Math 417 Homework 3

Due September 18

1. Define $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$. Show that $\sin'(z) = \cos(z)$, $\cos'(z) = -\sin(z)$, and $(\cos(z))^2 + (\sin(z))^2 = 1$ for all z.

2. Let D the set of all $z = re^{i\theta}$ such that $r > 0$ and $-\pi < \theta < \pi$. For some real $\alpha \neq 0$, on the set D define $f(re^{i\theta}) = r^\alpha e^{i\alpha\theta}$. Show that f is differentiable at every point in D with $f'(re^{i\theta}) = \alpha r^{\alpha-1} e^{i(\alpha-1)\theta}$.

3. Let D be as in problem 3. On D, define $g(re^{i\theta}) = i\theta + \ln(r)$. Show $g(z)$ is is differentiable at every point in D with $g'(z) = \frac{1}{z}$ for all z.

4. Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$, with $f'(z_0) \neq 0$. Show that the vectors $\nabla u(x_0, y_0)$ and $\nabla v(x_0, y_0)$ are perpendicular and have the same magnitude. Show that the same is true for $g(z) = \bar{z}$ at every point; hence the converse does not hold.

5. Let $u(x, y) = 2y^2 - 2x^2$. Find all functions $v(x, y)$ such that $u(x, y) + iv(x, y)$ is differentiable at every z in $\mathbf{C}$