

Stability of Oscillatory Integral Asymptotics in Two Dimensions

Michael Greenblatt

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1. Definitions and Background Information.

Let $S(x, y)$ be a smooth function defined on a neighborhood of the origin, and let $\phi(x, y)$ be a smooth cutoff function supported in a sufficiently small neighborhood of the origin. The main object we are looking at is the oscillatory integral

$$J_{S,\phi}(\lambda) = \int_{\mathbf{R}^2} e^{i\lambda S(x,y)} \phi(x, y) dx dy \quad (1.1)$$

Such integrals come up in a variety of settings in analysis and mathematical physics. For a given $S(x, y)$ we are interested in the asymptotic behavior of $|J_{S,\phi}(\lambda)|$ as $\lambda \rightarrow \infty$. As in the author's earlier paper [G1], in this paper we are focusing on how the asymptotic behavior is affected by perturbations of $S(x, y)$. We will always assume $\lambda > 0$, as multiplying λ by -1 just causes $J_{S,\phi}(\lambda)$ to be replaced by its complex conjugate.

If the Taylor series coefficients of $S(x, y)$ at the origin are all zero, then one can readily show directly that for no $\delta > 0$ is $|J_{S,\phi}(\lambda)|$ bounded by $C\lambda^{-\delta}$, so we do not consider such situations. We also may always assume that $S(0, 0) = 0$; in the integrand (1.1) one may always factor out a $e^{i\lambda S(0,0)}$, effectively replacing $S(x, y)$ by $S(x, y) - S(0, 0)$ and $|J_{S,\phi}(\lambda)|$ will not be affected. It also does no harm to assume that $\nabla S(0, 0) = 0$. For if $\nabla S(0, 0) \neq 0$, then in (1.1) one can integrate by parts arbitrarily many times to show that $J_{S,\phi}(\lambda)$ decays faster than any $C_n \lambda^{-n}$, and the same is true if S is replaced by a small perturbation of S .

Thus throughout this paper we will assume that the following conditions hold.

$$S(0, 0) = 0 \quad \nabla S(0, 0) = 0 \quad (1.2)$$

By resolution of singularities (see [AGV] ch. 7 for details), for a given real-analytic nonconstant $S(x, y)$ satisfying (1.2), there is a positive δ and an integer $p = 0$ or 1 such that for any $\phi(x, y)$ with sufficiently small support, as $\lambda \rightarrow \infty$ one has an asymptotic development

$$J_{S,\phi}(\lambda) = A_{S,\phi} \lambda^{-\delta} (\ln \lambda)^p + o(\lambda^{-\delta} (\ln \lambda)^p) \quad (1.3)$$

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This (δ, p) is optimal in the sense that $A_{S,\phi} \neq 0$ as long as $\phi(x, y)$ is nonnegative (or nonpositive) with $\phi(0, 0) \neq 0$. In [IM2] it is shown that (1.3) still holds for general smooth $S(x, y)$ so long as $S(x, y)$ can be put in Case 1 or Case 2 superadapted coordinates. (See Definitions 1.9 and 1.10 below.) They also show that in these situations $A_{S,\phi}$ is always a multiple of $\phi(0, 0)$. In the appendix, we will give explicit formulas for such $A_{S,\phi}$ which we will use in the proof of Theorem 2.3. These will be the same formulas that were shown to hold in the real-analytic case in [G2].

In the remaining smooth finite type situations, namely when $S(x, y)$ has Case 3 superadapted coordinates, one does not necessarily have asymptotics (1.3). However, in [G2] it is shown there is some $\delta > 0$ independent of ϕ such that for some constant $A_{S,\phi}$ one has

$$|J_{S,\phi}(\lambda)| \leq A_{S,\phi} \lambda^{-\delta} \quad (1.4)$$

This δ is such that for any $\delta' > \delta$, so long as $\phi(x, y)$ is nonnegative with $\phi(0, 0) \neq 0$ there is no constant $B_{S,\phi,\delta'}$ such that asymptotically one has

$$|J_{S,\phi}(\lambda)| \leq B_{S,\phi,\delta'} \lambda^{-\delta'} \quad (1.5)$$

In view of the above, we make the following definitions as in [G1].

Definition 1.1. The *oscillation index* of $S(x, y)$ at the origin is the δ for which (1.3) holds when $S(x, y)$ has Case 1 or 2 superadapted coordinates, or for which (1.4) – (1.5) holds when $S(x, y)$ has Case 3 superadapted coordinates.

By [G2] and [IM2], $p = 0$ in (1.3) when $S(x, y)$ has Case 1 superadapted coordinates or Case 3 superadapted coordinates in the real-analytic case, and $p = 1$ if it has Case 2 superadapted coordinates. In view of this and (1.4) – (1.5), we make the following definition.

Definition 1.2. The *multiplicity* of the oscillation index is defined to be 1 if $S(x, y)$ has Case 2 superadapted coordinates, and is defined to be zero otherwise.

Definition 1.3. $S(x, y)$ is said to be of *oscillatory type* $(-\delta, p)$ if $S(x, y)$ has oscillation index δ with multiplicity p .

We order the oscillatory types of phase functions lexicographically; we say that $(-\delta_1, p_1) < (-\delta_2, p_2)$ if $-\delta_1 < -\delta_2$ or if $-\delta_1 = -\delta_2$ and $p_1 < p_2$. Thus a smaller type corresponds to faster oscillatory integral decay.

Newton polygons and adapted coordinates.

We now provide some relevant definitions that will be used throughout the paper.

Definition 1.4. Assume that $S(x, y)$ is of finite type at the origin, and let $S(x, y) = \sum_{a,b} s_{ab} x^a y^b$ denote the Taylor expansion of $S(x, y)$ at the origin. For any (a, b) for which

$s_{ab} \neq 0$, let Q_{ab} be the quadrant $\{(x, y) \in \mathbf{R}^2 : x \geq a, y \geq b\}$. Then the *Newton polygon* $N(S)$ of $S(x, y)$ is defined as the convex hull of the union of all Q_{ab} .

A Newton polygon consists of finitely many (possibly zero) bounded edges of negative slope as well as an unbounded vertical ray and an unbounded horizontal ray.

Definition 1.5. The *Newton distance* $d(S)$ of $S(x, y)$ is defined to be $\inf\{x : (x, x) \in N(S)\}$.

The line $x = y$ comes up so frequently in this subject it has its own name.

Definition 1.6. The *bisectrix* is the line with equation $x = y$.

The connection between Newton polygons and the asymptotics of oscillatory integrals and sublevel set measures is as follows. It was shown by Varchenko [V] in the real-analytic case, and by Ikromov-Müller [IM1] in the smooth case, that for any $S(x, y)$ one can do a real-analytic (resp. smooth) coordinate change fixing the origin of the form $(x, y) \rightarrow (x - \phi(y), y)$ or $(x, y) \rightarrow (x, y - \phi(x))$, such that in the new coordinates the oscillation index δ of $S(x, y)$ satisfies $\delta = \frac{1}{d(S)}$. The multiplicity is equal to 1 if the coordinate change can be made such that afterwards the bisectrix intersects $N(S)$ at a vertex. The multiplicity is zero otherwise. Furthermore, in any coordinate system with the same origin, $d(S) \leq \frac{1}{\delta}$. Thus the coordinate systems where $d(S)$ are maximal are important to a discussion of asymptotics, and correspondingly we have the following definition.

Definition 1.7. A local coordinate system near $(0, 0)$ is called *adapted* if $d(S) = \frac{1}{\delta}$ in this coordinate system.

We now come to the definition of superadapted coordinate systems, which are a refinement of the notion of adapted coordinate systems.

Definition 1.8. For a given edge or vertex e of $N(S)$, let $S_e(x, y) = \sum_{(a,b) \in e} s_{ab} x^a y^b$ denote the sum of all terms of the Taylor expansion of $S(x, y)$ on e .

Definition 1.9. A local coordinate system near the origin is called *superadapted* if for any compact edge e containing the intersection $(d(S), d(S))$ of the bisectrix with $N(S)$, the functions $S_e(1, y)$ and $S_e(-1, y)$ both do not have zeroes of order $d(S)$ or greater other than possibly $y = 0$.

In [G2] it is shown that any smooth function can be put in superadapted coordinates. Also, as one would expect by symmetry, one can use the functions $S_e(x, 1)$ and $S_e(x, -1)$ in place of $S_e(1, y)$ and $S_e(-1, y)$. We select the y -variable here for definiteness.

As in [G2], we divide functions that are in superadapted coordinates into three cases:

Definition 1.10. A function $S(x, y)$ in superadapted coordinates is said to be in Case 1 if the bisectrix intersects the Newton polygon $N(S)$ in the interior of a compact edge. It is said to be in Case 2 if the bisectrix intersects $N(S)$ at a vertex, and in Case 3 if the bisectrix intersects $N(S)$ in the interior of the horizontal or vertical ray.

2. Theorems and a little history

A notable work on the effect of phase perturbations on the two-dimensional oscillation index and its multiplicity was given by Karpushkin in [K1]-[K2]. Here he proved estimates that were uniform in a strong way, for the real-analytic case. Specifically, Karpushkin's theorem is as follows. Let D_r denote the open disk in \mathbf{R}^2 of radius r centered at the origin, and E_r the open disk in \mathbf{C}^2 of radius r centered at the origin. For a function $f(x, y)$ real-analytic on D_r , let $\tilde{f}(z_1, z_2)$ denote the unique holomorphic extension of $f(x, y)$ to E_r . Then Karpushkin's theorem for oscillatory integrals is as follows.

Theorem: ([K1]-[K2]) Suppose $S(x, y)$ is real-analytic on D_r satisfying (1.2) with oscillatory type $(-\delta, p)$. Then there is an $\eta > 0$, an $s < r$, and a positive constant C_S depending on $S(x, y)$ such that if $f(x, y)$ is real-analytic on D_s and $\tilde{f}(z_1, z_2)$ extends to a continuous function on \bar{E}_s with $|\tilde{f}(z_1, z_2)| < \eta$ for all $(z_1, z_2) \in \bar{E}_s$ then for all $\lambda > 0$ and all $\phi \in C_c^\infty(D_s)$ one has the estimate

$$|J_{S+f, \phi}(\lambda)| \leq C_{S, \phi} (1 + \lambda)^{-\delta} \ln(1 + \lambda)^p \quad (2.1)$$

The proofs in [K1]-[K2] as well as in his other papers used ideas from singularity theory, notably the theory of versal deformations, which is a way of converting arbitrary real analytic perturbations of $S(x, y)$ into a finite list of canonical forms which then can be analyzed individually.

Another method for dealing with stability of oscillatory integrals and sublevel set measures was introduced in [PSSt], where a slightly weaker version of Karpushkin's theorem is proved. [PSSt] uses a method which is sometimes referred to as the method of algebraic estimates. Some higher-dimensional theorems are also proved in [PSSt]; an example of Varchenko in [V] shows the full analogues are not feasible. It should be mentioned that for the case of linear perturbations of smooth functions in two dimensions, stability is proven in [IKeM]. In addition, there is an extensive body of research concerning analogous stability issues for complex analytic functions in several variables.

Our first theorem describes how the oscillatory type decreases or remains the same in a given direction, modulo finitely values, and gives criteria on a direction for the type to remain the same.

Theorem 2.1. Suppose a smooth $S(x, y)$ satisfies (1.2) and is in superadapted coordinates. Let $f(x, y)$ be a smooth function with nonvanishing Taylor expansion at $(0, 0)$ and satisfying (1.2). Let $(-\frac{\delta}{2}, p)$ denote the oscillatory type of $S(x, y)^2 + f(x, y)^2$. There is a finite subset $I \subset \mathbf{R}$ such that if $t \notin I$ then the oscillatory type of $S(x, y) + tf(x, y)$ is $(-\delta, p)$. This

type is less than or equal to the oscillatory type of $S(x, y)$, with equality holding in the following situations.

a) $S(x, y)$ is in Case 1 superadapted coordinates. If $x + my = c$ denotes the equation of the edge of $N(S)$ intersecting the bisectrix, then equality holds iff $N(f) \subset \{(x, y) \in \mathbf{R}^2 : x + my \geq c\}$.

b) $S(x, y)$ is in Case 2 superadapted coordinates. Then equality holds iff there are $\infty \geq m_1 > m_2 \geq 0$ and $c_1, c_2 > 0$ such that the lines $x + m_1 y = c_1$ and $x + m_2 y = c_2$ intersect at the vertex $(d(S), d(S))$ of $N(S)$ and such that $N(f) \subset \{(x, y) : x + m_1 y \geq c_1\} \cap \{(x, y) : x + m_2 y \geq c_2\}$. When $m_1 = \infty$, the first line is taken to be $y = d(S)$.

c) $S(x, y)$ is in Case 3 superadapted coordinates. Let $y = d(S)$ or $x = d(S)$ denote the equation of the infinite ray of $N(S)$ intersecting the bisectrix. Then equality holds iff $N(f) \subset \{(x, y) : y \geq d(S)\}$ or $N(f) \subset \{(x, y) : x \geq d(S)\}$ respectively.

Some motivation for Theorem 2.1, courtesy [DK] where they considered the complex analytic case, is as follows. Let $\frac{\delta}{2}$ denote the oscillatory index of $S(x, y)^2 + f(x, y)^2$ as above, and let D_r denote the open disk of radius r centered at the origin. Then by [G2] for example, $\frac{\delta}{2}$ is the supremum of the μ for which $\int_{D_r} (S(x, y)^2 + f(x, y)^2)^{-\mu} dx dy$ is finite for small r (this is true for any function). However, by the Cauchy-Schwarz inequality, for $v_1^2 + v_2^2 = 1$ one has

$$|v_1 S(x, y) + v_2 f(x, y)|^{-2\mu} \geq (S(x, y)^2 + f(x, y)^2)^{-\mu} \quad (2.2)$$

Integrating (2.2) with respect to x and y over D_r shows that the oscillatory index of $v_1 S(x, y) + v_2 f(x, y)$ is at most δ . Scaling this, we see that the same is true for any $S(x, y) + t f(x, y)$.

On the other hand, for any $\mu < \delta$ one has

$$\int_{v_1^2 + v_2^2 = 1} \left(\int_{D_r} |v_1 S(x, y) + v_2 f(x, y)|^{-2\mu} dx dy \right) = \int_{D_r} \left(\int_{v_1^2 + v_2^2 = 1} |v_1 S(x, y) + v_2 f(x, y)|^{-2\mu} \right) dx dy$$

Rotating the inner integral and using that $\delta \leq 1$ this becomes

$$\begin{aligned} & \int_{D_r} \left(\int_{v_1^2 + v_2^2 = 1} v_1^{-\mu} (S(x, y)^2 + f(x, y)^2)^{-\mu} \right) dx dy \\ & = C_\mu \int_{D_r} (S(x, y)^2 + f(x, y)^2)^{-\mu} dx dy \end{aligned} \quad (2.3)$$

The right hand side of (2.3) is finite for any $\mu < \delta$, so the oscillation index of $v_1 S(x, y) + v_2 f(x, y)$ is greater than or equal to δ for almost all (v_1, v_2) on the circle. Scaling this, one gets that the oscillation index of $S(x, y) + tf(x, y)$ is greater than or equal to δ for almost all t as well. In view of the above, we conclude the oscillation index of $S(x, y) + tf(x, y)$ is equal to δ for almost all t .

Thus Theorem 2.1 shows the oscillation index of $S(x, y) + tf(x, y)$ is equal to twice that of $S(x, y)^2 + f(x, y)^2$ for not just almost all t , but for all but finitely t , and furthermore there is agreement with regards to the multiplicity. In addition, sufficient and necessary conditions on $f(x, y)$ are provided for the oscillatory type of the generic $S(x, y) + tf(x, y)$ to be the same as that of $S(x, y)$. In the course of our arguments we will see that the exceptional values of t for a given $f(x, y)$ are determined "effectively", meaning that they are explicitly determined through a finite sequence of finding roots r of polynomials of one variable and coordinate changes of the form $(x, y) \rightarrow (x, y + rx^m)$.

The functions where equality holds in Theorem 2.1 are important for the other theorems of this paper. Thus we make the following definition.

Definition 2.2. For a given $S(x, y)$, a smooth function $f(x, y)$ is said to be a *good direction* if equality holds in Theorem 2.1 for $f(x, y)$.

Our second theorem provides a smooth analogue of the uniform estimates of Karpushkin's theorem, but only in good directions. For the Case 1 and Case 2 statements, we will make use of the norm $|f|_{r,N}$ on perturbation functions defined by

$$|f|_{r,N} = \sum_{0 \leq \alpha, \beta \leq N} \sup_{(x,y) \in D_r} |\partial_x^\alpha \partial_y^\beta f(x, y)| \quad (2.4)$$

For the Case 3 statement, which concerns perturbations in good directions in the real-analytic case, we use the following variant of the norm $|f|_{r,N}$. Without loss of generality, assume the bisectrix intersects the horizontal ray. Then a good direction $f(x, y)$ can be written as $f(x, y) = y^d F(x, y)$ for some real-analytic $F(x, y)$. We define the norm $\|f\|_{r,N}$ by

$$\|f\|_{r,N} = \sum_{0 \leq \alpha, \beta \leq N} \sup_{(x,y) \in D_r} |\partial_x^\alpha \partial_y^\beta F(x, y)| \quad (2.5)$$

Theorem 2.2. Suppose a smooth $S(x, y)$ satisfies (1.2) and is in superadapted coordinates. Let $(-\delta, p)$ denote the oscillatory type of $S(x, y)$.

Suppose $S(x, y)$ is in Case 1 or Case 2 superadapted coordinates. Then there are $r, N > 0$ such that for some constants C and η the following holds. For all $\lambda > 2$, all $\phi \in C_c^\infty(D_r)$, and all good directions f with $|f|_{r,N} < \eta$ we have

$$|J_{S+f,\phi}(\lambda)| < C \|\phi\|_{C^1(D_r)} \lambda^{-\delta} (\ln \lambda)^p \quad (2.6)$$

If $S(x, y)$ is real-analytic and in Case 3 superadapted coordinates, then there are $r, N > 0$ such that for some constants C and η the following holds. For all $\lambda > 2$, all $\phi \in C_c^\infty(D_r)$,

and all f real-analytic on a neighborhood of \bar{D}_r such that f is a good direction with $\|f\|_{r,N} < \eta$ we have

$$|J_{S+f,\phi}(\lambda)| < C\|\phi\|_{C^1(D_r)}\lambda^{-\delta} \quad (2.7)$$

Remark 1. By Karpushkin's results, in the real-analytic situation (2.6) – (2.7) hold in all directions if one uses the magnitude of the complex extension of a perturbation function as its norm.

Remark 2. For general smooth $S(x, y)$ in Case 3 superadapted coordinates, it can be shown the Case 1 and 2 statement (2.6) still holds for $\phi(x, y)$ supported in some $D_s \subset D_r$, where s depends on f . The author does not know if the full statement holds in the general smooth Case 3 situation.

Our third and final theorem shows that in Case 1 for all smooth phases, and in Case 3 for real-analytic functions, for some $\alpha > 0$ the coefficient $A_{S,\phi}$ in (1.3) is locally an α -Lipschitz function of f in the norms $|f|_{r,N}$ and $\|f\|_{r,N}$ respectively, again when f is restricted to good directions. (In other directions the coefficient abruptly drops to zero.)

Theorem 2.3. If $S(x, y)$ is in Case 1 of superadapted coordinates and r, η and N are as in Theorem 2.2, then there are $\alpha, B > 0$ such that if f_1 and f_2 are good directions with $|f_1|_{r,N}, |f_2|_{r,N} < \eta$, then the leading coefficient of the asymptotics (1.3) satisfies

$$|A_{S+f_2,\phi} - A_{S+f_1,\phi}| \leq B|f_2 - f_1|_{r,N}^\alpha |\phi(0,0)| \quad (2.8)$$

Suppose $S(x, y)$ is in Case 3 of superadapted coordinates and r, η and N are as in Theorem 2.2. There are $\alpha, B > 0$ such that if $f_1(x, y)$ and $f_2(x, y)$ are good directions real-analytic on a neighborhood of \bar{D}_r with $\|f_1\|_{r,N}, \|f_2\|_{r,N} < \eta$, then

$$|A_{S+f_2,\phi} - A_{S+f_1,\phi}| \leq B\|f_2 - f_1\|_{r,N}^\alpha \|\phi\|_{L^\infty} \quad (2.9)$$

Remark 1. The Case 2 analogue of (2.8) – (2.9) is actually false as simple examples show; take $S(x, y) = x^d y^d$ and then perturb by $f_1(x, y) = \epsilon x^d y^d$ and $f_2(x, y) = \epsilon x^e y^f$ where $e < d$ and $f > d$. However, if one restricts to $f(x, y)$ such that $N(f) \subset N(S)$, then the analogue does hold; this is an immediate consequence of the formulas in [G2] and the appendix of this paper.

Remark 2. There is no smooth Case 3 version of (2.9) since one does not necessarily have an asymptotic expansion for a smooth Case 3 phase.

Remark 3. The proof of Theorem 2.3 will reveal that one can take f_1 or f_2 to be the zero function in (2.8) and (2.9).

Remark 4. One also has an analogue of Theorem 2.3 where ϕ is also allowed to change; one just bounds $|A_{S+f,\phi_2-\phi_1}|$ using Theorem 2.2 and adds the result to (2.8) or (2.9).

3. Preliminary lemmas and the proof of Theorem 2.1.

For a polynomial $f(x)$, let $\text{ord}_x(f)$ denote the order of a zero of f at x , with $\text{ord}_x(f) = 0$ if $f(x) \neq 0$. We start by proving a lemma concerning polynomials in one variable.

Lemma 3.1. Suppose $p(x)$ and $q(x)$ are polynomials in one variable, neither identically zero. Let \mathbf{R}_0 denote $\mathbf{R} - \{0\}$, and let $m = \sup_{x \in \mathbf{R}_0} \min(\text{ord}_x(p), \text{ord}_x(q))$. Then there is a finite $I \subset \mathbf{R}$ such that if $t \notin I$, then $\text{ord}_x(p + tq) \leq \max(1, m)$ for all $x \in \mathbf{R}_0$.

Proof. The result is immediate if $p(x)$ and $q(x)$ are multiples of each other, so we assume this is not the case. Let J be the set $\{x \in \mathbf{R} : p(x)q'(x) - p'(x)q(x) = 0, q(x) = 0, \text{ or } p(x) = 0\}$. J is finite since $p(x)q'(x) - p'(x)q(x)$ can't be identically zero; if it were by the quotient rule for derivatives $(\frac{p}{q})'(x)$ would be zero for all x , contradicting the assumption that $p(x)$ and $q(x)$ are not multiples of each other.

I claim that for all $t \in \mathbf{R}$ the function $p(x) + tq(x)$ has no zero of order greater than 1 on $\mathbf{R} - J$. For suppose $p(x) + tq(x)$ had a zero of order at least 2 at some $x_0 \in \mathbf{R} - J$. Then $p(x_0)q'(x_0) = -tq(x_0)q'(x_0) = p'(x_0)q(x_0)$. So $p(x_0)q'(x_0) - p'(x_0)q(x_0) = 0$, contradicting that $x_0 \in \mathbf{R} - J$.

Now suppose x_0 is in the finite set J . Since Lemma 3.1 and our proof so far are symmetric in p and q , without loss of generality we may assume that $\text{ord}_{x_0}(p) \leq \text{ord}_{x_0}(q)$ (which may be zero.) Write $p(x) = (x - x_0)^{\text{ord}_{x_0}(p)}r(x)$ and $q(x) = (x - x_0)^{\text{ord}_{x_0}(q)}s(x)$. Thus we have

$$p(x) + tq(x) = (x - x_0)^{\text{ord}_{x_0}(p)}(r(x) + t(x - x_0)^{\text{ord}_{x_0}(q) - \text{ord}_{x_0}(p)}s(x)) \quad (3.1)$$

Let K be a closed interval centered at x_0 such that $r(x) \neq 0$ on K . Then letting $u(x) = \frac{s(x)}{r(x)}$, on K we may write

$$p(x) + tq(x) = r(x)(x - x_0)^{\text{ord}_{x_0}(p)}(1 + t(x - x_0)^{\text{ord}_{x_0}(q) - \text{ord}_{x_0}(p)}u(x)) \quad (3.2)$$

Here $u(x_0) \neq 0$. Consider the case that $\text{ord}_{x_0}(q) > \text{ord}_{x_0}(p)$. Note that there is an interval K centered at x_0 such that for any t the factor $1 + t(x - x_0)^{\text{ord}_{x_0}(q) - \text{ord}_{x_0}(p)}u(x)$ has at most two zeroes on K , each with multiplicity one, and $x = x_0$ cannot be a zero. Thus $p(x) + tq(x)$ has a zero of order $\text{ord}_{x_0}(p)$ at $x = x_0$, and no other possible zeroes of multiplicity greater than 1 on K . This is better than what we need.

Now consider the case where $\text{ord}_{x_0}(q) = \text{ord}_{x_0}(p)$. Then we write $u(x) = u_0 + (x - x_0)^m v(x)$ for some $m > 0$ and some $v(x)$ with $v(x_0) \neq 0$ (again we use that p and q are not multiples of each other). Now we have

$$p(x) + tq(x) = r(x)(x - x_0)^{\text{ord}_{x_0}(p)}(1 + tu_0 + t(x - x_0)^m v(x))$$

If we exclude $t = -\frac{1}{u_0}$, then exactly as in the $\text{ord}_{x_0}(q) > \text{ord}_{x_0}(p)$ situation there is some K centered at x_0 such that $p(x) + tq(x)$ has a zero of order $\text{ord}_{x_0}(p)$ at $x = x_0$, and no other possible zeroes of multiplicity greater than 1 on K . Thus regardless of whether or

not $\text{ord}_{x_0}(q) > \text{ord}_{x_0}(p)$, excluding one possible value of t if necessary we can find an interval K centered at x_0 such that the zeroes of $p(x) + tq(x)$ on K are of order at most $\max(1, \text{ord}_{x_0}(p))$, which is at most $\max(1, m)$ as long as $x_0 \neq 0$. Since J is finite, we can cover $\mathbf{R}_0 \cap J$ with finitely many intervals K_i with this property. Since all zeroes of all $p(x) + tq(x)$ are of order 1 on $\mathbf{R}_0 - J$ we are done with the proof of the lemma.

Recall that for a smooth function $g(x, y)$ satisfying (1.2), being in superadapted coordinates means that if e is a compact edge of $N(g)$ intersecting the bisectrix then the polynomials $g_e(1, y)$ and $g_e(-1, y)$ do not have zeroes of order $d(g)$ or higher, except possibly at $y = 0$. In this situation, by [G2] we necessarily have $|J_{g,\phi}| \leq C_{g,\phi} \lambda^{-\delta} \ln(\lambda)^p$ where $(-\delta, p)$ is the type of g at the origin (even in Case 3 superadapted coordinates where asymptotics may not exist).

Suppose we consider functions $g(x, y)$ defined on the semidisk $D_r \cap \{x > 0\}$ that are of the form $s(x^{\frac{1}{n}}, y)$ for some smooth $s(x, y)$ on a neighborhood of the origin. Then we may form the Newton polygon of $g(x, y)$ as before, except instead of having exponents in $\mathbf{Z} \times \mathbf{Z}$, we consider $g(x, y)$'s Taylor expansion as having exponents in $\frac{1}{n}\mathbf{Z} \times \mathbf{Z}$. In this paper we will be only considering $g(x, y)$ whose Taylor expansions only contain terms of total degree two or more, so we will assume that throughout. Note that this implies that $d(g) \geq 1$.

For $\lambda > 0$ and a smooth compactly supported function $\phi(x, y)$ on $D_r \cap \{x \geq 0\}$ we look at the oscillatory integral

$$J_{g,\phi}^+(\lambda) = \int_{\{x>0\}} e^{i\lambda g(x,y)} \phi(x,y) dx dy \quad (3.3)$$

We define the type of g to be the minimal $(-\delta, p)$ such that one has an estimate $|J_{g,\phi}^+(\lambda)| \leq C_{g,\phi} \lambda^{-\delta} \ln(\lambda)^p$ valid for all $\phi(x, y)$ with sufficiently small support.

Lemma 3.2. Suppose the leftmost vertex of $N(g)$ is of the form $x^a y^b$ for integers a and b and that one of the following two situations holds.

- 1) $N(g)$ does not intersect the bisectrix at a vertex and for each compact edge e of $N(g)$ any zeroes of $g_e(1, y)$ are of order $< d(g)$, or of order 1 in the case that $d(g) = 1$.
- 2) $N(g)$ intersects the bisectrix at a vertex and for each compact edge e of $N(g)$ the zeroes of $g_e(1, y)$ are of order $\leq d(g)$.

Then g is of type $(-\frac{1}{d(g)}, p)$, where $p = 1$ if $N(g)$ intersects the bisectrix at a vertex and $p = 0$ otherwise. Furthermore for sufficiently small $r > 0$ and $0 < \epsilon < \frac{1}{2}$ there are constants $c_{g,r}$ and $c'_{g,r}$ such that

$$c_{g,r} \epsilon^{-\frac{1}{d(g)}} \ln(\epsilon)^p \leq |\{(x, y) \in D_r^+ : |g(x, y)| < \epsilon\}| \leq c'_{g,r} \epsilon^{-\frac{1}{d(g)}} \ln(\epsilon)^{p'} \quad (3.4)$$

Here D_r^+ are the points in D_r for which $x > 0$, and $p = p'$ unless $d(N) = 1$ and some $g_e(1, y)$ has a zero, in which case $p = 0$ and $p' = 1$.

Proof. We will not prove this lemma in detail since very similar arguments appear in [G2]; related arguments are also in section 4 of this paper. Briefly, the lemma is true for the following reasons. One may split up a neighborhood of the origin into "slivers" touching the origin and each sliver is a subset of $\{x > 0\}$ or $\{x < 0\}$. One finds upper bounds for $J_{g,\phi}^+(\lambda)$ or $|\{(x, y) \in D_r^+ : |g(x, y)| < \epsilon\}|$ by adding up estimates corresponding the $\{x > 0\}$ slivers, and the estimate for a given sliver is given by doing Van der Corput-type arguments in x or y and then integrating the result in the other variable. These Van der Corput arguments can almost always be made in the y direction or for a first derivative in the x -direction, and these arguments carry over directly to the current setting. The sole exception is for the sliver corresponding to the left-most vertex, so we require that vertex to be of the form $x^a y^b$ for integers a and b . Then the arguments of [G2] carry over for that sliver as well.

Sharpness of the exponents obtained in this way comes from directly estimating $J_{g,\phi}^+(\lambda)$ from below on certain rectangles determined by $N(g)$, for ϕ such that $\phi(0, 0) \neq 0$. Similarly, the lower bounds of (3.4) come from evaluating the measure of such rectangles arising from $N(g)$ on which $|g(x, y)| < \epsilon$. This concludes our discussion of the proof.

Now suppose $g(x, y)$ and $h(x, y)$ are two functions of the form $s(x^{\frac{1}{n}}, y)$, s smooth on a neighborhood of the origin, such that the exponents of their left-most vertices are integers. We direct our attention to $N(g + th)$ for various $t \in \mathbf{R}$. There is a finite $I \subset \mathbf{R}$ such that if $t \notin I$, then the vertices of $N(g + th)$ are all vertices of $N(g)$ or $N(h)$; one excludes the t for which a term in $g(x, y)$'s Taylor expansion corresponding a vertex of $N(g)$ is exactly canceled out by a term in $th(x, y)$'s Taylor expansion, or vice versa. In fact there is a single Newton polygon N which is equal to $N(g + th)$ for all $t \notin I$.

Note that if $t \notin I$ and $g + th$ satisfies the hypotheses of of Lemma 3.2, then $g + th$ is of type $(-\frac{1}{d(N)}, p)$ and equation (3.4) holds for $g + th$. For the purposes of proving Theorem 2.1, we are interested in knowing for which t does this happen. For this we have the following lemma.

Lemma 3.3. For a given compact edge e of N let $g^e(x, y)$ and $h^e(x, y)$ denote the sum of the terms of the Taylor expansions of g and h respectively whose exponents are on e . As above, let N be the common Newton polygon of $g + th$ for all but finitely many t .

a) Suppose N does not intersect the bisectrix at a vertex and for each compact edge e of N $\sup_{y_0 \in \mathbf{R}_0} \min(\text{ord}_{y_0}(g^e(1, y)), \text{ord}_{y_0}(h^e(1, y))) < d(N)$. Then there is a finite $J \subset \mathbf{R}$ such that if $t \notin J$, then any zero of any $(g + th)_e(1, y)$ besides $y = 0$ is of order less than $d(N)$ if $d(N) > 1$, and is of order 1 if $d(N) = 1$.

b) Suppose N intersects the bisectrix at a vertex and for each compact edge e of N

$\sup_{y_0 \in \mathbf{R}_0} \min(\text{ord}_{y_0}(g^e(1, y)), \text{ord}_{y_0}(h^e(1, y))) \leq d(N)$. Then there is a finite $J \subset \mathbf{R}$ such that if $t \notin J$, then any zero of any $(g + th)_e(1, y)$ other than $y = 0$ is of order less than or equal to $d(N)$.

c) Let $(-\frac{\delta}{2}, p)$ denote the type of $g(x, y)^2 + h(x, y)^2$. In both part a) and part b), if $t \notin J$ then $g(x, y) + th(x, y)$ is of type $(-\delta, p)$. Furthermore, $g(x, y)^2 + h(x, y)^2$ satisfies the hypotheses of Lemma 3.2, as does $g(x, y) + th(x, y)$ if $t \notin J$.

Proof. By Lemma 3.1, in the setting of part a) if $d(N) > 1$ there is necessarily a finite $J \subset \mathbf{R}$ such that if $t \notin J$, then for each compact e all zeroes of the polynomial $(g+th)_e(1, y)$ of order less than $d(N)$, while if $d(N) = 1$ any zeroes are of order 1. In the setting of part b) one can similarly remove a finite J such that the zeroes are of order at most $d(N)$. Thus by Lemma 3.2, in the settings of both part a) and part b) the type of $g + th$ is $(-\frac{1}{d(N)}, p)$ where $p = 1$ iff N intersects the bisectrix at a vertex. Furthermore, equation (3.4) holds.

We now examine the function $g(x, y)^2 + h(x, y)^2$. Note that the vertices of $N(g^2)$ are exactly $\{2v : v \text{ is a vertex of } N(g)\}$, and the edges of $N(g^2)$ are $\{2e : e \text{ is an edge of } N(g)\}$. (Here $2e$ denotes the dilate of e by a factor of 2 and similarly for $2v$). Thus $N(g^2)$ is $2N(g)$, the dilate by 2 of $N(g)$. Furthermore, if e is a compact edge of $N(g)$, then $(g^2)_{2e}(1, y) = (g_e(1, y))^2$. The analogous statements also hold for $N(h^2)$.

Thus the Newton polygon of $g^2 + h^2$ is exactly $2N$, and for any edge e of $2N$, $(g^2 + h^2)_e(1, y) = (g^e(1, y))^2 + (h^e(1, y))^2$. Since $(g^e(1, y))^2$ and $(h^e(1, y))^2$ are nonnegative functions, the maximum order of any zero of $(g^2 + h^2)_e(1, y)$ for $y \neq 0$ is given by $\sup_{y_0 \neq 0} \min(\text{ord}_{y_0}((g^e(1, y))^2, (h^e(1, y))^2) = 2 \sup_{y_0 \neq 0} \min(\text{ord}_{y_0}(g^e(1, y), h^e(1, y)))$. By assumption, this quantity is less than or equal to $2d(N) = d(2N)$ when $2N$ intersects the bisectrix at a vertex, and is less than $d(2N)$ when $2N$ does not. Thus by Lemma 3.2, the type of $g^2 + h^2$ is given by $(-\frac{1}{2d(N)}, p)$ and $g^2 + h^2$ satisfies (3.4) with $p = p'$ since $d(2N) > 1$. Since $g + th$ is of type $(-\frac{1}{d(N)}, p)$ for $t \notin J$, this completes the proof.

Proof of Theorem 2.1.

The argument will proceed as follows. We will consider $(x, y) \in D_r \cap \{x > 0\}$, as analogous arguments always work for $D_r \cap \{x < 0\}$. For a given f , we will either be able to apply Lemma 3.3 immediately, or through an iterative process we will construct a smooth $p(x)$ such that for some n , if $S'(x, y) = S(x, y - p(x^{\frac{1}{n}}))$ and $f'(x, y) = S(x, y - p(x^{\frac{1}{n}}))$, then $S'(x, y)$ and $f'(x, y)$ will satisfy the conditions of Lemma 3.3. Thus in these situations, except for finitely many t we will have $|J_{\phi, S'+tf'}(\lambda)^+| \leq C\lambda^{-\delta}(\ln(\lambda))^p$, where $(-\frac{\delta}{2}, p)$ denotes the type of $(S')^2 + (f')^2$, which by (3.4) is the same as the type of $S^2 + f^2$ when viewed as a smooth function on all of D_r (these coordinate changes do not affect the measures). Also, the coordinate changes are such that $J_{\phi, S'+tf'}^+(\lambda) = J_{\phi, S+tf}^+(\lambda)$, and if one adds the $x < 0$ and $x > 0$ portions of this, one gets $|J_{\phi, S+tf}(\lambda)| \leq C\lambda^{-\delta}(\ln(\lambda))^p$. That this type $(-\delta, p)$ is optimal follows from adding over the $x < 0$ and $x > 0$ portions the lower bounds of (3.4) and using that the oscillatory type of a smooth function $a(x, y)$ is equal

to the supremum of the $(-\theta, q)$ for which one has an estimate $|\{(x, y) \in D_r : |a(x, y)| < \epsilon\}| \geq c'_{g,r} \epsilon^\theta \ln(\epsilon)^q$. (This was proved in [G2]; for the real-analytic case, see [AGV] Ch. 7.)

In summary, to prove Theorem 2.1 it will suffice to show that either S and f satisfy the conditions of Lemma 3.3 to start with, or the derived S' and f' satisfy the conditions of Lemma 3.3. We will then identify when S has the same type as the generic $S + tf$; this will determine whether or not f is a good direction. It will transpire that good directions occur in some cases when S and f initially satisfy the conditions of Lemma 3.3, and never when one performs the coordinate changes. The former situations will correspond to the directions stipulated in Theorem 2.1.

We now begin the main argument. Let N be the polygon such that $N(S^2 + f^2) = 2N$. Let $I \subset \mathbf{R}_0$ such that if $t \in \mathbf{R}_0 - I$ then no vertices of $N(S)$ or $N(tf)$ get cancelled out in adding $S + tf$. So in particular, if $t \in \mathbf{R}_0 - I$, then as in the proof of Lemma 3.3 we have $2N(S + tf) = N(S^2 + f^2)$. If the bisectrix intersects N at a vertex v then either $N(f)$ or $N(S)$ has a vertex at v . In the first case each $f_e(1, y)$ has no zero of order greater than $d(N)$ besides $y = 0$, and in the latter case the same holds for each $S_e(1, y)$. Thus the conditions of Lemma 3.3 are satisfied, with f a good direction in the first case. If the bisectrix intersects N in the interior of one of the infinite rays, then the bisectrix intersects either $N(f)$ or $N(S)$ in the interior of this ray, and with the same Newton distance $d(N)$. In the former case each $f_e(1, y)$ has no zero of order $d(N)$ or higher, other than possibly $y = 0$, and in the latter case the same is true for each $S_e(1, y)$. Thus once again the hypotheses of Lemma 3.3 are immediately satisfied, with f a good direction in the former case. So from now on we may assume the bisectrix intersects N in the interior of a compact edge which we denote by e_0 .

As before, let $S^{e_0}(x, y)$ and $f^{e_0}(x, y)$ respectively denote the sum of the terms of S 's and f 's Taylor expansion that are on e_0 ; in either case there may be one, or even no terms. Suppose $S^{e_0}(x, y)$ has at least one term and each exponent (a, b) appearing in $S^{e_0}(x, y)$ satisfies $b < a$. Then if b_0 denotes the maximum of such b , $S^{e_0}(1, y)$ is a polynomial of degree $b_0 < d(N)$. Hence the zeroes of $S^{e_0}(1, y)$ are of order at most b_0 , and thus $\min_{y_0 \in \mathbf{R}_0} (\text{ord}_{y_0}(S^{e_0}(1, y)), \text{ord}_{y_0}(f^{e_0}(1, y))) < d(N)$. Furthermore, if e' is an edge of N below the bisectrix, a nonzero $f^{e'}(1, y)$, $S^{e'}(1, y)$ has degree $< d(N)$, while if e' is above the bisectrix then a nonzero $f^{e'}(x, 1)$ or $S^{e'}(x, 1)$ has degree $< d(N)$. Thus these polynomials have no zeroes of degree $d(N)$ or larger. By mixed homogeneity of $f^{e'}(x, y)$ and $S^{e'}(x, y)$, in the latter two cases the same must be true for $f^{e'}(1, y)$ or $S^{e'}(1, y)$, other than $y = 0$. Thus the conditions of Lemma 3.3 are satisfied. Furthermore, since S is in superadapted coordinates and the bisectrix intersects $N(S)$ above the line containing e_0 , the type of S is greater than the type of the generic $S + tf$ and hence f is not a good direction.

The argument of the previous paragraph still applies if each exponent (a, b) satisfies $b > a$ instead of $a < b$, since one can reverse the roles of the x and y variables in this situation. We can also replace $S^{e_0}(x, y)$ by $f^{e_0}(x, y)$ and the above argument still gives

that Lemma 3.3 applies. This time the generic $S + tf$ is of the same type as S iff $S^{e_0}(x, y)$ has terms both above and below the bisectrix. So f is a good direction when this happens.

Thus we may restrict our attention from now on to the situation where the bisectrix intersects N in the interior of a compact edge e_0 , and where $f^{e_0}(x, y)$ or $S^{e_0}(x, y)$ are either the zero polynomial, or are nonzero and have exponents that aren't all strictly above or strictly below the bisectrix. Suppose first $S^{e_0}(x, y)$ is nonzero. Since its exponents aren't all strictly above or below the bisectrix, either $N(S) \cap e_0 = \{(d(N), d(N))\}$, or $N(S)$ has a compact edge e intersecting the bisectrix such that $e \subset e_0$. In the former case, $S^{e_0}(1, y)$ has no zeroes at all other than $y = 0$, and in the latter case since S is in superadapted coordinates $S^e(1, y)$ has no zeroes of degree $d(N)$ or more other than possibly $y = 0$. Any $f^{e'}(1, y)$ or $S^{e'}(1, y)$ for an edge e' of N not intersecting the bisectrix will have zeroes of order less than $d(N)$ similarly to two paragraphs ago. Thus the conditions of Lemma 3.3 are satisfied. If $N(S)$ has a vertex on the bisectrix, then S is of type $(-\frac{1}{d(N)}, 1)$ while the generic $S + tf$ is of type $(-\frac{1}{d(N)}, 0)$. Hence f is not a good direction. Otherwise, S is of type $(-\frac{1}{d(N)}, 0)$ so, f is a good direction.

Thus it remains to consider the case where $S^{e_0}(x, y)$ is the zero polynomial, so that $(S + tf)_{e_0}(x, y) = tf_{e_0}(x, y)$. If each zero of $f_{e_0}(1, y)$ is of order $< d(f) = d(N)$, then $S + tf$ is in superadapted coordinates and like in the above cases Lemma 3.3 applies. Note that f is not a good direction here as $d(S) > d(S + tf)$. So we may devote our attention to where $S^{e_0}(x, y)$ is the zero polynomial and $f_{e_0}(1, y)$ has a zero of order at least $d(f) = d(N)$; in essence this situation is the crux of the proof.

Let the equation of the line containing e_0 be denoted by $x + m_f y = c_f$. If the bisectrix intersects $N(S)$ in the interior of a compact edge, we denote the equation of this edge by $x + m_S y = c_S$. Otherwise, let $x + m_S y = c_S$ denote any line l with m_S and c_S rational such that $N(S) \cap l = (d(S), d(S))$. Switching the roles of the x and y axes if necessary, we may assume $m_S \leq m_f$.

Let r be a zero of $f_{e_0}(1, y)$ of order at least $d(N)$. We do a coordinate change now, letting $S_1(x, y) = S(x, y + rx^{m_f})$ and $f_1(x, y) = f(x, y + rx^{m_f})$. (This is how the fractional powers of x come in; we need that $m_S \leq m_f$ and m_f need not be an integer).

First we show that $d(S_1) \leq d(S)$. To see this, let w denote the edge or vertex of $N(S)$ such that the supporting line of $N(S)$ of slope $-\frac{1}{m_f}$ intersects $N(S)$ at w , and let z denote the edge or vertex of $N(S_1)$ such that the supporting line of $N(S_1)$ of slope $-\frac{1}{m_f}$ intersects $N(S_1)$ at z . Since $m_S \leq m_f$, either w is a vertex of $N(S)$ or w is an edge of $N(S)$ intersecting or below the bisectrix. If w is an edge or vertex lying entirely below the bisectrix then the coordinate change doesn't affect any of $N(S)$ on the bisectrix or higher, and thus $d(S_1) = d(S)$. If w is a vertex of $N(S)$ on or above the bisectrix, then the coordinate change results in an edge z extending from w to the x -axis. So $d(S_1) \leq d(S)$ here. If w is an edge whose upper vertex is $(d(S), d(S))$, the coordinate change converts

w to an edge z whose upper vertex is also $(d(S), d(S))$. (z is an edge and not a vertex here since S is in superadapted coordinates.) Thus $d(S_1) = d(S)$. Lastly, if w is an edge of $N(S)$ intersecting the bisectrix in its interior then $(S_1)_z(1, y) = S_w(1, y + r)$. Since S is in superadapted coordinates, $S_w(1, y)$ has no zeroes of order $d(S)$ or higher (including at $y = 0$), so the same is true for $S_w(1, y + r)$. Thus the bisectrix intersects $N(S_1)$ at $(d(S), d(S))$, which is in the interior of z . Hence $d(S_1) = d(S)$ here as well.

Note that the above argument shows that $(S_1)_z(x, y)$ necessarily has at least one term whose exponents lie below the bisectrix. As a consequence, all supporting lines of $N(S_1)$ of slope greater than $-\frac{1}{m_f}$ intersect $N(S_1)$ below the bisectrix only, a fact that will come in handy later on.

We now examine the effect of the coordinate change on $N(f)$. Since f has an edge e_0 of slope $-\frac{1}{m_f}$, f_1 will have an edge or vertex which we call e' that is on the same line as e_0 , such that $(f_1)_{e'}(1, y) = f_{e_0}(1, y + r)$. So since f_{e_0} has a zero of order $d(f)$ or greater at $y = r$, $(f_1)_{e'}(1, y)$ has a zero of order $d(f)$ or greater at $y = 0$. Hence e' lies entirely on or above the bisectrix. For any smooth function $g(x, y)$, it is not hard to show (as is shown in [G2] and elsewhere) that the maximum possible number of roots of any $g_e(x, y)$, counted according to multiplicity, is $2d(g)$. Hence in the case at hand, the maximum total number of roots of $f_{e_0}(1, y)$ is $2d(f)$. Since $f_{e_0}(1, y)$ has a zero of order $d(f)$ or greater at $y = r$, the maximum possible order of a zero of $(f_1)_{e'}(1, y) = f_{e_0}(1, y + r)$ besides $y = 0$ is $2d(f) - d(f) = d(f)$, with equality possible only if the bottom endpoint of e' is on the bisectrix.

Let (a, b) denote the lower vertex of e' , and let $\infty \geq m > m_f$ be minimal such that the line l containing (a, b) of slope $-\frac{1}{m}$ is horizontal or intersects either $N(f_1) - \{(a, b)\}$ or $N(S_1)$. If $m = \infty$, then $N(S_1) \subset N(f_1)$ and the bisectrix intersects $N(f_1)$ in the horizontal ray, possibly at the vertex of this ray. In this case the conditions of Lemma 3.3 hold for the following reasons. If e is an edge of $N(S_1)$ above l with slope less than $-\frac{1}{m_f}$, the polynomials $S_1^e(1, y)$ and $f_1^e(1, y)$ are the same as $S^e(1, y)$ and $f^e(1, y)$ since the coordinate change did not affect the terms of the polynomials on edges above e_0 . So the terms of $S^e(1, y)$ and $f^e(1, y)$ have integral coordinates and lie wholly on or above the bisectrix. Thus their zeroes are of order less than $d(f_1)$ with equality possible only if the lower endpoint of e is on the bisectrix. If $N(f_1)$ has an edge of slope $-\frac{1}{m_f}$, it will be the edge we called e' before. Recall that $(f_1)_{e'}(1, y) = f_{e_0}(1, y + r)$, whose zeroes we saw were of order at most $d(f) \leq d(f_1)$ where equality can hold here only if (a, b) is a vertex of $N(f_1)$. We conclude that we may apply Lemma 3.3. f is not a good direction here; by above $N(S_1)$ has a vertex (v_1, v_2) below the bisectrix, so $v_2 < d(S_1)$. Since $N(S_1) \subset N(f_1)$ this vertex lies on or above the horizontal ray of $N(f_1)$. Hence for the generic t , $d(S_1 + tf_1) \leq d(f_1) \leq v_2 < d(S_1)$ and f is not a good direction.

So from now on assume m is finite. Except for finitely many t , $N(S_1 + tf_1)$ will have an edge k on the line l , and $(S_1 + tf_1)_k(x, y) = (S_1)^k(x, y) + t(f_1)^k(x, y)$. Let N_1 denote the Newton polygon of $S_1 + tf_1$ for all but finitely many t , and $d(N_1)$ its Newton

distance. We first consider the case where $(S_1)^k(x, y)$ is nonzero. By above all supporting lines of $N(S_1)$ of slope greater than $-\frac{1}{m_f}$ intersect $N(S_1)$ entirely below the bisectrix. So since $-\frac{1}{m} > -\frac{1}{m_f}$, $(S_1)^k(x, y)$ is either $(S_1)_e(x, y)$ for some edge e lying wholly below the bisectrix, or is of the form cx^vy^w for some vertex (v, w) of $N(S)$ lying below the bisectrix. Thus in either case, since (a, b) is on or above the bisectrix, the bisectrix intersects N_1 in the interior of k or at the upper vertex of k .

We are now in a position to apply Lemma 3.3. The polynomials $(S_1)^e(1, y)$ and $(f_1)^e(1, y)$ for e below k are all below the bisectrix, so are all of degree less than $d(N_1)$ and thus have zeroes of order less than $d(N_1)$. The same is true for $(S_1)^k(1, y)$ since the exponents of the terms of $(S_1)^k(x, y)$ are all below the bisectrix. For any edge e of N_1 above k with slope $-\frac{1}{m_f}$ or smaller, the polynomials $S_1^e(1, y)$ and $f_1^e(1, y)$ have zeroes of small enough order to apply Lemma 3.3 similarly to two paragraphs ago. We conclude that we may apply Lemma 3.3 to S_1 and f_1 . Note that because k intersects $N(S_1)$ below the bisectrix, one has $d(S_1 + tf_1) < d(S_1)$ for the generic t and thus f is not a good direction.

Thus it remains to consider the situation where $S_1^k(x, y)$ is zero. In this case l intersects $N(f_1) - \{(a, b)\}$ but not $N(S_1)$. Thus the edge k is also an edge of $N(f_1)$. If the lower vertex (a', b') of k is on or above the bisectrix, we replace (a, b) by (a', b') and then repeat the argument of the previous three paragraphs. Since the y -coordinates are integers, this can only happen finitely many times and eventually (a', b') will be below the bisectrix. Denote by k' the analogue of k once (a', b') is below the bisectrix. Then if k' intersects the bisectrix at its upper vertex, we may immediately apply Lemma 3.3 and we are done. Since this upper vertex is not on $N(S)$, f is not a good direction in this situation. If k' does not intersect the bisectrix at its upper vertex, we are back to the situation of the sixth paragraph of this proof, where we had $S^{e_0}(x, y)$ being the zero polynomial, with the bisectrix intersecting N in the interior of a compact edge e_0 coming entirely from f , so that $(S + tf)_{e_0}(x, y) = tf_{e_0}(x, y)$. The only difference is that instead of dealing with smooth functions we are now dealing with smooth functions of $x^{\frac{1}{\alpha}}$ and y , where α is the difference between the y coordinates of the upper and lower vertices of e_0 .

Thus we may iterate the above process. If the procedure ends after finitely many iterations, we eventually are in one of the cases already handled and we are done. Suppose the now that procedure does not ever end. Let f_l , S_l , and the Newton polygon N_l correspond to the l th stage of the iteration, and let $d(N_l)$ denote the Newton distance. Let $(x, y) \rightarrow (x, y + r_l x^{m_l})$ denote the coordinate change going from the l th to $l+1$ st stage, and let (a_l, b_l) be the analogue of the lower vertex (a, b) above. Since b_l is nonincreasing with l and has integral values, eventually b_l stops changing at some $l = l_0$ and stays at some value b_0 . This means a_l stays at some value a_0 as well. Furthermore, since the b_l are nonincreasing, we are dealing with smooth functions of $x^{\frac{1}{b_l}}$ and y throughout. As a result, the slopes of the edges intersecting the bisectrix, increasing with each iteration, will all be of the form $-\frac{nb_l}{p}$ with n and p integers such that $1 \leq n \leq b$. Because these slopes increase with each iteration and n is constrained to be between 1 and b , these slopes go to zero as l goes to infinity. Since the vertex (v_1, v_2) of $N(S_1)$ below the bisectrix will still be there

in every iteration, b_0 satisfies $b_0 \leq v_2 < d(S_1)$.

We now use Borel's theorem (see [H]) to find an $s(x)$ which is a smooth function of $x^{\frac{1}{b_1}}$ such that the Taylor expansion of $s(x)$ at $x = 0$ is $\sum_{l=l_0}^{\infty} r_l x^{m_l}$. We perform the coordinate change $(x, y) \rightarrow (x, y + s(x))$. Define $\tilde{f}(x, y) = f_{l_0}(x, y + s(x))$ and $\tilde{S}(x, y) = S_{l_0}(x, y + s(x))$. Then $N(\tilde{f})$ has an infinite ray with vertex (a_0, b_0) , $N(\tilde{S}) \subset N(\tilde{f})$, and $d(\tilde{S}) = d(S_1)$. Now we may apply Lemma 3.3. Since $d(\tilde{S} + t\tilde{f}) \leq d(\tilde{f}) = b_0 < d(S_1) = d(\tilde{S})$, \tilde{f} is not a good direction here.

We have now exhausted all the ways in which the algorithm may proceed. It is worth mentioning that in order to apply Lemma 3.3 as the iteration proceeds, we need first that the leftmost vertices of the Newton polygons arising have integral coordinates, which is automatic given the definition of the coordinate changes here, and secondly that the polynomials coming from the new edges still satisfy the condition on the order of their zeroes. To see this, first observe that the edges not entirely on or above the bisectrix can be dealt with like before; the degree of any $S_l^e(x, y)$ and $f_l^e(x, y)$ appearing will be at most $d(N_l)$, with equality only possible if the upper vertex of e is on the bisectrix. Thus the conditions needed for Lemma 3.3 will hold. For an edge e' on or above the bisectrix, write its upper and lower endpoints as (a'', b'') and (a''', b''') respectively, and then $b'' \leq b$, where b is as before, and we saw $b \leq 2d(f) < 2d(N_l)$. Since (a''', b''') is on or above the bisectrix, we also have $b''' \geq d(N_l)$. Hence $b'' - b''' < d(N_l)$. This means any zero $f_{e'}(1, y)$ other than $y = 0$ is of order at most $b'' - b''' < d(N_l)$. Thus we may apply Lemma 3.3.

Thus we are done other than listing the good directions; that is, the f for which $S + tf$ is of the same type as S for all but finitely many t . The only f that were good directions in the above argument came near the beginning. If the bisectrix intersected $N(S)$ at a vertex or inside one of the rays, this only happened in the first paragraph of the main argument in the case when $d(f) = d(N)$. These correspond to the Newton polygon configurations given in Theorem 2.1. If the bisectrix intersected $N(S)$ inside the interior of a compact edge e , this only happened in the third and fourth paragraphs of the main argument, when $N(f)$ lay wholly on or above the line containing e , which is the Newton polygon configuration stipulated by Theorem 2.1 as well. This completes the proof.

4. Proof of Theorem 2.2.

We first do the case where $S(x, y)$ is in Case 1 superadapted coordinates. Let e denote the edge of $N(S)$ intersecting the bisectrix in its interior, and let $x + my = c$ denote the equation of this edge. We focus our attention on (x, y) inside a small rectangle $T_r = [-r, r] \times [r^m, r^m]$, where r is viewed as a fixed number sufficiently small for our arguments to proceed. We do a finite Taylor expansion of $S(x, y)$ in the x variable, obtaining

$$S(x, y) = \sum_{0 \leq j \leq c} \frac{\partial_x^j S(0, y)}{j!} x^j + E(x, y) \quad (4.1)$$

We then do a partial Taylor expansion in the y variable of a given term $\frac{\partial_x^j S(0, y)}{j!}$ up to the

y^k power, where k is maximal such that $j + mk \leq c$. Then (4.1) becomes

$$S(x, y) = \sum_{0 \leq j+mk \leq c} \frac{\partial_x^j \partial_y^k S(0, 0)}{j!k!} x^j y^k + F(x, y) \quad (4.2)$$

Here the error term $F(x, y)$ satisfies the estimate

$$|F(x, y)| \leq C(|x|^{c+1} + |y|^{\frac{c}{m}+1}) \|S\|_{C^n(T_r)} \quad (4.3)$$

Here C is a constant depending only on the edge e and $n = \max(\frac{c}{m}, c) + 1$, although the exact value of n is not important for us. Since the edge e of $N(S)$ has equation $x + my = c$, (4.2) can be rewritten as

$$S(x, y) = S_e(x, y) + F(x, y) \quad (4.4)$$

Let $d = d(S)$. Suppose $1 \leq l < d$. Then $\partial_l S(x, y)$ has an edge or vertex e_l on the line $x + my = c - ml$, and as in (4.4) we may write

$$\partial_l^y S(x, y) = (\partial_l^y S)_{e_l}(x, y) + F_l(x, y) \quad (4.5)$$

This time the error estimate is

$$|F_l(x, y)| \leq C(|x|^{c+1-lm} + |y|^{\frac{c}{m}-l+1}) \|S\|_{C^{n+l}(T_r)} \quad (4.6)$$

Next, we divide T_r into four parts along the curves $y = \pm|x|^m$. The estimates for each of the parts is done the same way, so we focus on the $0 < x < r$, $-x^m < y < x^m$ region, which we denote by U_r . Note that on U_r , since $|y| < x^m$, the error estimates (4.4) and (4.6) become

$$|F(x, y)| \leq Cx^{c+\min(1,m)} \|S\|_{C^n(T_r)} \quad (4.7a)$$

$$|F_l(x, y)| \leq Cx^{c-lm+\min(1,m)} \|S\|_{C^{n+l}(T_r)} \quad (4.7b)$$

Suppose $1 \leq l < d$ is such that $S_e(1, y)$ has a zero of order l at some $y = y_0$. Then since $S_e(x, y)$ is a mixed homogeneous function, $S_e(x, y)$ has a zero of order l in the y direction at every point on the curve $y = y_0 x^m$ for $x > 0$, and there is some $\epsilon > 0$ such that on the wedge $W_{y_0, \epsilon, r} = \{(x, y) \in U_r : (y_0 - \epsilon)x^m \leq y \leq (y_0 + \epsilon)x^m\}$ one has estimates

$$C_1 x^{c-lm} < (\partial_l^y S)_{e_l}(x, y) < C_2 x^{c-lm} \quad (4.8)$$

Given the error estimate (4.7b), if x is sufficiently small, depending on $\|S\|_{C^{n+l}(T_r)}$, on $W_{y_0, \epsilon, r}$ one has estimates

$$C'_1 x^{c-lm} < (\partial_l^y S)(x, y) < C'_2 x^{c-lm} \quad (4.9)$$

Suppose now $0 < |y_0| < 1$ is such that $S_e(x, y)$ does not have a zero at y_0 . Then along the curve $y = y_0 x^m$, by mixed homogeneity $S_e(x, y)$ is equal to ax^c for some nonzero a . Thus $\partial_x[S_e(x, y_0 x^m)] = acx^{c-1}$ is nonzero. But by the chain rule

$$\partial_x[S_e(x, y_0 x^m)] = (\partial_x S)_{e_1}(x, y_0 x^m) + my_0 x^{m-1} (\partial_y S)_{e_1}(x, y_0 x^m) \quad (4.10)$$

Here $(\partial_x S)_{e_1}$ defined similarly to $(\partial_y S)_{e_1}$ except with the roles of the x and y variables reversed. Thus always either $(\partial_x S)_{e_1}(x, y_0 x^m)$ or $(\partial_y S)_{e_1}(x, y_0 x^m)$ or nonzero. Therefore, by mixed homogeneity of $(\partial_x S)_{e_1}$ and $(\partial_y S)_{e_1}$, either $(\partial_x S)_{e_1}(x_0, 1) \neq 0$ for $x_0 = y_0^{\frac{1}{m}}$ or $(\partial_y S)_{e_1}(1, y_0) \neq 0$. In the latter case we have some $W_{y_0, \epsilon, r}$ such that (4.9) holds with $l = 1$, and in the former case, reversing the roles of the x and y axes there is some analogous $W_{x_0, \epsilon, r}$ on which we have

$$C_1 x^{c-1} < \partial_x S(x, y) < C_2 x^{c-1} \quad (4.11)$$

In the case where $y_0 = 0$, then we do a coordinate change $(x, y) \rightarrow (x, y + \frac{1}{2}x^m)$. Then y_0 becomes $\frac{1}{2}$ instead of 0, and now either (4.9) holds for $l = 1$ or (4.11) holds. The error estimates still hold and the possible introduction of fractional powers of x does not interfere with our arguments.

Thus for any y , there is a small interval $[y_0 - \epsilon, y_0 + \epsilon]$ or $[x_0 - \epsilon, x_0 + \epsilon]$ with $x_0 = y_0^{\frac{1}{m}}$ such that on the associated wedge $W_{y_0, \epsilon, r}$ or $W_{x_0, \epsilon, r}$ either (4.9) holds for some $1 \leq l < d$, or (4.11) holds. (We use the fact that $S(x, y)$ is in superadapted coordinates here to ensure l never has to be d or higher). By compactness, we can cover T_r with finitely many such wedges W_i .

Suppose now that $f(x, y)$ is a perturbation function whose Newton polygon lies entirely on or above the line $x + my = c$. Then (4.4) – (4.7) hold for f in place of S . Hence there are constants C_3 and C_4 independent of f such that if $\|f\|_{C^{n+d}}$ is sufficiently small, on a given W_i one of the following two equations holds.

$$C_3 x^{c-lm} < \partial_i^y (S + f) < C_4 x^{c-lm} \quad (4.12a)$$

$$C_3 x^{c-1} < \partial_x (S + f) < C_4 x^{c-1} \quad (4.12b)$$

We now proceed as follows. If (4.12a) holds, we perform a Van der Corput-type argument in the y direction and integrate the result with respect to x , and if (4.12b) holds we perform it in the x direction and integrate the result with respect to y . In the former case, Lemma 2.0 of [G2] gives

$$\left| \int_{W_i} e^{i\lambda S(x, y)} \phi(x, y) dx dy \right| \leq C \lambda^{-\frac{1+m}{(c-lm)+ml}} \quad (4.13a)$$

Since $d = \frac{c}{1+m}$, (4.13a) is the same as

$$\left| \int_{W_i} e^{i\lambda S(x, y)} \phi(x, y) dx dy \right| \leq C \lambda^{-\frac{1}{d}} \quad (4.13b)$$

In the proof of Lemma 2.0, the constant C is of the form $\|\phi\|_{C^1(W_i)}$ times a function of C_3 and the supremum of finitely many derivatives of the phase $S + f$. In particular, we can write it as $C' \|\phi\|_{C^1(W_i)}$ where C' is uniform over all perturbations in question.

In the case where (4.12b) holds, one applies Lemma 2.0 in the x direction, obtaining

$$\left| \int_{W_i} e^{i\lambda S(x,y)} \phi(x,y) dx dy \right| \leq C \lambda^{-\frac{1+\frac{1}{m}}{(\frac{c}{m}-\frac{1}{m})+\frac{1}{m}}} \quad (4.14)$$

The exponent on the right-hand side here is once again $-\frac{m+1}{c} = -\frac{1}{d}$, so (4.13b) holds once again with C of the form $C' \|\phi\|_{C^1(W_i)}$. This completes the proof of Theorem 2.2 when $S(x,y)$ is in Case 1 superadapted coordinates.

We now move to the case when $S(x,y)$ is in Case 2 superadapted coordinates. In this situation, there is a finite collection of pairs (l_1, l_2) of lines containing (d, d) such that the Newton polygon of any good direction $f(x,y)$ is contained in the wedge above (d, d) determined by l_1 and l_2 for one of these pairs. It suffices to prove the uniform estimates for a given pair. Write the equations of the lines l_1 and l_2 as $x + m_1 y = c_1$ and $x + m_2 y = c_2$. Let m_3 be any rational number between m_1 and m_2 . Let l_3 be the line of slope $-\frac{1}{m_3}$ containing (d, d) , and write its equation as $x + m_3 y = c_3$. Thus $N(S)$ intersects l_3 at the one point (d, d) . This time, we let V_r be the rectangle $[-r, r] \times [-r^{m_3}, r^{m_3}]$, and divide it into four parts via the curves $y = \pm|x|^{m_3}$. We focus on the part where $0 < x < r$ and $-x^{m_3} < y < x^{m_3}$, which we denote by W_r , as the other three regions are dealt with the same way.

We examine the expansion (4.2) with m and c replaced by m_3 and c_3 respectively. For some $a \neq 0$ we get

$$S(x,y) = ax^d y^d + F(x,y) \quad (4.15)$$

Here $|F(x,y)| \leq C|x|^{c_3+\min(1,m_3)} \|S\|_{C^n(V_r)}$ as in (4.7a). Similarly, taking the d th y derivative gives

$$\partial_y^d S(x,y) = ad!x^d + F_l(x,y) \quad (4.16)$$

This time we have

$$|F_l(x,y)| \leq Cx^{c_3-dm_3+\min(1,m_3)} \|S\|_{C^{n+d}(V_r)} \quad (4.17a)$$

Since (d, d) is on l_3 , one has $c_3 - dm_3 = d$, and the above becomes

$$|F_l(x,y)| \leq Cx^{d+\min(1,m_3)} \|S\|_{C^{n+d}(V_r)} \quad (4.17b)$$

Thus if r is sufficiently small, depending on $\|S\|_{C^{n+d}(V_r)}$, we have

$$|\partial_y^d S(x,y)| > \frac{|a|}{2} x^d \quad (4.18)$$

If the perturbation function f is such that $|f|_{r,n+d}$ is sufficiently small, then looking at (4.16) – (4.17) with f in place of S gives

$$|\partial_y^d f(x,y)| < \frac{|a|}{4} x^d \quad (4.19)$$

Combining with (4.18) leads to

$$|\partial_y^d(S + f)(x, y)| > \frac{|a|}{4}x^d \quad (4.20)$$

If we apply the Van der Corput lemma in the y direction, and integrate the result in x (see the proof of Lemma 4.1 of [G2] for a very similar calculation), one obtains

$$\left| \int_{W_r} e^{i\lambda S(x, y)} \phi(x, y) dx dy \right| \leq C \|\phi\|_{C_1(V_r)} \lambda^{-\frac{1+m}{d+m+d}} \ln \lambda \quad (4.21)$$

Since $-\frac{1+m}{d+m+d}$ is just $-\frac{1}{d}$, (4.21) is the desired estimate and we are done with the proof of Theorem 2.2 when $S(x, y)$ is in Case 2 superadapted coordinates.

We now move to the case where $S(x, y)$ is in Case 3 superadapted coordinates. Without loss of generality we may assume the bisectrix intersects $N(S)$ in the interior of a horizontal ray. We denote the lower left vertex of $N(S)$ by (c, d) , where $0 \leq c < d = d(S)$, and we let $ax^c y^d$ be the corresponding term of the Taylor series of $S(x, y)$. If r is sufficiently small, which we may assume, then on D_r we have

$$|\partial_x^c S(x, y)| > c! \frac{|a|}{2} |y|^d \quad (4.22a)$$

For $c = 0$ we may also assume that r is small enough so that we have

$$|\partial_y^d S(x, y)| > d! \frac{|a|}{2} \quad (4.22b)$$

The only conditions we need on r is that (4.22a) – (4.22b) holds, so in what follows we always assume r is some fixed number such that (4.22a) – (4.22b) are satisfied. Suppose $f(x, y)$ is an admissible direction. Then $f(x, y)$ is of the form $F(x, y)y^d$ for some real-analytic $F(x, y)$. The norm $\|\cdot\|_{r, N}$ is such that if η is sufficiently small and N is sufficiently large, then if $\|f\|_{r, N} < \eta$ we have

$$|\partial_x^c F(x, y)| < c! \frac{|a|}{4} \quad (4.23a)$$

If $c = 0$ then if η is sufficiently small we also must have

$$|\partial_y^d (F(x, y)y^d)| < d! \frac{|a|}{4} \quad (4.23b)$$

Thus in view of (4.22), we have

$$|\partial_x^c (S + f)(x, y)| > c! \frac{|a|}{4} |y|^d \quad (4.24a)$$

While if $c = 0$ we also have

$$|\partial_y^d(S+f)(x,y)| > d! \frac{|a|}{4} \quad (4.24b)$$

We use the Van der Corput lemma in conjunction with (4.24a) – (4.24b) to get the desired upper bounds on $|J_{S+f,\phi}(\lambda)|$. When $c = 0$ we use (4.24b), obtaining

$$\left| \int e^{i\lambda(S(x,y)+f(x,y))} \phi(x,y) dy \right| \leq C \lambda^{-\frac{1}{d}} \quad (4.25)$$

Integrating this with respect to x gives the desired uniform bounds. Suppose now $c > 0$. Then the Van der Corput lemma applied to (4.24a) gives

$$\left| \int e^{i\lambda(S(x,y)+f(x,y))} \phi(x,y) dx \right| \leq C \max(1, \lambda^{-\frac{1}{c}} |y|^{-\frac{d}{c}}) \quad (4.26)$$

As a result,

$$|J_{S+f,\phi}(\lambda)| \leq C \int_0^1 \max(1, \lambda^{-\frac{1}{c}} y^{-\frac{d}{c}}) dy \quad (4.27)$$

The quantity $\lambda^{-\frac{1}{c}} y^{-\frac{d}{c}}$ is equal to 1 when $y = \lambda^{-\frac{1}{d}}$. Since the exponent $\frac{d}{c}$ is greater than 1, the integrals over the two sides of this breaking point will be comparable. The integral over $y < \lambda^{-\frac{1}{d}}$ is just $\lambda^{-\frac{1}{d}}$, so we conclude that

$$|J_{S+f,\phi}(\lambda)| \leq C' \lambda^{-\frac{1}{d}} \quad (4.28)$$

These are the desired uniform estimates and we are done.

5. Proof of Theorem 2.3.

We first consider the case where $S(x,y)$ is in Case 1 superadapted coordinates. Since $|A_{\phi,S+f}|$ is uniformly bounded for $|f|_{r,N} < \eta$ by Theorem 2.2, it suffices to show that there is some $\delta_0 > 0$ such that if f_1 and f_2 satisfies the hypotheses of Theorem 2.3 and $|f_2 - f_1|_{r,N} < \delta_0$, then the estimate (2.8) holds. This is what we will prove.

By (A.3), $A_{\phi,S+f}$ is a linear combination of $B_{\phi,S+f}$ and $B_{\phi,-S-f}$, so it suffices to prove (2.8) with $B_{\phi,S+f_i}$ in place of $A_{\phi,S+f_i}$ ($B_{\phi,-S-f_i}$ uses the same argument with S replaced by $-S$, and f_i replaced by $-f_i$). Furthermore, by [G2] we have

$$B_{\phi,S} = (m+1)^{-1} \phi(0,0) \int_{-\infty}^{\infty} (S_e^+(1,y)^{-\frac{1}{d}} + S_e^+(-1,y)^{-\frac{1}{d}}) dy \quad (5.1)$$

Here $d = d(S)$, e denotes the edge of $N(S)$ intersecting the bisectrix in its interior, and $S_e(x,y)^+$ denotes $\max(S_e(x,y), 0)$. The $S_e^+(1,y)^{-\frac{1}{d}}$ and $S_e^+(-1,y)^{-\frac{1}{d}}$ integrals are dealt with the same way, so we focus our attention on the $S_e^+(1,y)^{-\frac{1}{d}}$ integral. Thus if we define

$$D_S = \int_{-\infty}^{\infty} S_e^+(1,y)^{-\frac{1}{d}} dy \quad (5.2)$$

Then it suffices to show that for some constants B and α , whenever $|f_2|_{r,N}, |f_1|_{r,N} < \eta$ we have

$$|D_{S+f_2} - D_{S+f_1}| \leq B|f_2 - f_1|_{r,N}^\alpha \quad (5.3)$$

Write $S(1, y) = \sum_{i=m}^n c_i y^i$. Since $S(x, y)$ is in superadapted coordinates, $n > d$ and $m < d$. We break $D_S = D_S^1 + D_S^2$, where D_S^1 is the integral over $|y| < 1$ and D_S^2 is the integral over $|y| \geq 1$. We change variables from y to $\frac{1}{y}$ in D_S^2 , and obtain

$$D_S^2 = \int_{|y| \leq 1} y^{-2} \left(\sum_{i=m}^n c_i y^{-i} \right)^+^{-\frac{1}{d}} dy \quad (5.4)$$

$$= \int_{|y| \leq 1} \left(\sum_{i=m}^n c_i y^{2d-i} \right)^+^{-\frac{1}{d}} dy \quad (5.5)$$

We compare this expression with

$$D_S^1 = \int_{|y| \leq 1} \left(\sum_{i=m}^n c_i y^i \right)^+^{-\frac{1}{d}} \quad (5.6)$$

The highest power of y appearing in (5.5) is $2d - m > d$, and the lowest power is $2d - n < d$. Furthermore, perturbations of S in the $|\cdot|_{r,N}$ norm will correspond to perturbations of the polynomial $\sum_{i=m}^n c_i y^{2d-i}$ in the $|\cdot|_{r,N}$ norm; $(S+f)_e(x, y)$ depends solely on finitely many terms of the Taylor expansion of $S+f$ at the origin. Thus finding bounds of the form (5.3) for $|D_{S+f_2}^2 - D_{S+f_1}^2|$ are equivalent to finding bounds of this form for $|D_{S+f_2}^1 - D_{S+f_1}^1|$, so we restrict our attention to bounding the latter.

Since any zero of $S_e(1, y)$ is of order less than d , we can divide $[-1, 1] = \cup_{i=1}^K I_i$, where I_i is a closed interval on which for some $\epsilon > 0$ and some $0 \leq l < d$ depending on i we have an estimate

$$\left| \frac{d^l S_e(1, y)}{dy^l} \right| > 2\epsilon \quad (5.7)$$

For a given perturbation function $f(x, y)$, we let $f_e(x, y)$ denote the sum of terms of f 's Taylor expansion lying on e , where $f_e(x, y)$ can now be zero. Shrinking the η and raising the N of Theorem 2.2 if necessary, we may assume that for all f with $|f|_{N,r} < \eta$ on I_i we have

$$\left| \frac{d^l f_e(1, y)}{dy^l} \right| < \epsilon$$

As a result we have

$$\left| \frac{d^l (S+f)_e(1, y)}{dy^l} \right| > \epsilon \quad (5.8)$$

Thus if $l \geq 1$, one may apply the measure Van der Corput lemma (see [C]) in the y direction to obtain that there is a constant C depending only on S such that for all λ the following holds.

$$|\{y \in I_i : |(S+f)_e(1, y)| < \lambda\}| < C\lambda^{\frac{1}{l}} \quad (5.9)$$

Next, we write $D_S^1 = \sum_{i=1}^K E_S^i$, where E_S^i denotes the portion of the integral over I_i . We will give a bound of the form (5.3) for each $|E_{S+f_2}^i - E_{S+f_1}^i|$. Note that

$$E_{S+f_2}^i - E_{S+f_1}^i \leq \int_{I_i} |((S+f_2)_e(1,y))^+)^{-\frac{1}{d}} - ((S+f_1)_e(1,y))^+)^{-\frac{1}{d}}| \quad (5.10)$$

Denote $|f_2 - f_1|_{r,N}$ by δ . We write (5.10) as $J_1 + J_2$, where J_1 is the integral over y for which $|(S+f_1)_e(1,y)| < \delta^a$ for some small a to be determined by our arguments, and J_2 is the integral over y where $|(S+f_1)_e(1,y)| \geq \delta^a$. Note that if $l = 0$ in (5.8) – (5.9), then if $\delta < \epsilon^{\frac{1}{a}}$, which we may assume, by (5.8) $|(S+f_1)_e(1,y)|$ is never less than δ^a and therefore $J_1 = 0$. Thus when analyzing J_1 we may always assume $l \geq 1$. By (5.10) we have

$$|J_1| \leq \int_{|(S+f_1)_e(1,y)| < \delta^a} |(S+f_2)_e(1,y)|^{-\frac{1}{d}} + \int_{|(S+f_1)_e(1,y)| < \delta^a} |(S+f_1)_e(1,y)|^{-\frac{1}{d}} \quad (5.11)$$

Since $|f_2 - f_1|_{r,N} = \delta$, $|(f_2 - f_1)_e(1,y)| < C\delta$ for all $y \in I_i$. Therefore $|(S+f_2)_e(1,y)| \leq |(S+f_1)_e(1,y)| + |(f_1 - f_2)_e(1,y)| < \delta^a + C\delta < 2\delta^a$. Hence for any y in the domain of the first integral, we have $|(S+f_2)_e(1,y)| < \delta + \delta^a < 2\delta^a$. Thus (5.11) is bounded by

$$|J_1| \leq \int_{|(S+f_2)_e(1,y)| < 2\delta^a} |(S+f_2)_e(1,y)|^{-\frac{1}{d}} + \int_{|(S+f_1)_e(1,y)| < \delta^a} |(S+f_1)_e(1,y)|^{-\frac{1}{d}} \quad (5.12)$$

We bound the first term of (5.12) as the second term is done the same way. For any $g(y)$, by the relation between L^p norms and distribution functions, applied to $\frac{1}{|g(y)|}$, we have

$$\int_{|g(y)| < \lambda_0} |g(y)|^{-\frac{1}{d}} dy = \frac{1}{d} \int_0^{\lambda_0} \lambda^{-\frac{1}{d}-1} |\{y : |g(y)| < \lambda\}| d\lambda \quad (5.13)$$

Thus the first term of (5.12) is bounded by

$$\frac{1}{d} \int_0^{2\delta^a} \lambda^{-\frac{1}{d}-1} |\{y : |(S+f_2)_e(1,y)| < \lambda\}| d\lambda \quad (5.14)$$

Inserting (5.9) (recalling $l \geq 1$), we have that (5.14) is bounded by

$$\frac{1}{d} \int_0^{2\delta^a} \lambda^{-\frac{1}{d} + \frac{1}{l} - 1} d\lambda \quad (5.15)$$

Since $l < d$, the exponent in (5.15) is greater than -1 , so (5.15) is bounded by $C\delta^{a'}$ for some $a' > 0$. This is the estimate that we seek since $|f_2 - f_1|_{r,N} = \delta$.

It remains to bound J_2 , the portion of (5.10) where $|(S+f_1)_e(1,y)| \geq \delta^a$. In this case, $|(S+f_2)_e(1,y) - (S+f_1)_e(1,y)| = |(f_2 - f_1)_e(1,y)| \leq C\delta < \frac{1}{2}\delta^a$. So if the integrand in J_2 is nonzero the terms are both positive. Thus J_2 becomes

$$\int_{(S+f_1)_e(1,y) \geq \delta^a} |((S+f_2)_e(1,y))^{-\frac{1}{d}} - ((S+f_1)_e(1,y))^{-\frac{1}{d}}| \quad (5.16)$$

By the mean value theorem, for each y there is a t_y with $|t_y| \leq |(f_1 - f_2)_e(1, y)| < C\delta$ such that the integrand of (5.16) is equal to $\frac{1}{d}|(f_2 - f_1)_e(1, y)((S + f_1)_e(1, y) + t_y)^{-\frac{1}{d}-1}|$. Since $|t_y| < C\delta$ and $(S + f_1)_e(1, y) \geq \delta^a$, $|(S + f_1)_e(1, y) + t_y|$ is at least $\frac{1}{2}(S + f_1)_e(1, y)$. So since $|(f_2 - f_1)_e(1, y)| < C\delta$, we conclude that the integrand in (5.16) is bounded by $C\delta(S + f_1)_e(1, y)^{-\frac{1}{d}-1}$. Thus (5.16) is bounded by

$$C\delta \int_{|(S+f_1)_e(1,y)| \geq \delta^a} |(S + f_1)_e(1, y)|^{-\frac{1}{d}-1} dy \quad (5.17)$$

If $l = 0$, then the integrand in (5.17) is bounded below by (5.8) and thus (5.17) gives a bound of $C\delta$, better than what we need. If $l > 0$, we again use the relation between L^p norms and distribution functions, this time in the form

$$\int_{(S+f_1)_e(1,y) \geq \delta^a} |(S+f_1)_e(1,y)|^{-\frac{1}{d}-1} dy = \left(1 + \frac{1}{d}\right) \int_{\delta^a}^{\infty} \lambda^{-\frac{1}{d}-2} |\{y : |(S+f_1)_e(1,y)| < \lambda\}| d\lambda \quad (5.18)$$

Substituting (5.9) in this time gives that (5.17) is bounded by

$$C\delta \int_{\delta^a}^{\infty} \lambda^{\frac{1}{l}-\frac{1}{d}-2} d\lambda \quad (5.19)$$

Since $l \geq 1$, the exponent here is less than -1 , and we may integrate. Thus (5.19) becomes $C\delta^{1+a(\frac{1}{l}-\frac{1}{d}-1)}$. As long as a is sufficiently small this is bounded by say $C\delta^{\frac{1}{2}}$ and we are done with the proof for the case of Case 1 superadapted coordinates.

We now proceed to Case 3 superadapted coordinates in the real-analytic case. Without loss of generality, we assume that the bisectrix intersects $N(S)$ in the interior of a horizontal ray. We let $a(x)y^d$ denote the sum of the terms of $S(x, y)$'s Taylor expansion on this ray, so that $a(x)$ is real-analytic and $d = d(S)$. In [G2] it is proven that (A.3) holds in the real-analytic situation for Case 3 superadapted coordinates, so like above it suffices to prove (2.9) with $B_{\phi, S}$ in place of $A_{\phi, S}$. Explicit formulas for $B_{\phi, S}$ are given in [G2]. If d is even, we have

$$B_{\phi, S} = 2 \int_{-\infty}^{\infty} (a(x)^+)^{-\frac{1}{d}} \phi(x, 0) dx \quad (5.20a)$$

While if d is odd, we have

$$B_{\phi, S} = \int_{-\infty}^{\infty} |a(x)|^{-\frac{1}{d}} \phi(x, 0) dx \quad (5.20b)$$

For an allowable perturbation function $f(x, y)$, we may write $f(x, y) = F(x)y^d + O(y^{d+1})$, with $F(x)$ real-analytic, so that when d is even we have

$$B_{\phi, S+f_2} - B_{\phi, S+f_1} = 2 \int_{-\infty}^{\infty} (((a(x)+F_2(x))^+)^{-\frac{1}{d}} - ((a(x)+F_1(x))^+)^{-\frac{1}{d}}) \phi(x, 0) dx \quad (5.21a)$$

While if d is odd we have

$$B_{\phi, S+f_2} - B_{\phi, S+f_1} = \int_{-\infty}^{\infty} (|(a(x) + F_2(x))|^{-\frac{1}{d}} - |(a(x) + F_1(x))|^{-\frac{1}{d}}) \phi(x, 0) dx \quad (5.21b)$$

Thus if d is even we have

$$|B_{\phi, S+f_2} - B_{\phi, S+f_1}| \leq 2\|\phi\|_{L^\infty} \int_{-r}^r |((a(x) + F_2(x))^+)^{-\frac{1}{d}} - ((a(x) + F_1(x))^+)^{-\frac{1}{d}}| dx \quad (5.22a)$$

And if d is odd we have

$$|B_{\phi, S+f_2} - B_{\phi, S+f_1}| \leq \|\phi\|_{L^\infty} \int_{-r}^r | |(a(x) + F_2(x))|^{-\frac{1}{d}} - |(a(x) + F_1(x))|^{-\frac{1}{d}} | dx \quad (5.22b)$$

Recall r is the radius of the disk we are working in. Let ax^c denote the term of lowest order of $a(x)$. Then we may assume that r is sufficiently small that $|\frac{d^c(a(x)+F(x))}{dx^c}| > \frac{|a|}{2}$ on the disk for all allowable perturbations. Thus we may argue as in Case 1 with just one I_i , using this inequality in place of (5.8) and using the fact that $|F_2(x) - F_1(x)| \leq \|f_2 - f_1\|_{r, N}$. The argument for (5.22a) is exactly the same as in Case 1, while in (5.22b) one makes minor modifications due to the fact one is no longer taking the positive parts of the functions in question. We omit the details for brevity and we are done.

Appendix. Formulas for the coefficient of the principal term of the asymptotics.

In this appendix, we show that the formulas for the coefficients $A_{S, \phi}$ given in the real-analytic case in [G2] carry over to the smooth situation when $S(x, y)$ is in Case 1 or Case 2 superadapted coordinates. When $S(x, y)$ has a Morse critical point at the origin, explicit formulas are well-known for the general smooth case (see [S] p.344-347), so throughout we will assume $S(x, y)$ does not have a Morse critical point at the origin. In [G2], one defines

$$I_{S, \phi}(\epsilon) = \int_{\{(x, y): 0 < S(x, y) < \epsilon\}} \phi(x, y) dx dy \quad (A.1)$$

It is shown in [G2] that for Case 1 or Case 2 in the smooth case, and in Case 3 for the real-analytic case only, that if $S(x, y)$ is in superadapted coordinates then as $\epsilon \rightarrow 0$ one has asymptotics

$$I_{S, \phi}(\epsilon) = B_{\phi, S} \epsilon^\delta |\ln(\epsilon)|^p + o(\epsilon^\delta |\ln(\epsilon)|^p) \quad (A.2)$$

Here $(-\delta, p)$ is the oscillatory type of S and explicit formulas for $B_{\phi, S}$ are shown in [G2]. In the real-analytic case, one can use well-known methods (cf Ch 7 of [AGV]) to get explicit formulas for the $A_{\phi, S}$ in terms of these formulas for $B_{\phi, S}$. Namely, in Case 1 and Case 3 superadapted coordinates we have

$$A_{\phi, S} = \frac{\Gamma(\frac{1}{d})}{d} (e^{i\frac{\pi}{2d}} B_{\phi, S} + e^{-i\frac{\pi}{2d}} B_{\phi, -S}) \quad (A.3)$$

In Case 2, one has

$$A_{\phi,S} = -\frac{\Gamma(\frac{1}{d})}{d}(e^{i\frac{\pi}{2d}}B_{\phi,S} + e^{-i\frac{\pi}{2d}}B_{\phi,-S}) \quad (A.4)$$

In this section, we will explain why formulas (A.3) and (A.4) still hold for $S(x, y)$ if it is in Case 1 or Case 2 superadapted coordinates. In both cases our arguments resemble those used in [IM2] for the Case 1 situation. (Case 2 is analyzed differently in that paper.)

We first suppose $S(x, y)$ is in Case 1 superadapted coordinates. As in section 4, we denote by e the edge of $N(S)$ intersecting the bisectrix in its interior, and the equation of the line containing this edge by $x + my = c$. Again we take a small enough rectangle $[-r, r] \times [-r^m, r^m]$ and divide it into 4 regions via the curves $y = \pm|x|^m$. As before we focus our attention on the the region U_r consisting of points where $0 < x < r$ and $-x^m < y < x^m$ as the other three regions are dealt with similarly. We also again use the wedges W_i used in section 4. Recall each wedge W_i is of the form $\{(x, y) : 0 < x < r, a_i x^m < y < b_i x^m\}$, and on a given wedge we either have an estimate $\partial_y S(x, y) > Cx^{c-lm}$ for some $1 \leq l < d(S)$, or an estimate $\partial_x S(x, y) > Cx^{c-1}$ (possibly after a coordinate change of the form $(x, y) \rightarrow (x, y + \frac{1}{2}x^m)$ in the case where the wedge was centered along the x -axis.)

Let $\epsilon > 0$ be a small number, to be determined by our arguments. We write $W_i = X_i \cup Y_i$, where

$$X_i = \{(x, y) : 0 < x < \lambda^{-\epsilon}, a_i x^m < y < b_i x^m\} \quad (A.5a)$$

$$Y_i = \{(x, y) : \lambda^{-\epsilon} < x < r, a_i x^m < y < b_i x^m\} \quad (A.5b)$$

We first estimate $\int_{Y_i} e^{i\lambda S(x,y)} \phi(x, y) dx dy$. To do this, if $\partial_y S(x, y) > Cx^{c-lm}$ on W_i then we use the Van der Corput lemma for measures (see [C]) in the y direction and integrate the result with respect to x , and if $\partial_x S(x, y) > Cx^{c-1}$ we use the Van der Corput lemma for measures in the x direction and integrate the result with respect to y . In the former case, for a given x we get the estimate

$$\left| \int_{a_i x^m}^{b_i x^m} e^{i\lambda S(x,y)} \phi(x, y) dy \right| \leq C\lambda^{-\frac{1}{l}} |x|^{-\frac{\epsilon}{l}+m} \quad (A.6)$$

Thus we have

$$\left| \int_{Y_i} e^{i\lambda S(x,y)} \phi(x, y) dx dy \right| \leq C\lambda^{-\frac{1}{l}} \int_{\lambda^{-\epsilon}}^r |x|^{-\frac{\epsilon}{l}+m} dx \quad (A.7)$$

Since $l < d$, we may let $\delta > 0$ be the minimal possible value of $\frac{1}{l} - \frac{1}{d(S)}$. Thus if we choose ϵ small enough that the x integral here is bounded by $C\lambda^{\frac{\delta}{2}}$, then since $-\frac{1}{l} + \frac{\delta}{2} \leq -\frac{1}{d(S)} - \frac{\delta}{2}$ we have

$$\left| \int_{Y_i} e^{i\lambda S(x,y)} \phi(x, y) dx dy \right| \leq C'\lambda^{-\frac{1}{d(S)} - \frac{\delta}{2}} \quad (A.8)$$

Hence the Y_i integral is of order of magnitude less than $\lambda^{-\frac{1}{d(S)}}$. We now turn to the X_i integral. Here we write $S(x, y) = S_N(x, y) + (S(x, y) - S_N(x, y))$, where $S_N(x, y)$ is the sum

of the terms of the Taylor expansion of $S(x, y)$ to a high enough order N to be determined as a function of ϵ . Then

$$e^{i\lambda S(x, y)} = e^{i\lambda S_N(x, y)} e^{i\lambda(S(x, y) - S_N(x, y))} \quad (A.9)$$

$$= e^{i\lambda S_N(x, y)} + e^{i\lambda S_N(x, y)} (e^{i\lambda(S(x, y) - S_N(x, y))} - 1) \quad (A.10)$$

Since on X_i we have $0 < x < \lambda^{-\epsilon}$ and $|y| < x^m$, by choosing N sufficiently large (depending on ϵ) we can ensure that on X_i we have

$$|S(x, y) - S_N(x, y)| \leq C_N \lambda^{-3} \quad (A.11)$$

By the mean value theorem this in turn implies that

$$|e^{i\lambda S_N(x, y)} (e^{i\lambda(S(x, y) - S_N(x, y))} - 1)| \leq C'_N \lambda^{-2} \quad (A.12)$$

Thus we have

$$\int_{X_i} e^{i\lambda S(x, y)} \phi(x, y) dx dy = \int_{X_i} e^{i\lambda S_N(x, y)} \phi(x, y) dx dy + O(\lambda^{-2}) \quad (A.13)$$

As a result,

$$\int_{W_i} e^{i\lambda S(x, y)} \phi(x, y) dx dy = \int_{X_i} e^{i\lambda S_N(x, y)} \phi(x, y) dx dy + O(\lambda^{-\frac{1}{d(S)} - \frac{\delta}{2}}) \quad (A.14)$$

Furthermore, by applying the Y_i integral argument to S_N in place of S for N sufficiently large, we have

$$\int_{X_i} e^{i\lambda S_N(x, y)} \phi(x, y) dx dy = \int_{W_i} e^{i\lambda S_N(x, y)} \phi(x, y) dx dy + O(\lambda^{-\frac{1}{d(S)} - \frac{\delta}{2}}) \quad (A.15)$$

We conclude that

$$\int_{W_i} e^{i\lambda S(x, y)} \phi(x, y) dx dy = \int_{W_i} e^{i\lambda S_N(x, y)} \phi(x, y) dx dy + O(\lambda^{-\frac{1}{d(S)} - \frac{\delta}{2}}) \quad (A.16)$$

Adding this over all W_i and over all 4 pieces of $[-r, r] \times [-r^m, r^m]$, we get that if N is sufficiently large, then

$$J_{S, \phi}(\lambda) = J_{S_N, \phi}(\lambda) + O(\lambda^{-\frac{1}{d(S)} - \frac{\delta}{2}}) \quad (A.17)$$

But $J_{S_N, \phi}(\lambda)$ has asymptotics with an $O(\lambda^{-\frac{1}{d(S)}})$ term, whose formula is given explicitly in terms of finitely many terms of the Taylor expansion of $S_N(x, y)$. Since these terms are the same for S and S_N if N is sufficiently large, these formulas will hold for $S(x, y)$ as well. This completes the proof for Case 1 superadapted coordinates.

We now move on to Case 2 superadapted coordinates. We argue similarly to the Case 1 situation, using the Case 2 wedges of Section 4. Specifically, we let m be such

that the line $x + my = c$ intersects $N(S)$ at the point $(d(S), d(S))$ only, and we divide the rectangle $[-r, r] \times [-r^m, r^m]$ into 4 regions V_r via the curves $y = \pm|x|^m$. As in section 4, we focus our attention on W_r , the region where $0 < x < r$ and $-r^m < y < r^m$ as the other three regions are done the same way. As in (4.18), on W_r we have an estimate

$$|\partial_y^d S(x, y)| > Cx^d \quad (\text{A.18})$$

Similar to in Case 1, for small $\epsilon > 0$ we subdivide into regions X and Y defined by

$$X = \{(x, y) : 0 < x < \lambda^{-\epsilon}, -x^m < y < x^m\} \quad (\text{A.19a})$$

$$Y = \{(x, y) : \lambda^{-\epsilon} < x < r, -x^m < y < x^m\} \quad (\text{A.19b})$$

One may use the Van der Corput lemma in the y direction to obtain

$$\left| \int_{-x^m}^{x^m} e^{i\lambda S(x, y)} \phi(x, y) dy \right| \leq C\lambda^{-\frac{1}{d}} |x|^{-1} \quad (\text{A.20})$$

Thus we have

$$\left| \int_Y e^{i\lambda S(x, y)} \phi(x, y) dx dy \right| \leq C\lambda^{-\frac{1}{d}} \int_{\lambda^{-\epsilon}}^r |x|^{-1} dx \quad (\text{A.21})$$

$$\leq C\epsilon \ln(\lambda) \lambda^{-\frac{1}{d}} \quad (\text{A.22})$$

So while this does contribute to the main term of the asymptotics, it does so in a way that shrinks linearly with ϵ . As for the X integral, one can argue exactly as in the Case 1 situation and say that if one replaces S by S_N in $\int_X e^{i\lambda S(x, y)} \phi(x, y) dx dy$, the difference is bounded in absolute value by $C_\epsilon \lambda^{-2}$. (Since N is a function of ϵ so is the constant here.) Furthermore, the argument used to show (A.22) works for S_N in place of S , so $|\int_Y e^{i\lambda S_N(x, y)} \phi(x, y) dx dy|$ is also bounded by (A.22). We conclude that

$$\left| \int_{W_r} e^{i\lambda S(x, y)} \phi(x, y) dx dy - \int_{W_r} e^{i\lambda S_N(x, y)} \phi(x, y) dx dy \right| \leq C_\epsilon \lambda^{-2} + C\epsilon \ln(\lambda) \lambda^{-\frac{1}{d}} \quad (\text{A.23})$$

Adding (A.23) over all four squares, we conclude that

$$|J_{S, \phi}(\lambda) - J_{S_N, \phi}(\lambda)| \leq C_\epsilon \lambda^{-2} + C\epsilon \ln(\lambda) \lambda^{-\frac{1}{d}} \quad (\text{A.24})$$

Like in Case 1, the formulas of [G2] are such that for N sufficiently large, the formula applied to S is the same as the formula applied to S_N . Thus we have shown that the formula for the leading term of the asymptotics in the real-analytic case also holds in the smooth case modulo a term bounded by $C\epsilon \ln(\lambda) \lambda^{-\frac{1}{d}}$ as $\lambda \rightarrow \infty$. Letting ϵ go to zero shows that the leading terms are in fact the same, and we are done.

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