theorem of [G1], if U is a small enough neighborhood of the origin and ϵ_0 denotes the supremum of the numbers ϵ for which $\int_U |S|^{-\epsilon}$ is finite, then $d(S) \leq \frac{1}{\epsilon_0}$, with $d(S) = \frac{1}{\epsilon_0}$ in cases 1, 2, and 3. Hence if one is in cases 1, 2, or 3, one is in adapted coordinates.

Case 2 has some special features for which the following preliminary lemma will be useful. Related lemmas occur in [PSSt] and [V].

Lemma 7.1. Suppose the bisectrix intersects $N(S)$ at a vertex (d,d) but is not in superadapted coordinates. Correspondingly, let e be a compact edge of $N(S)$ containing (d,d) such that $S_e(1,y)$ or $S_e(-1,y)$ has a zero of order d or greater. If (d,d) is the upper vertex of e, then $S_e(x,y)$ is of the form $cx^\alpha(\frac{y}{x^m} - r)^d$ for positive integers α, m and some nonzero c, r. If (d,d) is the lower vertex of e, then $S_e(x,y)$ has the analogous form $c'y^{\alpha'}(\frac{x}{y^{m'}} - r')^d$. with $c', r' \neq 0$ and α', m' positive integers.

Proof. We first consider the case where (d,d) is the upper vertex of e. Write the equation of e as $t_1 + mt_2 = \alpha$. We will show that these values of m and α work. Note that if $s_{ab}x^ay^b$ appears in $S_e(x,y)$ then $a + mb = \alpha$. We factor out x^α, writing $S_e(x,y) = x^\alpha T_e(x,y)$. Each term of $T_e(x,y)$ is now of the form $t_{ab}x^{a-\alpha}y^b$ with $(a-\alpha)+mb = 0$ or $(a-\alpha) = -mb$. Thus we have

$$t_{ab}x^{a-\alpha}y^b = t_{ab}(\frac{y}{x^m})^b \tag{7.1}$$

Conequently for a polynomial $P(z)$, we can write

$$S_e(x,y) = x^\alpha P(\frac{y}{x^m}) \tag{7.2}$$

Plugging in $x = 1$ or -1, we see that $P(y)$ is a polynomial of degree d with a real zero r of order d or greater. Therefore we must have $P(y) = c(y-r)^d$ for some $c \neq 0$. Hence $S_e(x,y) = cx^\alpha(\frac{y}{x^m} - r)^d$. Since x can only appear to integer powers, m must be an integer and therefore α is as well. Also, since it comes from an edge $S_e(x,y)$ contains multiple terms. Hence $r \neq 0$ and we are done with the case where (d,d) is the upper vertex of e.

The case where (d,d) is the upper vertex of e is done similarly. Since $S_e(1,y)$ or $S_e(-1,y)$ has a zero $r \neq 0$ of order d or more, $S_e(x,y)$ has zeroes of order d along some curve $y = rx^m$. Hence $S_e(x,1)$ or $S_e(-x,1)$ has a zero not at the origin of order d or more and the above argument applies, reversing the roles of the x and y variables.

Lemma 7.2. Suppose one is not in superadapted coordinates. Suppose e is an edge of $N(S)$ intersecting the bisectrix in its interior with equation $t_1 + mt_2 = \alpha$ for $m \geq 1$ such that $S_e(1, y)$ has a zero $r \neq 0$ of order $k \geq d(S)$. Then m is an integer and both $S_e(1, y)$ and $S_e(-1, y)$ have a zero of order k not at the origin.

Proof. Exactly as (7.2), there is some polynomial $Q(y)$ such that for $x > 0$ we have

$$S_e(x, y) = x^\alpha Q(\frac{y}{x^m}) \tag{7.2$'$}$$

Plugging in $x = 1$, we see that $Q(y) = S_e(1, y)$.

We now show that m must in fact be an integer. To see this, note that if m were not an integer, then the degrees of the powers of y appearing in $S_e(1, y)$ would have to be separated by at least 2. Hence $S_e(1, y)$ would have to be of the form $y^\beta R(y^c)$ for some $\beta \geq 0$, $c \geq 2$, where R is a polynomial. Next, since $(d(S), d(S))$ is on $N(S)$, we have $\alpha = (1+m)d(S)$. Since $m > 1$ when $m \geq 1$ is not an integer, the maximum possible value of y on the line $t_1 + mt_2 = (1+m)d(S)$ for $t_1, t_2 \geq 0$ is $\frac{m+1}{m} d(S) < 2d(S)$. Thus the degree of $y^\beta R(y^c)$ is less than $2d(S)$, and hence the degree of $R(y)$ is less than $\frac{2d(S)}{c} \leq d(S)$. Hence the zeroes of $R(y)$ are of order less than $d(S)$, implying the zeroes of of $S_e(1, y) = y^\beta R(y^c)$ other than $y = 0$ are of order less than $d(S)$. This contradicts our assumption that $S_e(1, y)$ has a zero $r \neq 0$ of order $k \geq d(S)$ and we conclude that m is an integer.

Note that since m is an integer so is α. Consequently by (7.2$'$) if m is even, then $S_e(1, y) = \pm S_e(-1, y)$, while if m is odd one has $S_e(1, y) = \pm S_e(-1, -y)$. Hence in either case both $S_e(1, y)$ and $S_e(-1, y)$ have a zero of order k not at the origin. This completes the proof of Lemma 7.2.

The next lemma is the crux of this section. To set it up, suppose $S(x, y)$ is not in superadapted coordinates and the bisectrix intersects the interior of an edge e. Then since $S_e(1, y)$ or $S_e(-1, y)$ has a zero $r \neq 0$ of order $k \geq d(S)$, $S_e(x, y)$ has zeroes of order k at any point on a curve of the form $y = rx^m$. Hence $S_e(x, 1)$ or $S_e(x, -1)$ has a zero of order k away from the origin. Thus we may switch the roles of the x and y axes if we want and assume e has equation $t_1 + mt_2 = \alpha$ for $m \geq 1$; by Lemma 7.2 m is an integer and $S_e(1, y)$ has a zero $r \neq 0$ of order k.

Lemma 7.3. Suppose $S(x, y)$ is not in superadapted coordinates and the bisectrix intersects the interior of an edge e. As described above, switching the x and y axes if necessary, write the equation of e as $t_1 + mt_2 = \alpha$ for an integer $m \geq 1$ and assume $S_e(1, y)$ has a zero $r \neq 0$ of order $k \geq d(S)$. Then there is a coordinate change of the form $(x, y) \to (x, y + a(x))$ such that $a(x)$ is smooth with $a(0) = 0$, after which one is either case 1 or 3 of superadapted coordinates, or the following more general version of case 2:

Case 2'. The bisectrix intersects $N(S)$ at a vertex (d, d).

Proof. Let $Q(y) = S_e(1, y)$, and let (p, q) denote the upper vertex of the edge e; necessarily

$q > d(S)$. We will find a smooth function $a(x)$ such that $S'(x,y) = S(x, y + a(x))$ is in one of the following two mutually exclusive categories.

Category 1: $S'(x,y)$ is either in case 1, case 2', or case 3.

Category 2: The bisectrix intersects the interior of an edge e' of $N(S')$ with equation $t_1 + m't_2 = \alpha'$, $m' > m \geq 1$, such that the upper vertex (p', q') of e' satisfies $q' < q$ and such that $S'(x,y)$ is not in case 1. (In particular by Lemma 7.2 $S'_{e'}(1,y)$ has a zero of order $\geq d(S')$).

Lemma 7.3 will then follow; there can be at most q iterations of category 2.

We first consider the case where $k < q$. The function $Q(y+r)$ has a root at $y = 0$ of order k. We choose $a(x) = rx^m$ and define $S'(x,y) = S(x, y + a(x)) = S(x, y + rx^m)$. Note that $t_1 + mt_2 = \alpha$ is a supporting line of $N(S')$ as it was for $N(S)$, and that there is an edge E of $N(S')$ on this line whose upper vertex is (p, q). Observe that $S'_E(x,y) = S_e(x, y + rx^m) = x^\alpha Q(\frac{y}{x^m} + r)$. Since Q has a zero of order k at r, the lowest power of y appearing in $S'_E(x,y)$ is y^k and therefore E's lower vertex is at a point (j, k) for some j. Since both vertices of E have y-coordinates at least $d(S)$, they are both in the portion of the line $t_1 + mt_2 = \alpha$ on or above $(d(S), d(S))$. Thus the edge E lies wholly on or above the bisectrix. If the bisectrix intersects $N(S')$ at a vertex or inside the horizontal or vertical rays, one is in Category 1. Otherwise, it must intersect $N(S')$ in the interior of an edge e' whose upper vertex is either (j, k) or a lower vertex. And because $t_1 + mt_2 = \alpha$ is a supporting line for $N(S')$ and e' lies below E, e' will have equation $t_1 + m't_2 = \alpha'$ for some $m' > m \geq 1$. Thus we are either in case 1 superadapted coordinates (which is in Category 1) or we are in Category 2. Hence when $k < q$, $S'(x,y)$ is in either Category 1 or 2 and we are done.

It remains to consider the situation where r is a zero of $Q(y)$ of order q. In this case we have $Q(y) = c(y - r)^q$ for some c. For a large integer n we expand $S(x,y)$ as

$$S(x,y) = cx^\alpha(\frac{y}{x^m} - r)^q + T_n(x,y) + E_n(x,y) \tag{7.3}$$

Here the polynomial $T_n(x,y)$ are the terms of S's Taylor expansion with exponents less than n. For all $0 \leq \beta, \gamma < n$ one has

$$|\frac{\partial^{\beta+\gamma} E_n}{\partial x^\beta \partial y^\gamma}(x,y)| < C(|x|^{n-\beta} + |y|^{n-\gamma}) \tag{7.4}$$

Note that

$$S(x, x^m y) = cx^\alpha(y - r)^q + x^{\alpha+1}T'_n(x,y) + E_n(x, x^m y) \tag{7.5}$$

Here $T'_n(x,y)$ is also a polynomial. We define $s(x,y) = \frac{S(x,x^m y)}{x^\alpha}$, so that

$$s(x,y) = c(y - r)^q + xT'_n(x,y) + x^{-\alpha}E_n(x, x^m y) \tag{7.6}$$

We claim that the function $s(x,y)$ is smooth on a neighborhood of $(0,r)$. Off the y-axis smoothness holds because $S(x,y)$ is smooth. One can show that a given derivative of $s(x,y)$ exists when $x = 0$ and equals that of $c(y-r)^q + xT_n'(x,y)$ for large enough n by examining the difference quotient of a one-lower order derivative of (7.6), inductively assuming this lower-order derivative exists and has the right value when $x = 0$. Equation (7.4) ensures that the difference quotient of the lower derivative of $x^{-\alpha}E_n(x, x^m y)$ tends to zero as x goes to zero. We conclude that $s(x,y)$ is smooth on a neighborhood of $(0,r)$.

We next use the smooth implicit function theorem on $\frac{\partial^{q-1}s}{\partial y^{q-1}}$ and find a smooth function $k(x)$ defined in a neighborhood of $x = 0$ such that $k(0) = r$ and $\frac{\partial^{q-1}s}{\partial y^{q-1}}(x, k(x)) = 0$. Transferring this back to $S(x,y)$ we have

$$\frac{\partial^{q-1}S}{\partial y^{q-1}}(x, x^m k(x)) = 0 \tag{7.7}$$

Thus if we let $a(x) = x^m k(x)$ and $S'(x,y) = S(x, y + x^m k(x))$, for all x we consequently have

$$\frac{\partial^{q-1}S'}{\partial y^{q-1}}(x, 0) = 0 \tag{7.8}$$

Thus for every a the Taylor series coefficient $S'_{a\,q-1}$ is zero.

Next, since $t_1 + mt_2 = \alpha$ is a supporting line for $N(S)$, this line is also a supporting line for $N(S')$ and intersects $N(S')$ at the single vertex (p,q). If $S'(x,y)$ is in Category 1 we have nothing to prove, so we may assume we are not in Category 1. Let e' denote the edge of $N(S')$ intersecting the bisectrix and denote its equation by $t_1 + m't_2 = \alpha'$. Since e' lies within the set $t_1 + mt_2 \geq \alpha$ and is no higher than the vertex (p,q) of $N(S')$ that is on the supporting line $t_1 + mt_2 = \alpha$, we have $m' > m \geq 1$. If the upper vertex (p', q') of e' satisfies $q' < q$, one is in Category 2 and we are done. So we assume this upper vertex is (p,q) itself.

If $S'_{e'}(1,y)$ has a real zero $r' \neq 0$ of order $k < q$, one is in the situation above (7.3); there is a smooth $b(x)$ such that $S'(x, y + b(x)) = S(x, y + a(x) + b(x))$ is in Category 1 or 2 as needed. The only other possibility is that $S'_{e'}(1,y)$ has a single zero $r' \neq 0$ of order q. But this cannot happen. For this would imply $S'_{e'}(x,y) = c'x^{\alpha'}(\frac{y}{x^{m'}} - r')^q$ has a nonvanishing y^{q-1} term. Consequently, for some a the Taylor series coefficient $S'_{a\,q-1}$ would be nonzero, contradicting (7.8). Thus the case where $S'_{e'}(1,y)$ has a single zero of order q does not occur, and we are done with the proof of Lemma 7.3.

The final step of the proof of the existence of superadapted coordinates is the following.

Lemma 7.4. Suppose one is in case 2'; that is, the bisectrix intersects $N(S)$ at a vertex (d,d). Then there exists a smooth coordinate change fixing the origin after which one is in case 2 of superadapted coordinates.

Proof. Suppose the bisectrix intersects $N(S)$ at a vertex (d,d) but $S(x,y)$ is not in superadapted coordinates. Then (d,d) is on an edge e of $N(S)$ such that $S_e(1,y)$ or $S_e(-1,y)$ has a zero $r \neq 0$ of order d or greater. By Lemma 7.1, $S_e(x,y)$ is of the form $cx^\alpha(\frac{y}{x^m} - r)^d$ or $cy^\alpha(\frac{x}{y^m} - r)^d$ for positive integers α, m and some nonzero c, r, the first corresponding to the case where e lies below (d,d) and the second corresponding to where it lies above (d,d). Switching axes if necessary, assume that $S_e(x,y) = cx^\alpha(\frac{y}{x^m} - r)^d$. One can argue as in (7.3) – (7.8) to obtain a function of the form $T(x,y) = S(x, y + a(x))$ such that the bisectrix intersects $N(T)$ at the vertex (d,d), but such that as in the last paragraph of the proof of Lemma 7.3 if (d,d) is the upper vertex of an edge e' of $N(T)$ then $T_{e'}(1,y)$ does not have a zero of order d (or greater). We also must have that $T_{e'}(-1,y)$ has no zero of order d or greater; for if it did by Lemma 7.1 we could write $T_{e'}(x,y)$ in the form $cx^\alpha(\frac{y}{x^m} - r)^d$, which would imply $T_{e'}(1,y)$ also has such a zero, a contradiction.

There still remains the possibility that (d,d) is the lower vertex of a compact edge f of $N(T)$ such that $T_f(1,y)$ has a zero of order order d or greater. By Lemma 7.1, if this happens $T_f(x,y)$ is of the form $cy^a(\frac{x}{y^g} - r)^d$ where g is an integer. Again using the argument from (7.3) onwards, this time reversing the roles of the x and y variables, there is a smooth coordinate change of the form $\beta : (x,y) \rightarrow (x - y^g h(x,y), y)$ such that if one denotes $T \circ \beta$ by S', then if (d,d) is the lower vertex of an edge f' of $N(S')$ then $S'_{f'}(x,1)$ and $S'_{f'}(x,-1)$ do not have any zeroes of order d or greater other than $x = 0$. This means $S'_{f'}(1,y)$ and $S'_{f'}(-1,y)$ also have no such zero. For if one of the functions did, $S'_{f'}(x,y)$ would have zeroes of order d along some curve $y = sx^n$, which would imply either $S'_{f'}(x,1)$ or $S'_{f'}(x,-1)$ had zeroes of order d or greater away from the origin, a contradiction.

Furthermore, the slope of e' is of the form $-\frac{1}{m'}$ for m' a positive integer and $g > \frac{1}{m'}$. As a result, e' is an edge of $N(S')$ containing (d,d) and the coordinate change β did not change any of the terms of $T_{e'}(x,y)$. Thus $S'_{e'}(x,y) = T_{e'}(x,y)$ and all zeroes of $S'_{e'}(1,y)$ and $S'_{e'}(-1,y)$ other than $y = 0$ have order less than d. Hence we are in superadapted coordinates and the proof of Theorem 7.4 is complete.

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