Fourier Transforms of Irregular Local Hypersurface Measures in $\mathbb{R}^3$

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1. Background and Theorem Statements.

In this paper, we prove sharp estimates for Fourier transforms of local hypersurface measures in $\mathbb{R}^3$ with possibly singular density functions. These estimates are for both the normal and tangential components of the Fourier transform. The densities allowed are those from a reasonably general class of functions that can be constructed from real-analytic functions. The tools used are Van der Corput-type lemmas and a resolution of singularities algorithm for real-analytic functions in two variables which provides appropriate coordinate systems to effectively use such Van der Corput lemmas.

Let $f(x_1, x_2)$ be a real-analytic function on a neighborhood of the origin such that $f(0,0) = 0$ and $\nabla f(0,0) = (0,0)$. The surfaces we will look at will be a portion of the graph of $f(x_1, x_2)$ near the origin. Let $q_1(x_1, x_2),..., q_N(x_1, x_2)$ be real-analytic functions on a neighborhood of $(0,0)$, not identically zero and let $D_0$ be a disk centered at the origin such that $f(x_1, x_2)$ and all $q_i(x_1, x_2)$ are defined on $D_0$. Define $E = \cap_{j=1}^N \{(x_1, x_2) \in D_0 : q_j(x_1, x_2) > 0\}$. For $D \subset D_0$ a smaller disk centered at the origin, we will examine the Fourier transform of the part of the graph of $f(x_1, x_2)$ above $E \cap D$, weighted by a function $\phi(x_1, x_2)$ to be described momentarily. Note that by taking $N = 1$ and $q_1(x_1, x_2) = 1$, this includes the case where $E \cap D = D$.

We look at the surface measure (inverse) Fourier transform given by

$$ U(\lambda_0, \lambda_1, \lambda_2) = \int_{E \cap D} e^{i \lambda_0 f(x_1, x_2) + i \lambda_1 x_1 + i \lambda_2 x_2} \phi(x_1, x_2) \, dx_1 \, dx_2 \quad (1.1) $$

We assume $E \cap D \neq \emptyset$ to avoid trivialities. The function $\phi(x_1, x_2)$ here is as follows. For some real-analytic functions $\{h_i(x_1, x_2)\}_{i=1}^M$ on $D_0$ we assume that

$$ \phi(x_1, x_2) = \alpha(x_1, x_2) \prod_{i=1}^M |h_i(x_1, x_2)|^{\gamma_i} \quad (1.2) $$

Here $\alpha(x_1, x_2)$ is $C^1$ on $D$ except at $(0,0)$ and for some constant $A$ satisfies

$$ |\alpha(x_1, x_2)| \leq A \quad (1.3) $$

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\[ |\nabla \alpha(x_1, x_2)| \leq A(x_1^2 + x_2^2)^{-\frac{1}{2}} \] (1.4)

The prototypical \( \alpha(x_1, x_2) \) would just be a smooth function, but since our arguments carry through whenever (1.3) – (1.4) hold we assume this more general form. The only assumption we make on the exponents \( \gamma_i \) is that \( \int_{E \cap D_0} \prod_{j=1}^M |h_i(x_1, x_2)|^{\gamma_i} \) is finite, so that (1.1) is well-defined.

Integrating over sets \( E \cap D \) in (1.1) allows us to consider the Fourier transforms of analytic "pieces" of surface measures. For example if one just wants to estimate the Fourier transform of the measure corresponding to the portion of the surface inside some analytic three-dimensional region, one can use a partition of unity to reduce the integral in question into finitely many terms of the form (1.1). Another example would be if one wants to estimate the Fourier transform of a surface measure where one wants several different weights on several different parts of the surface. If the different parts can be defined via real-analytic functions, then one can again write the integral in question as the sum of finitely many terms of the form (1.1).

Our theorem statements will make use of the following fact about \( g(x_1, x_2) = \chi_E(x_1, x_2) \prod_{i=1}^M |h_i(x_1, x_2)|^{\gamma_i}. \)

**Lemma 1.1.** (Lemma 1.2 of [G6]) Suppose \( v = (v_1, v_2) \) is a unit vector in \( \mathbb{R}^2 \), and let \( v^\perp = (v_2, -v_1) \) be the orthogonal unit vector. There exist \( \delta_v, c_v > 0 \) and a \( e_v = 0 \) or 1 such that if \( c < c_v \) then there are \( a_{v,c}, b_{v,c} > 0 \) such that for \( 0 < r < c \) one has

\[
 a_{v,c} r^{\delta_v} |\ln r|^{e_v} \leq \int_{\{(x_1, x_2); |(x_1, x_2) \cdot v| < r, |(x_1, x_2) \cdot v^\perp| < c\}} g(x_1, x_2) \, dx_1 \, dx_2 \leq b_{v,c} r^{\delta_v} |\ln r|^{e_v}
\] (1.5)

Let \( \delta = \inf_v \delta_v. \) In [G6] it is shown that \( \delta > 0 \), and that if \( \delta < \frac{1}{2} \) then \( \delta_v \) is the same for all but at most finitely many directions \( v \). If there are exceptional \( v \), then \( \delta_v \) is smaller for the remaining directions. Thus if \( \delta < \frac{1}{2} \), then \( \delta = \delta_v \) for at least one \( v \). Let \( e = 0 \) or 1 denote the maximal \( e_v \) for such directions. Then \( (\delta, e) \) can be viewed as corresponding to the slowest possible decay rate in \( r \) of (1.5) in any direction.

It is worth mentioning that in [G6] it was shown that the exceptional directions must satisfy the following. Let \( H_j(x_1, x_2) \) denote the homogeneous polynomial given by the sum of the terms of lowest degree in the Taylor expansion of \( h_j(x_1, x_2) \) at the origin. The zero set of \( H_j(x_1, x_2) \) is either the origin or a finite union of lines passing through the origin. Let \( l_1, ..., l_p \) be the (possibly empty) set of lines that are either part of the zero set of some \( H_j(x_1, x_2) \) or are tangent to \( E \cap D \) at the origin. In [G6] it was shown that if \( v \) is an exceptional direction, then \( v \) is perpendicular to some \( l_i \).

Our theorems will have the most desirable statements when the \( h_i(x_1, x_2) \) and \( q_j(x_1, x_2) \) are compatible with \( f(x_1, x_2) \) on \( E \cap D \) in a certain sense:
Lemma 1.1. If \( b \)

\[ \text{Suppose the conclusions of Theorem 3.2 hold for } f(x_1, x_2), \text{ the } h_i(x_1, x_2), \text{ and the } q_j(x_1, x_2) \text{ as described after Corollary 3.3, there is some domain } D_i \text{ for which } \alpha_i = s_i \text{ and } \beta_i = 0. \]

That is:

\( a \) The coordinate change \( \eta_i(x_1, x_2) \) is of the form \((x_1, x_2) \rightarrow (\pm x_1, \pm x_2 + p_i x_1 + l_i x_1^{s_i} + o(x_1^{s_i})) \) or \((x_1, x_2) \rightarrow (x_1 + p_i x_2 + l_i x_2^{s_i} + o(x_2^{s_i}), \pm x_2)\), where \( s_i > 1 \) and \( l_i \neq 0 \).

\( b \) There are positive constants \( c_1 \) and \( c_2 \) such that on \( D'_i \) one has

\[ c_1 x_1^{s_i} < |f_i(x_1, x_2)| < c_2 x_1^{s_i} \quad (1.6) \]

If the functions \( \{h_i(x_1, x_2)\}_{j=1}^M \) and \( \{q_j(x_1, x_2)\}_{j=1}^N \) are not incompatible with \( f(x_1, x_2) \), we say they are compatible with \( f(x_1, x_2) \).

Why such a condition is needed will be explained below from (1.16) onwards. The most common situation where the compatibility condition holds would be when resolving the singularities of \( f(x_1, x_2) \) on a neighborhood of the origin causes the singularities of each \( h_i(x_1, x_2) \) and each \( q_j(x_1, x_2) \) to be resolved as well. For as shown in [G1], in such situations one always has \( \alpha_i > s_i \). An example of when this occurs is when \( M = 2 \) with \( h_1(x_1, x_2) = x_1^2 + x_2^2 \), \( h_2(x_1, x_2) = f(x_1, x_2) \), and \( E \cap D = D \). This was the situation considered in [G2].

Our theorem on Fourier transform decay when the compatibility condition holds is as follows.

**Theorem 1.3.** Suppose \( \alpha(x_1, x_2) \) satisfies (1.3)–(1.4) and \( \{h_i(x_1, x_2)\}_{i=1}^M \), \( \{q_j(x_1, x_2)\}_{j=1}^N \) are compatible with \( f(x_1, x_2) \). Then if the radius of \( D \) is sufficiently small, the following holds.

Let \( d\mu_h \) denote the measure \( \prod_{i=1}^M |h_i(x_1, x_2)|^{\gamma_i} \, dx_1 \, dx_2 \). Suppose \( \epsilon, \ell > 0 \) are such that for some \( C_{\epsilon, \ell} > 0 \), for all \( 0 < t < \frac{1}{2} \) one has \( \mu_h((x_1, x_2) \in E \cap D : |f(x_1, x_2)| < t) \leq C_{\epsilon, \ell} t^{\ell} |\ln t|^{\ell} \).

\( a \) If \( \epsilon < \frac{1}{3} \) we have an estimate

\[ |U(\lambda_0, \lambda_1, \lambda_2)| \leq C_{f, \mu_h, A, E \cap D, \epsilon, \ell} (1 + |\lambda_0|)^{-\epsilon} \ln(2 + |\lambda_0|) \ell \quad (1.7a) \]

If \( \epsilon = \frac{1}{3} \), we get an additional factor of \( \ln(2 + |\lambda_0|) \) on the right-hand side of (1.7a).

If \( \epsilon > \frac{1}{3} \) then one at least has

\[ |U(\lambda_0, \lambda_1, \lambda_2)| \leq C_{f, \mu_h, A, E \cap D} (1 + |\lambda_0|)^{-\frac{\epsilon}{3}} \quad (1.7b) \]

\( b \) Let \( |\lambda'| \) denote the magnitude of the vector \( \lambda' = (\lambda_1, \lambda_2) \). Let \( (\delta, e) \) be as defined after Lemma 1.1. If \( \delta < \frac{1}{3} \) there is a \( C_{f, \mu_h, A, E \cap D} > 0 \) such that one has the estimate

\[ |U(\lambda_0, \lambda_1, \lambda_2)| \leq C'_{f, \mu_h, A, E \cap D} (1 + |\lambda'|)^{-\delta} \ln(2 + |\lambda'|)^e \quad (1.8a) \]
If $\delta = \frac{1}{3}$ then the same estimate holds with an additional factor of $\ln(2 + |\lambda'|)$.

If $\delta > \frac{1}{3}$ one at least gets an estimate

$$|U(\lambda_0, \lambda_1, \lambda_2)| \leq C_{f, \mu_h, A, E \cap D} (1 + |\lambda'|)^{-\frac{3}{2}}$$

(1.8b)

c) If $\alpha(x_1, x_2)$ is nonnegative and bounded below by a positive constant on a neighborhood of the origin, then if $\delta < \frac{1}{4}$ the exponent $\delta$ in (1.8a) cannot be improved. Similarly, if one denotes by $\epsilon_0$ the supremum of the $\epsilon$ for which one has an estimate $\mu_h(\{(x_1, x_2) \in E \cap D : |f(x_1, x_2)| < t\}) \leq C_{\epsilon, I} t^\epsilon \ln t$, then if $\epsilon_0 < \frac{1}{3}$ the exponent in the right-hand side of (1.7a) cannot be taken greater than $\epsilon_0$.

Although we won’t show it here, resolution of singularities can be used to show that $\epsilon_0$ is always positive. Statements such as the $\epsilon_0 < \frac{1}{3}$ case of Theorem 1.3a) are sometimes referred to as ”stability theorems” for oscillatory integrals: the optimal estimate one can obtain in (1.1) when $\lambda_1 = \lambda_2 = 0$, which can be shown using resolution of singularities to be the optimal estimate of the form (1.7a), is shown by Theorem 1.3a) to be preserved under linear perturbations when $\epsilon_0 < \frac{1}{3}$. This was the type of problem considered in [G2].

When the compatibility condition doesn’t hold, one can break the surface into finitely many pieces such that for a given piece, a version of Theorem 1.3 will hold after an appropriate linear transformation:

**Theorem 1.4.** Suppose the compatibility condition does not hold. Then for sufficiently small $D$ one can write $D = \cup_{i=1}^K A_i$ such that if $S_i$ is the surface $\{(x_1, x_2, f(x_1, x_2)) : (x_1, x_2) \in E \cap A_i\}$, with the same weight $d\mu_h$ as before, then for each $i$, there is a map of the form $L_i(x_1, x_2, x_3) = (\tilde{L}_i(x_1, x_2), x_3 + a_1 x_1 + a_2 x_2)$, $\tilde{L}_i$ an invertible linear map, such that the surface $L_i(S_i)$ with weight $\tilde{L}_i^{-1} d\mu_h$ satisfies the following variant of Theorem 1.3.

a) If $L_i$ is the identity map, then the compatibility condition holds when restricted to $E \cap A_i$ and therefore $L_i(S_i) = S_i$ satisfies the conclusions of Theorem 1.3.

b) If $L_i$ is not the identity map, then $L_i(S_i)$ with weight $\tilde{L}_i^{-1} d\mu_h$ satisfies the conclusions of Theorem 1.3a)-1.3b), except in part a) one uses a certain $f(\tilde{L}_i^{-1}(x_1, x_2)) + r_i(x_1)$ in place of $f(x_1, x_2)$. Here $r_i(x_1)$ is a real analytic function of some $x_1^{N_i}$, where $N_i$ is a positive integer. The function $r_i(x_1)$ vanishes to an integral order $s_i \geq 2$ at $x_1 = 0$.

It is worth pointing out that unlike in Theorem 1.3, the analogue of $\epsilon_0$ in Theorem 1.4b) may be zero, such as in the example of (1.16) – (1.18).

**Example.** Suppose there is just one $h_i(x_1, x_2)$, given by $h_1(x_1, x_2) = f(x_1, x_2)$. Then $\phi(x_1, x_2)$ can be $|f(x_1, x_2)|^\rho$ multiplied by a smooth $\alpha(x_1, x_2)$ supported in $D_0$ and equal to 1 on a neighborhood of the origin. The compatibility condition holds here trivially.
Focusing on part a) of Theorem 1.3, using resolution of singularities (see [G4] for details) it can be shown that \(|\{(x_1, x_2) \in E \cap D : |f(x_1, x_2)| < \epsilon\}|\) has asymptotics of the form \(At^s \ln t^m\) plus smaller terms as \(t \to 0\). Here \(m = 0\) or \(1\). Thus if \(D\) is such that \(\alpha(x_1, x_2) = 1\) on \(D\), then \(\mu_h(\{(x_1, x_2) \in E \cap D : |f(x_1, x_2)| < \epsilon\})\) is given by

\[
\int_{\{(x_1, x_2) \in E \cap D : |f(x_1, x_2)| < \epsilon\}} |f(x_1, x_2)|^\rho \, dx_1 \, dx_2
\]

(1.9)

\[
= \sum_{k=0}^\infty \int_{\{(x_1, x_2) \in E \cap D : 2^{-k-1}t \leq |f(x_1, x_2)| < 2^{-k}t\}} |f(x_1, x_2)|^\rho \, dx_1 \, dx_2
\]

(1.10)

\[
\leq \sum_{k=0}^\infty C(2^{-k}t)^s (\ln(2^{-k}t))^m (2^{-k}t)^\rho
\]

(1.11)

Assuming \(\rho > -s\), this is bounded by

\[
C' t^{\rho + s} \ln t^m
\]

(1.12)

Hence Theorem 1.3 part a) holds with \((\epsilon, l) = (\rho + s, m)\). The case where \(\rho = 0\) is the well-known case of scalar oscillatory integrals, studied in [V] [IkKeM] [IkM], as well as in the author’s earlier work and elsewhere. For example, the paper [V] shows that when \(\rho = 0\), part a) of Theorem 1.3 holds and is sharp when \(\epsilon_0 < \frac{1}{3}\). Some generalizations appear in [G2].

There has been quite a bit of work done previously estimating the decay rate of Fourier transforms of hypersurface measures in \(\mathbb{R}^3\), often when \(E \cap D = D, k = 1\), and \(h_1(x_1, x_2) = 1\), and \(\alpha(x_1, x_2)\) is smooth. So \(\phi(x_1, x_2)\) is a smooth compactly supported function here. Then Theorems 1.3 and 1.5 follow from [G1] in this situation. The general stability results of Karpushkin [K1][K2] also imply part a) of Theorem 1.3 (and more) in this situation. Stronger estimates and generalizations to smooth \(f(x_1, x_2)\) are shown in [D] [IkKeM] [IkM]. The latter papers use the early work [V], where nice geometric connections to the Newton polygon of \(f(x_1, x_2)\) in certain “adapted” coordinate systems are proven. Further cases of part a) follow from [PrY] and [G2]. Other results on the decay of the Fourier transforms of two-dimensional hypersurfaces are proven in [ESa]. If one adds an appropriate nondegeneracy condition on the phase, many of these two-dimensional results extend to higher dimensions; we refer to [CKaN] [DeNSar] [G3] [V] for examples. Other oscillatory integrals related to surface measure Fourier transforms were analyzed in [Gr].

**Sharpness of the conditions** \(\delta < \frac{1}{3}\) and \(\epsilon_0 < \frac{1}{3}\).

Consider the situation where \(E \cap D = \{(x_1, x_2) \in D : x_1 > 0, x_1^3 < x_2 < 2x_1^3\}\), \(f(x_1, x_2) = x_1^3\), and there are two \(h_i(x_1, x_2)\), given by \(h_1(x_1, x_2) = x_1\) and \(h_2(x_1, x_2) = x_2 - x_1^3\). We make no restrictions on \(\gamma_1\) for now, and let \(\gamma_2 = -1 + \eta\) for some small \(\eta\). Assume \(\alpha(x_1, x_2)\) is identically equal to 1 on \(D\). Then in the case at hand we have

\[
U(\lambda_0, \lambda_1, \lambda_2) = \int_{E \cap D} x_1^{\gamma_1} (x_2 - x_1^3)^{-1+\eta} e^{i\lambda_0 x_1^2 + i\lambda_1 x_1 + i\lambda_2 x_2} \, dx_1 \, dx_2
\]

(1.13)
Changing variables from \((x_1, x_2)\) to \((x_1, x_2 + x_3^1)\), we get

\[
U(\lambda_0, \lambda_1, \lambda_2) = \int_{\{(x_1, x_2) \in \tilde{D} : x_1 > 0, 0 < x_2 < x_3^1\}} x_1^{\gamma_1} x_2^{-1+\eta} e^{i\lambda_0 x_1^2 + i\lambda_1 x_1 + i\lambda_2 x_2^1 + i\lambda_2 x_2} \, dx_1 \, dx_2
\]

(1.14)

Here \(\tilde{D}\) is the disk \(D\) in the transformed coordinates. We look at \(U(\lambda_0, \lambda_1, \lambda_2)\) on rays \((\lambda_0, c_1 \lambda_0, c_2 \lambda_0)\) for fixed \(c_1\) and \(c_2\). Then (1.14) becomes

\[
U(\lambda_0, c_1 \lambda_0, c_2 \lambda_0) = \int_{\{(x_1, x_2) \in \tilde{D} : x_1 > 0, 0 < x_2 < x_3^1\}} x_1^{\gamma_1} x_2^{-1+\eta} e^{i\lambda_0 (x_1^2 + c_1 x_1 + c_2 x_2^1)} e^{i\lambda_0 c_2 x_2} \, dx_1 \, dx_2
\]

(1.15)

When \(\eta\) is very small, the \(x_2^{-1+\eta}\) factor ensures that one gets very little decay in (1.15) due to the \(c_2 x_2\) term in the exponential; the behavior is driven by the \(x_1\) integral for fixed values of \(x_2\). Let \(r\) denote the inradius of the disk-like \(\tilde{D}\). If one chooses \(c_1\) and \(c_2\) so that \(x_1^2 + c_1 x_1 + c_2 x_2^1\) has a stationary point of order 3 at some \(x_0\) satisfying \(\frac{\pi}{4} < x_0 < r\), then the best estimate one can get is \(|U(\lambda_0, c_1 \lambda_0, c_2 \lambda_0)| \leq C|\lambda_0|^{-\frac{1}{3}-\eta'}\), where \(\eta'\) approaches zero as \(\eta\) approaches zero. On the other hand, one may choose \(\gamma_1\) so that the exponent \(\epsilon_0\) in Theorem 1.3c) is any given value greater than \(\frac{1}{3}\). Thus the exponent \(\frac{1}{3}\) in Theorem 1.3a) cannot be improved in general. Similarly, one can choose \(\gamma_1\) so that the exponent \(\delta_{\epsilon}\) of Lemma 1.1 in the \(v = (0, 1)\) direction is any given value greater than \(\frac{1}{3}\), and \(\delta_{\epsilon}\) will only be greater in other directions. Hence the exponent \(\frac{1}{3}\) in Theorem 1.3b) can also not be improved in general.

The role of the compatibility condition.

Consider the situation where \(E \cap D = D\), \(\alpha(x_1, x_2) = 1\) on \(D\), \(f(x_1, x_2) = x_1^m\) for some \(m \geq 2\), and there is one \(h_i(x_1, x_2)\), given by \(h_1(x_1, x_2) = x_2 - x_1^m\). We let \(\gamma_1 = -1+\eta\) for some small \(\eta > 0\). Then we have

\[
U(\lambda_0, \lambda_1, \lambda_2) = \int_D \left|x_2 - x_1^m\right|^{-1+\eta} e^{i\lambda_0 x_1^m + i\lambda_1 x_1 + i\lambda_2 x_2} \, dx_1 \, dx_2
\]

(1.16)

Changing variables from \((x_1, x_2)\) to \((x_1, x_2 + x_1^m)\) leads to

\[
U(\lambda_0, \lambda_1, \lambda_2) = \int_D \left|x_2\right|^{-1+\eta} e^{i\lambda_0 x_1^m + i\lambda_1 x_1 + i\lambda_2 x_2 + i\lambda_2 x_2^m} \, dx_1 \, dx_2
\]

(1.17)

So in particular

\[
U(\lambda_0, 0, -\lambda_0) = \int_D \left|x_2\right|^{-1+\eta} e^{i\lambda_0 x_2} \, dx_1 \, dx_2
\]

(1.18)

If \(\eta\) is very small, the (1.18) will have very slow decay in \(|\lambda_0|\); the exponent of the decay rate goes to zero as \(\eta\) tends to zero. Thus both Theorem 1.3a) and 1.3b) will be violated in this instance.

The same general phenomenon will occur whenever the compatibility condition does not hold. If one has a domain \(D_i\) on which \(|f(x_1, x_2)| \sim x_1^m\) and a map of the form
\((x_1, x_2) \rightarrow (x_1, x_2 + p_i x_1 + l_i x_1^{s_i} + o(x_1^{s_i}))\) is used on \(D_i\) in the resolution of singularities process, where \(l_i \neq 0\), then there will be a certain ratio \(\frac{|\lambda_0|}{|\lambda_2|}\) for which one gets a similar sort of cancellation as in the above example. It can also be shown that if there is an \(|h_i(x_1, x_2)|^{\eta}\) of the form \(|x_2 - l_i x_1^{s_i}|^{-1+\eta}\) for sufficiently small \(\eta > 0\), then if one replaces \(E \cap D\) by \(E \cap D \cap \{(x_1, x_2) : x_1 > 0, p_i x_1 + (l_i - \rho) x_1^{s_i} < x_2 < p_i x_1 + (l_i + \rho) x_1^{s_i}\}\) for sufficiently small \(\rho\), then Theorem 1.3 will not hold for the resulting surface. Whether or not the theorem holds for the original surface depends on the integral over the rest of \(E \cap D\).

Although it is unclear what the sharp exponents would be when \(\delta > \frac{1}{3}\) or \(\epsilon_0 > \frac{1}{3}\), it turns out there is a partial version of Theorem 1.3b) with a relatively simple form that applies in these situations, regardless of whether or not the compatibility condition holds.

**Theorem 1.5.** If the radius of \(D\) is sufficiently small, the following holds. Suppose \(1 \leq p \leq \infty\) is such that \(\prod_{i=1}^{M} |h_i(x_1, x_2)|^{\eta} \in L^p(E \cap D)\). Let \(p'\) be the exponent conjugate to \(p\); that is, \(\frac{1}{p} + \frac{1}{p'} = 1\). Then if \(p \neq 2\) there is a \(C_{f, \mu s, \lambda, \epsilon, \gamma} > 0\) such that one has the estimate

\[
|U(\lambda_0, \lambda_1, \lambda_2)| \leq C_{f, \mu s, \lambda, \epsilon, \gamma} (1 + |\lambda'|)^{-\min(\frac{1}{p}, \frac{1}{2})}
\]

If \(p = 2\) then the same estimate holds with an additional factor of \((\ln(2 + |\lambda'|))^2\).

### 2. Van der Corput Lemmas.

We start with the well-known Van der Corput lemma (see p. 334 of [S]).

**Lemma 2.1.** Suppose \(h(x)\) is a \(C^k\) function on the interval \([a, b]\) with \(|h^{(k)}(x)| > A\) on \([a, b]\) for some \(A > 0\). Let \(\phi(x)\) be \(C^1\) on \([a, b]\). If \(k \geq 2\) there is a constant \(c_k\) depending only on \(k\) such that

\[
\left| \int_a^b e^{ih(x)} \phi(x) \, dx \right| \leq c_k A^{-\frac{k}{2}} \left( |\phi(b)| + \int_a^b |\phi'(x)| \, dx \right)
\]

If \(k = 1\), the same is true if we also assume that \(h(x)\) is \(C^2\) and \(h'(x)\) is monotone on \([a, b]\).

We will also make use of the following variant of Lemma 2.1 for \(k = 1\).

**Lemma 2.2.** Suppose the hypotheses of Lemma 2.1 hold with \(k = 1\), except instead of assuming that \(h'(x)\) is monotone on \([a, b]\) we assume that \(|h''(x)| < \frac{B}{(b-a)} A\) for some constant \(B > 0\). Then we have

\[
\left| \int_a^b e^{ih(x)} \phi(x) \, dx \right| \leq A^{-1} \left( \int_a^b |\phi'(x)| \, dx + (B + 2) \sup_{[a,b]} |\phi(x)| \right)
\]

**Proof.** We write \(e^{ih(x)} = (h'(x)e^{ih(x)})^{\frac{1}{h'(x)}}\) and integrate by parts in the integral being
estimated, integrating the $h'(x)e^{ih(x)}$ factor to $e^{ih(x)}$ and differentiating $\frac{\phi(x)}{h'(x)}$. We obtain

$$\int_a^b e^{ih(x)} \phi(x) \, dx = e^{ih(b)} \frac{\phi(b)}{h'(b)} - e^{ih(a)} \frac{\phi(a)}{h'(a)} - \int_a^b e^{ih(x)} \frac{d}{dx} \left( \frac{\phi(x)}{h'(x)} \right) \, dx$$

$$= e^{ih(b)} \frac{\phi(b)}{h'(b)} - e^{ih(a)} \frac{\phi(a)}{h'(a)} - \int_a^b e^{ih(x)} \frac{\phi'(x)}{h'(x)} \, dx + \int_a^b e^{ih(x)} \frac{\phi(x)h''(x)}{(h'(x))^2} \, dx$$

(2.3)

The condition that $|h'(x)| > A$ ensures that each of the two boundary terms is bounded in absolute value by $A^{-1} \sup_{[a,b]} |\phi(x)|$. As for the first integral term, taking absolute values of the integrand and inserting $|h'(x)| > A$ gives that this term is bounded in absolute value by $A^{-1} \int_a^b |\phi'(x)| \, dx$. In the second integral term, we use that $|h''(x)| < \frac{B}{(b-a)} A$ and $|h'(x)| > A$, resulting in

$$\left| \frac{\phi(x)h''(x)}{(h'(x))^2} \right| \leq B \frac{A^{-1}}{(b-a)} |\phi(x)|$$

$$\leq B \frac{A^{-1}}{(b-a)} \sup_{[a,b]} |\phi(x)|$$

(2.4)

Thus the second integral term is bounded by $BA^{-1} \sup_{[a,b]} |\phi(x)|$. Adding the bounds for the different terms gives us the bounds on the right-hand side of (2.2) and we are done.

In section 4 we will also use the following two-dimensional mixed-derivative version of the Van der Corput Lemma from [G2].

**Lemma 2.3.** Let $I_1$ and $I_2$ be closed intervals of lengths $l_1$ and $l_2$ respectively, and for some strictly monotone functions $f_1(x)$ and $f_2(x)$ on $I_1$ with $f_1(x) \leq f_2(x)$ let $R = \{(x,y) \in I_1 \times I_2 : f_1(x) \leq y \leq f_2(x)\}$ (Note $R$ might just be $I_1 \times I_2$). Suppose for some $k \geq 2$, $P(x,y)$ is a $C^k$ function on $R$ such that for each $(x,y) \in R$ one has

$$|\partial_{xy} P(x,y)| > M \quad \text{and} \quad \partial_y^k P(x,y) \neq 0$$

(2.5)

Further suppose that $\Psi(x,y)$ is a function on $R$ that is $C^1$ in the $y$ variable for fixed $x$, such that

$$|\Psi(x,y)| < N \quad \forall x, y \quad \text{and} \quad \int_{\{y:(x,y) \in R\}} |\partial_y \Psi(x,y)| \, dy < N \quad \forall x$$

(2.6)

If $R' \subset R$ such that the intersection of $R'$ with each vertical line is either empty or is a set of at most $l$ intervals, then the following estimate holds.

$$\left| \int_{R'} e^{iP(x,y)} \Psi(x,y) \, dx \, dy \right| < C_{kl} N \left( \frac{l_1 l_2}{M} \right)^{\frac{1}{2}}$$

(2.7)
3. Resolution of Singularities in two dimensions.

We will make use of the real-analytic case of the resolution of singularities theorem of [G2] (derived from a related theorem in [G5]), which goes as follows. Let \( S(x, y) = \sum_{\alpha, \beta} S_{\alpha \beta} x^\alpha y^\beta \) be a real-analytic function on a neighborhood of the origin, not identically zero, satisfying \( S(0, 0) = 0 \).

Divide the \( xy \) plane into eight triangles by slicing the plane using the \( x \) and \( y \) axes and two lines through the origin, one of the form \( y = mx \) for some \( m > 0 \) and one of the form \( y = mx \) for some \( m < 0 \). One must ensure that these two lines are not ones on which the function \( \sum_{\alpha+\beta=0} S_{\alpha \beta} x^\alpha y^\beta \) vanishes other than at the origin, where \( o \) denotes the order of the zero of \( S(x, y) \) at the origin. After reflecting about the \( x \) and/or \( y \) axes and/or the line \( y = x \) if necessary, each of the triangles becomes of the form \( T_b = \{(x, y) \in \mathbb{R}^2 : x > 0, 0 < y < bx\} \) (modulo an inconsequential boundary set of measure zero). The version of the real-analytic case of Theorem 2.1 of [G2] that is pertinent here is the following.

**Theorem 3.1.** (Theorem 2.1 of [G2]) Let \( T_b = \{(x, y) \in \mathbb{R}^2 : x > 0, 0 < y < bx\} \) be as above. Abusing notation slightly, use the notation \( S(x, y) \) to denote the reflected function \( S(\pm x, \pm y) \) or \( S(\pm y, \pm x) \) corresponding to \( T_b \). Then there is a \( a > 0 \) and a positive integer \( N \) such that if \( F_a \) denotes \( \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq a, 0 \leq y \leq bx\} \), then one can write \( F_a = \bigcup_{i=1}^n cl(D_i) \), such that for to each \( i \) there is a \( k_i(x) = p_i x \) or \( k_i(x) = p_i x + l_i x^{s_i} + \ldots \) with \( k_i(x^N) \) real-analytic, \( l_i \neq 0 \), and \( s_i > 1 \), such that after a coordinate change of the form \( \eta_i(x, y) = (x, \pm y + k_i(x)) \), the set \( D_i \) becomes a set \( D_i' \) on which the function \( S \circ \eta_i(x, y) \) approximately becomes a monomial \( d_i x^{\alpha_i} y^{\beta_i} \), \( \alpha_i \) a nonnegative rational number and \( \beta_i \) a nonnegative integer in the following sense.

a) \( D_i' = \{(x, y) : 0 < x < a, g_i(x) < y < G_i(x)\} \), where \( g_i(x^N) \) and \( G_i(x^N) \) are real-analytic. If we expand \( G_i(x) = H_i x^{M_i} + \ldots \), then \( M_i \geq 1 \) and \( H_i > 0 \).

b) Suppose \( \beta_i = 0 \). Then \( g_i(x) = 0 \). The set \( D_i' \) can be constructed such that for any predetermined \( \eta > 0 \) there is a \( d_i \neq 0 \) such that on \( D_i' \), for all \( 0 \leq l \leq \alpha_i \) one has

\[
|\partial_x^l (S \circ \eta_i)(x, y) - d_i \alpha_i (\alpha_i - 1) \ldots (\alpha_i - l + 1) x^{\alpha_i - l}| < \eta |d_i| x^{\alpha_i - l} \quad (3.1)
\]

c) If \( \beta_i > 0 \), then \( g_i(x) \) is either identically zero or \( g_i(x) \) can be expanded as \( h_i x^{m_i} + \ldots \) where \( h_i > 0 \) and \( m_i > M_i \). The \( D_i' \) can be constructed such that such that for any predetermined \( \eta > 0 \) there is a \( d_i \neq 0 \) such that on \( D_i' \), for all \( 0 \leq l \leq \alpha_i \) and all \( 0 \leq m \leq \beta_i \) one has

\[
|\partial_x^l \partial_y^m (S \circ \eta_i)(x, y) - \alpha_i (\alpha_i - 1) \ldots (\alpha_i - l + 1) \beta_i (\beta_i - 1) \ldots (\beta_i - m + 1) d_i x^{\alpha_i - l} y^{\beta_i - m}| \\
\leq \eta |d_i| x^{\alpha_i - l} y^{\beta_i - m} \quad (3.2)
\]
In [G1] it was shown that one can may do the constructions so that \( s_i \leq M_i \) for all \( i \) whenever \( k_i(x) \) is not of the form \( p_i x \).

It should be pointed out that the development this theorem was influenced by the philosophy of [PS] where one divides a neighborhood of the origin into wedges on which \( S(x,y) \) and its derivatives are well-behaved, as well as the general philosophy of resolution of singularities where one does changes of variables to monomialize a given real-analytic function of interest.

For the purposes of proving Theorem 1.3, we will need to simultaneously resolve the singularities of the phase \( f(x_1, x_2) \), its \( x_2 \) derivative \( \frac{\partial f}{\partial x_2}(x_1, x_2) \), the functions \( q_j(x_1, x_2) \) used to define \( E \cap D \), and the functions \( h_j(x_1, x_2) \) used in the definition of the weight function \( \phi(x_1, x_2) \). As is well-known in the subject of resolution of singularities, one can often simultaneously resolve the singularities of several functions by simply resolving the singularities of their product. This is the case here as well, as can be seen from the following theorem from [G6].

**Theorem 3.2.** Suppose \( S_1(x, y), \ldots, S_k(x, y) \) are real-analytic functions on a neighborhood of the origin with \( S_j(0, 0) = 0 \) for each \( j \). Let \( D'_i, \alpha_i, \) and \( \beta_i \) be as in Theorem 3.1 applied to \( \prod_{j=1}^k S_j(x, y) \). Then one can further divide each \( D'_i \) into finitely many pieces \( D'_{il} \), such that on each \( D'_{il} \) an additional coordinate change of the form \( (x, y) \to (x, y - c_{il} x^{M_{il}}) \) or \( (x, y - c_{il} x^{m_{il}}) \), \( c_{il} \geq 0 \), will result in each \( S_j(x, y) \) satisfying the conclusions of Theorem 3.1, with one difference: The domains \( D'_{il} \) with \( \beta_i = 0 \) can now only be assumed to have the same form as the domains where \( \beta_i > 0 \). That is, \( D'_{il} \) has the form \( \{(x, y) : 0 < x < a, g_{il}(x) < y < G_{il}(x)\} \), where \( g_{il}(x^{N}) \) and \( G_{il}(x^{N}) \) are real-analytic for some positive integer \( N, G_{il}(x) = H_{il} x^{M_{il}} + \ldots, \) and \( g_{il}(x) \) is identically zero or is of the form \( h_{il} x^{m_{il}} + \ldots \) where \( 1 \leq M_{il} < m_{il} \) and \( h_{il}, H_{il} > 0 \).

We also will make use of the following corollary to Theorem 3.2 that was proven in [G6].

**Corollary 3.3.** For any given \( K \), however large, for any predetermined \( \eta > 0 \) the \( D'_i \) can be constructed so that (3.1) and (3.2) hold for all \( \alpha_i, \beta_i < K \).

**Preliminaries for the theorem proofs.**

We first divide a small rectangle centered at the origin into 8 triangles using two lines \( x_2 = mx_1 \) and the \( x_1 \) and \( x_2 \) axes as in the beginning of this section. We then perform the appropriate reflection if necessary so that we are working on a set of the form \( \{(x_1, x_2) : 0 < x_1 < a, 0 < x_2 < bx_1\} \). Replacing \( f(x_1, x_2) \), the \( q_j(x_1, x_2) \), and the \( h_i(x_1, x_2) \) by the appropriate reflected functions as necessary, we apply Theorem 3.2 to simultaneously resolve the singularities of \( f(x_1, x_2) \), \( \frac{\partial f}{\partial x_2}(x_1, x_2) \), the \( q_j(x_1, x_2) \), and the \( h_i(x_1, x_2) \). We let \( D_i \) and \( D'_i \) be the domains as in Theorem 3.2 when applied in this way.
Lemma 3.4. Suppose the compatibility condition does not hold and $D_i$ is one of the above domains such that in the notation of Theorem 3.1 and 3.2 we have $\alpha_i = s_i$ and $\beta = 0$. Then $s_i \geq 2$ is an integer, and if we write $f_i = f \circ \eta_i$, on $D'_i$ one has $f_i(x_1, x_2) = d_i x_1^{s_i} + o(x_1^{s_i})$. In fact, this holds on a larger set $\{(x_1, x_2) : 0 < x_1 < a, |x_2| < x_1^{s_i'}\}$ for some $s_i' < s_i$.

Proof. Without loss of generality we may assume the coordinate change function $\eta_i(x_1, x_2)$ is of the form $(x_1, x_2 + p_i x_1 + l_i x_1^{s_i} + o(x_1^{s_i}))$ as the other reflected cases are done the same way. To simplify the exposition we restrict to the case where $p_i = 0$ as the general case will follow in the same way after changing coordinates from $(x_1, x_2)$ to $(x_1, x_2 + p_i x_1)$. Hence we assume $\eta_i(x_1, x_2) = (x_1, x_2 + l_i x_1^{s_i} + o(x_1^{s_i}))$ for some $s_i > 1$ and some $l_i \neq 0$.

Since the compatibility condition fails, in the final coordinates we have that $f_i(x_1, x_2) \sim x_1^{s_i}$ along any curve in $D'_i$ containing the origin. We examine any such curve of the form $x_1 \to (x_1, x_2 + s_i^+ \rho p(x_1^{s_i}))$ for a polynomial $p$, some large integer $N$, and $\rho > 0$. Such a curve exists because always $s_i \leq M_i$ in Theorems 3.1-3.2. In the original coordinates this curve becomes of the form $x_1 \to (x_1, l_i x_1^{s_i} + x_1^{s_i} + \rho p(x_1^{s_i}))$.

Let $f(x_1, x_2) = \sum_{a,b} f_{a,b} x_a^b$ denote the Taylor expansion of $f(x_1, x_2)$ at the origin. Let $t$ denote the minimum value of $a + s_i b$ amongst the nonzero terms in the Taylor expansion. Then along the above curve (in the original coordinates) one has

$$f(x_1, x_2) = \left( \sum_{a+s_i b = t} f_{a,b} \right) x_1^t + o(x_1^t) \quad (3.3)$$

Since $f(x_1, x_2) \sim x_1^{s_i}$ along this curve, one has that $t \leq s_i$. But $t$ is also equal to $a + s_i b$ for some nonnegative integers $a$ and $b$. Hence either $b = 0$ or $(a, b) = (0, 1)$. But one cannot have $(a, b) = (0, 1)$ since the gradient of $f$ is zero at the origin. So the only possibility is that $b = 0$. Hence (3.3) becomes

$$f(x_1, x_2) = f_{s_i,0} x_1^{s_i} + o(x_1^{s_i}) \quad (3.4)$$

In particular, $s_i \geq 2$ is an integer. Since by definition at least one term of $\sum_{a+s_i b = t} f_{a,b}$ is nonzero, we must have $f_{s_i,0} \neq 0$. And since $f(x_1, x_2) \sim x_1^{s_i}$ along this curve, we have $t = s_i$.

Restated in the language of Newton polygons, line $a + s_i b = s_i$ intersects the Newton polygon of $f(x_1, x_2)$ at the one point $(s_i, 0)$. Hence $(s_i, 0)$ is a vertex of the Newton polygon, and all other vertices lie strictly above the line. This means there’s a $s_i' < s_i$ such that the term $f_{s_i,0} x_1^{s_i'}$ dominates the Taylor expansion of $f$ on an entire set of the form $F = \{(x_1, x_2) : 0 < x_1 < a, |x_2| < x_1^{s_i'}\}$. That is, one has $f(x_1, x_2) = f_{s_i,0} x_1^{s_i'} + o(x_1^{s_i'})$ on this set.

After the coordinate change $\eta_i(x_1, x_2) = (x_1, x_2 + l_i x_1^{s_i} + o(x_1^{s_i}))$ into the coordinates of $D'_i$, one still has $f_i(x_1, x_2) = f_{s_i,0} x_1^{s_i'} + o(x_1^{s_i'})$ on the transformed $F$. Since $s_i > s_i'$,
this transformed set is very similar to $F$. In particular it contains the set $F'$ analogous to
$F$ where $s_i'$ is replaced by any $s_i''$ satisfying $s_i' > s_i'' > s_i$. Note that that $D_i'$ is a subset
of $F'$. Combining (3.1) and (3.4) we have that $d_i = f_{s_i,0}$ and the proof of the lemma is
complete.

4. Preliminaries for the Theorem proofs.

We start by writing the integral $U(\lambda_0, \lambda_1, \lambda_2)$ in (1.1) as $\sum_i U_i(\lambda_0, \lambda_1, \lambda_2)$, where

$$U_i(\lambda_0, \lambda_1, \lambda_2) = \int_{D_i} e^{i\lambda_0 f(x_1, x_2) + i\lambda_1 x_1 + i\lambda_2 x_2} \phi(x_1, x_2) \, dx_1 \, dx_2 \quad (4.1)$$

Here $D_i$ is a domain coming from the resolution of singularities process as described above
Lemma 3.4. Technically, $f(x_1, x_2)$ should be $f(\pm x_1, \pm x_2)$ or $f(\pm x_2, \pm x_1)$ and so on, but
this does not affect the statements or proofs here. We do not have to intersect $D_i$ with
$E$ in the new coordinates in (4.1); since every $q_j(x_1, x_2)$ is comparable to a monomial on
$D_i$ it is either positive everywhere on $D_i$ or negative everywhere on $D_i$. If every $q_j(x_1, x_2)$
is positive on $D_i$, we integrate over all of $D_i$, and otherwise we integrate over none of
$D_i$. Thus we ignore the latter $D_i$ and focus our attention on bounding (4.1) for the $i$ for which
each $q_j(x_1, x_2)$ is positive on $D_i$.

We will show that each $U_i(\lambda_0, \lambda_1, \lambda_2)$ satisfies the estimates of whichever theorem
we are proving; since there are finitely many of them this will suffice. We now apply the
resolution of singularities coordinate change $\eta_i$ of Theorem 3.2, obtaining

$$U_i(\lambda_0, \lambda_1, \lambda_2) = \int_{D_i'} e^{i\lambda_0 f_i(x_1, x_2) + i\lambda_1 x_1 + i\lambda_2 x_2 + i\lambda_3 k_i(x_1)} \phi_i(x_1, x_2) \, dx_1 \, dx_2 \quad (4.2)$$

Here $D_i'$ is $D_i$ in the new coordinates as in Theorem 3.2, $f_i$ denotes $f \circ \eta_i$, and $\phi_i$ denotes
$\phi \circ \eta_i$. The function $k_i(x_1)$ is the $x_2$ translation caused by the variable change when $x_1$ is
fixed. To simplify notation, we will always assume the $\pm i\lambda_2 x_2$ term is of the form $+i\lambda_2 x_2$.
Let $h_{ij}(x_1, x_2)$ denote the function $h_j \circ \eta_i(x_1, x_2)$; that is, the function $h_j(x_1, x_2)$ in the new
coordinates. By Theorem 3.2 one can write each $h_{ij}(x_1, x_2)$ as $d_{ij}x_1^{\alpha_{ij}}x_2^{\beta_{ij}}$ plus a smaller
error term for some $d_{ij} \neq 0$, with similar expressions for its various partial derivatives.
Since $\phi(x_1, x_2) = \alpha(x_1, x_2) \prod_{j=1}^M |h_j(x_1, x_2)|^{-\gamma_j}$, where $\alpha$ satisfies (1.3) – (1.4), we therefore
may write

$$\phi_i(x_1, x_2) = \alpha(x_1, x_2 + k_i(x_1)) \prod_{j=1}^M |x_1^{\alpha_{ij}}x_2^{\beta_{ij}} + \text{error}|^{-\gamma_j} \quad (4.3)$$

Thus we have

$$|\phi_i(x_1, x_2)| \leq C|x_1^{\alpha_{i1}}x_2^{\beta_{i1}}|^{-\gamma_1}...|x_1^{\alpha_{iM}}x_2^{\beta_{iM}}|^{-\gamma_M} \quad (4.4)$$

Using Corollary 3.3 on the $h_j(x_1, x_2)$ we also have

$$|\partial_{x_1} \phi_i(x_1, x_2)| \leq C \frac{1}{x_1}|x_1^{\alpha_{i1}}x_2^{\beta_{i1}}|^{-\gamma_1}...|x_1^{\alpha_{iM}}x_2^{\beta_{iM}}|^{-\gamma_M} \quad (4.5)$$
\[ |\partial_{x_2} \phi_i(x_1, x_2)| \leq C \frac{1}{x_2} |x_1^\alpha x_2^\beta|^\gamma |x_1^\alpha_1 x_2^\beta_1| |\cdots| x_1^\alpha_M x_2^\beta_M |^\gamma_M \] (4.6)

We now divide the integral (4.2) dyadically in both \( x_1 \) and \( x_2 \). Letting \( I_{lm} = [2^{-l-1}, 2^{-l}] \times [2^{-m-1}, 2^{-m}] \), we write \( U_i = U_i(\lambda_0, \lambda_1, \lambda_2) = \sum_{l,m} U_{ilm}(\lambda_0, \lambda_1, \lambda_2) \), where

\[ U_{ilm}(\lambda_0, \lambda_1, \lambda_2) = \int_{D_i \cap I_{lm}} e^{i \lambda_0 f_i(x_1, x_2) + i \lambda_1 x_1 + i \lambda_2 x_2 + i \lambda_i k_i(x_1)} \phi_i(x_1, x_2) \ dx_1 \ dx_2 \] (4.7)

It is convenient of our arguments to combine the linear term in \( k_i(x_1) = p_i x_1 + l_i x_1^{s_i} + o(x_1^{s_i}) \) with the \( x_1 \) in the \( \lambda_1 x_1 \) term in (4.7). So we let \( \lambda_3 = \lambda_1 + p_i \lambda_2 \), write \( K_i(x_1) = k_i(x_1) - p_i x_1 \), and (4.7) becomes

\[ U_{ilm}(\lambda_0, \lambda_1, \lambda_2) = \int_{D_i \cap I_{lm}} e^{i \lambda_0 f_i(x_1, x_2) + i \lambda_3 x_1 + i \lambda_2 x_2 + i \lambda_2 K_i(x_1)} \phi_i(x_1, x_2) \ dx_1 \ dx_2 \] (4.8)

Note that \( K_i(x_1) \) may be the zero function. From now on, we will use the notation \( C \) to denote a constant that depends on \( f, \alpha, \mu_h, A, E \cap D, \epsilon \), and \( l \) as permitted in our theorems.

We will estimate \( |U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \) through the use of Van der Corput lemmas in the coordinate systems provided by the resolution of singularities theorems. The key estimate that allows us to prove Theorem 1.3 is the following.

**Theorem 4.1.** Suppose the compatibility condition holds. Then for each \( i \) one has the following. If \( k_i(x_1) \) is linear, so that there is no \( l_i x_1^{s_i} \) term, we have

\[ |U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_2 x_2|^{-\frac{3}{2}}, |\lambda_0(x_1^{\alpha_i} x_2^{\beta_i})|^{-\frac{1}{2}}) \prod_{j=1}^M (x_1^{\alpha_i} x_2^{\beta_i})^\gamma_j \ dx_1 \ dx_2 \] (4.9a)

If \( k_i(x_1) \) is not linear, we have

\[ |U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_2 x_1^{s_i}|^{-\frac{3}{2}}, |\lambda_0(x_1^{\alpha_i} x_2^{\beta_i})|^{-\frac{1}{2}}) \prod_{j=1}^M (x_1^{\alpha_i} x_2^{\beta_i})^\gamma_j \ dx_1 \ dx_2 \] (4.9b)

**Proof.**

**Case 1.** \( \alpha_i > 0 \) and \( \beta_i = 0 \).

We first assume \( k_i(x_1) \) is not linear. We will use Lemma 2.1 for either second or third \( x_1 \)-derivatives and then integrate the result in \( x_2 \). Letting \( P(x_1, x_2) \) be the phase function in (4.7), we have

\[ \frac{\partial^2 P}{\partial x_1^2}(x_1, x_2) = \lambda_0 \frac{\partial f_i}{\partial x_1^2}(x_1, x_2) + \lambda_2 K_i''(x_1) \]

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\[ \frac{\partial^3 P}{\partial x_1^3}(x_1, x_2) = \lambda_0 \frac{\partial^3 f_i}{\partial x_1^3}(x_1, x_2) + \lambda_2 K'_i''(x_1) \] 

(4.10)

Since \( K_i \) is a real-analytic function of \( x_1^{\frac{1}{3}} \) for some large \( N \), we may write \( K_i(x_1) = l_i x_1^{s_i} + O(x_1^{s_i+\eta}) \) for some small \( \eta > 0 \), such that we have

\[ K'_i''(x_1) = l_i s_i (s_i - 1) x_1^{s_i-2} + O(x_1^{s_i-2+\eta}) \]

\[ K''_i(x_1) = l_i s_i (s_i - 1)(s_i - 2) x_1^{s_i-3} + O(x_1^{s_i-3+\eta}) \] 

(4.11)

In addition, by Corollary 3.3 we can write

\[ \frac{\partial^2 f_i}{\partial x_1^2}(x_1, x_2) = d_i \alpha_i (\alpha_i - 1)x_1^{\alpha_i-2} + \text{error} \]

\[ \frac{\partial^3 f_i}{\partial x_1^3}(x_1, x_2) = d_i \alpha_i (\alpha_i - 1)(\alpha_i - 2)x_1^{\alpha_i-3} + \text{error} \] 

(4.12)

Here the error term can be assumed to be of absolute value less than any \( \delta |d_i|x_1^{\alpha_i-2} \) and \( \delta |d_i|x_1^{\alpha_i-3} \) respectively.

Since we are assuming the compatibility condition holds, we must have \( s_i \neq \alpha_i \). So the 2 by 2 matrix \( M \) with rows \((\alpha_i(\alpha_i-1), s_i(s_i-1))\) and \((\alpha_i(\alpha_i-1)(\alpha_i-2), s_i(s_i-1)(s_i-2))\) has determinant \( \alpha_i(\alpha_i-1)s_i(s_i-1)(s_i-\alpha_i) \neq 0 \). Thus there is a constant \( \mu \) such that \( \|Mv\| \geq \mu \|v\| \) for all vectors \( v \in \mathbb{R}^2 \). In particular, if \( M_p \) denotes the \( p \)th row of \( M \), then given \( v \) for either \( p = 1 \) or \( 2 \) we have \( |M_p \cdot v| > \frac{\mu}{\sqrt{2}} \|v\| \) So if we let \( v = (\lambda_0 d_i x_1^{\alpha_i-2}, \lambda_2 l_i x_{s_i}^{s_i-2}) \) we see that for each \( x_1 \) we either have the \( p = 1 \) case

\[ |\lambda_0 d_i \alpha_i (\alpha_i - 1)x_1^{\alpha_i-2} + \lambda_2 l_i s_i (s_i - 1)x_1^{s_i-2}| \geq \frac{\mu}{2} (|\lambda_0 d_i \alpha_i (\alpha_i - 1)x_1^{\alpha_i-2}| + |\lambda_2 l_i s_i (s_i - 1)x_1^{s_i-2}|) \] 

(4.13)

Or the \( p = 2 \) case

\[ |\lambda_0 d_i \alpha_i (\alpha_i - 1)(\alpha_i - 2)x_1^{\alpha_i-3} + \lambda_2 l_i s_i (s_i - 1)(s_i - 2)x_1^{s_i-3}| \]

\[ \geq \frac{\mu}{2} (|\lambda_0 d_i \alpha_i (\alpha_i - 1)(\alpha_i - 2)x_1^{\alpha_i-3}| + |\lambda_2 l_i s_i (s_i - 1)(s_i - 2)x_1^{s_i-3}|) \] 

(4.14)

Note also that by looking at one higher derivative, one sees that there exists a \( c_1 > 0 \) such that if one of the two inequalities (4.13) \(- (4.14) \) holds for \( x_1 = x' \), it holds for all \( x_1 \in [(1 - c_1)x', (1 + c_1)x'] \) if one replaces \( \frac{\mu}{2} \) by \( \frac{\mu}{4} \). Furthermore, given (4.11) and (4.12), if \( \delta \) and \( x_1 \) are small enough, which we may assume, for all \((x_1, x_2) \in D' \cap I_{lm} \) with \( x_1 \in [(1 - c_1)x', (1 + c_1)x'] \) at least one of the two following inequalities holds.

\[ \frac{\partial^2 P}{\partial x_1^2}(x_1, x_2) \geq \frac{\mu}{8} (|\lambda_0 d_i \alpha_i (\alpha_i - 1)x_1^{\alpha_i-2}| + |\lambda_2 l_i s_i (s_i - 1)x_1^{s_i-2}|) \] 

(4.15a)
\[ |\frac{\partial^3 P}{\partial x_1^3}(x_1, x_2)| \geq \frac{\mu}{8} (|\lambda_0 d_\alpha (\alpha_i - 1)(\alpha_i - 2)x_1^{\alpha_i - 3}| + |\lambda_2 s_i (s_i - 1)(s_i - 2)x_1^{s_i - 3}|) \] (4.15b)

There is a technical issue in (4.15a) – (4.15b) worth mentioning. We have to be sure that the error terms in (4.12) can be made small enough such that (4.15a) or (4.15b) hold. However, this can be ensured during the resolution of singularities process by simply making the wedges narrower; a compactness argument then gives that the error terms are as small as needed.

Equations (4.15a) and (4.15b) can be summarized by the statement that for any \(x_0\) there is a constant \(\mu' > 0\) such that for either \(p = 2\) or \(p = 3\), for all \((x_1, x_2) \in D'_i \cap I_{im}\) with \(x_1 \in [(1 - c_1)x_0, (1 + c_1)x_0]\) we have

\[
|\frac{\partial^p P}{\partial x_1^p}(x_1, x_2)| \geq \mu' (|\lambda_0| 2^{lp 2^{-l_{\alpha_i}}} + |\lambda_2| 2^{lp 2^{-l_{s_i}}}) \] (4.16)

Next, we will bound a given \(|U_{x_0}(\lambda_0, \lambda_1, \lambda_2)|\), where

\[
U_{x_0}(\lambda_0, \lambda_1, \lambda_2) = \int_{D'_i \cap [(1 - c_1)x_0, (1 + c_1)x_0] \times [2^{-m-1}, 2^{-m}]} e^{i\lambda_0 f_i(x_1, x_2) + i\lambda_1 x_1 + i\lambda_2 x_2 + i\lambda_2 K_i(x_1)} \times \phi_i(x_1, x_2) \, dx_1 \, dx_2 \] (4.17)

Since \(D'_i \cap I_{im}\) is the union of boundedly many sets \(D'_i \cap [(1 - c_1)x_0, (1 + c_1)x_0] \times [2^{-m-1}, 2^{-m}]\), a bound for \(|U_{i_{lm}}(\lambda_0, \lambda_1, \lambda_2)|\) is given by a constant times the bound for \(|U_{x_0}(\lambda_0, \lambda_1, \lambda_2)|\).

We apply Lemma 2.1 for \(p\)th derivatives in the \(x_1\) direction in (4.17), using (4.16), and then integrate the result in the \(x_2\) variable, using (4.4) – (4.6) on the amplitude function. The result is

\[
|U_{x_0}(\lambda_0, \lambda_1, \lambda_2)| \leq C 2^{-l - m} \min(|\lambda_0|^{-\frac{1}{p}} (2^{i \alpha_i}), |\lambda_2|^{-\frac{1}{p}} (2^{i \beta_j} \gamma_j)) \prod_{j=1}^{M} (2^{-j \alpha_{ij} + k \beta_{ij}}) 
\]

\[
\leq C \int_{I_{lm}} \min((|\lambda_0|x_{1,1}^{\alpha_i})^{-\frac{1}{2}}, (|\lambda_2|x_{1,1}^{s_i})^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_{1,1}^{\alpha_{ij}}x_{2,1}^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \] (4.18)

Combining this with the estimate one obtains by simply taking absolute values of the original integrand and integrating, we therefore have

\[
|U_{x_0}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, (|\lambda_0|x_{1,1}^{\alpha_i})^{-\frac{1}{2}}, (|\lambda_2|x_{1,1}^{s_i})^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_{1,1}^{\alpha_{ij}}x_{2,1}^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \] (4.19)

This estimate is worse when \(p = 3\), so for any \(x_0\) we have

\[
|U_{x_0}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, (|\lambda_0|x_{1,1}^{\alpha_i})^{-\frac{1}{3}}, (|\lambda_2|x_{1,1}^{s_i})^{-\frac{1}{3}}) \prod_{j=1}^{M} (x_{1,1}^{\alpha_{ij}}x_{2,1}^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \] (4.20)
This is uniform in $x_0$, so adding over boundedly many $x_0$ also gives

$$|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, (|\lambda_0|x_1^{\alpha_i})^{-\frac{1}{2}}, (|\lambda_2|x_1^{\alpha_i})^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}}x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \tag{4.21}$$

This is the same as (4.9b), so we are done with the Case 1 argument for the case where $k_i(x_1)$ is not linear.

Suppose now $k_i(x_1)$ is linear. In place of (4.15a), by (3.1) one has

$$\left|\frac{\partial^2 P}{\partial x_1^2}(x_1, x_2)\right| \geq \frac{\mu}{8}(|\lambda_0d_i\alpha_i(\alpha_i - 1)x_1^{\alpha_i-2}| \tag{4.22}$$

Then the steps from (4.16) – (4.18) lead to

$$|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, (|\lambda_0|x_1^{\alpha_i})^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}}x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \tag{4.23}$$

In order to prove (4.9a), we may assume that for a constant $C$ of our choosing we have $|\lambda_2x_2| > C|\lambda_0x_1^{\alpha_i}|$, as otherwise (4.23) implies (4.9a). When this inequality holds, we may apply the Van der Corput lemma for first derivatives (Lemma 2.2) in the $x_2$ variable as the estimates given by Corollary 3.3 ensure that $|\partial_{x_2}P(x_1, x_2)| > c|\lambda_2|$ and $|\partial_{x_2}^2P(x_1, x_2)| < c'|\lambda_2|^{-\frac{1}{2}}$ in (4.8). Thus Lemma 2.2 implies that

$$|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} |\lambda_2x_2|^{-1} \prod_{j=1}^{M} (x_1^{\alpha_{ij}}x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \tag{4.24}$$

Combining with the estimate obtained by simply taking absolute values and integrating, this implies

$$|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_2x_2|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}}x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2$$

This in turn is bounded by

$$|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_2x_2|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}}x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \tag{4.25}$$

Combining (4.25) with (4.23) gives (4.9a) and we are done for the situation where $k_i(x_1)$ is linear. This completes the proof of Theorem 4.1 for the case when $\alpha_i > 0$ and $\beta_i = 0$.

**Case 2.** $\alpha_i > 0$ and $\beta_i > 0$. 

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Let $P(x_1, x_2)$ again denote the phase function $\lambda_0 f_1(x_1, x_2) + \lambda_3 x_1 + \lambda_2 x_2 + \lambda_2 K_1(x_1)$ in (4.8). By Theorem 3.2 one can write $f_i(x_1, x_2)$ in the form $d_i x_1^{\alpha_i} x_2^{r_i}$ plus a smaller error term for some $d_i \neq 0$, with a corresponding expression for its various partial derivatives. As a result \( \frac{\partial^2 P}{\partial x_1 \partial x_2}(x_1, x_2) \) is of the form $\alpha \beta \lambda_0 d_i x_1^{\alpha_i-1} x_2^{\beta_i-1}$ plus a smaller error term. Hence if we are on a sufficiently small neighborhood of the origin, which we may assume, for each $l$ and $m$, on $D'_l \cap I_{lm}$ we have

\[
\left| \frac{\partial^2 P}{\partial x_1 \partial x_2}(x_1, x_2) \right| \geq C|\lambda_0|2^{-l(\alpha_i-1) - m(\beta_i-1)} \tag{4.26}
\]

We now use the mixed-derivative Van der Corput lemma, Lemma 2.3, in conjunction with (4.4) – (4.6) to bound the amplitude and (4.26) to provide lower bounds for the mixed derivative. The rotation done before applying the resolution of singularities algorithm can be used to ensure that some $k^*h$th $x$ derivative is nonzero as is needed in Lemma 2.3. The result is

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_0|^{-\frac{1}{2}} (2^{-l(\alpha_i-1)-m(\beta_i-1)}) (2^{-\frac{1}{2}l}) \prod_{j=1}^{M} (2^{-\alpha_{ij} - m \beta_{ij}})^{\gamma_j} \tag{4.27}
\]

Since $h_j(x_1, x_2) \sim x_1^{\alpha_{ij}} x_2^{\beta_{ij}}$ on $I_{lm}$, we may write (4.27) in the more convenient form

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_0|^{-\frac{1}{2}} \int_{I_{lm}} \left( x_1^{\alpha_{ij}} x_2^{\beta_{ij}} \right)^{-\frac{1}{2}} \prod_{j=1}^{M} \left( x_1^{\alpha_{ij}} x_2^{\beta_{ij}} \right)^{\gamma_j} dx_1 dx_2 \tag{4.28}
\]

By simply taking absolute values of the integrand and integrating, one also has

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \prod_{j=1}^{M} \left( x_1^{\alpha_{ij}} x_2^{\beta_{ij}} \right)^{\gamma_j} dx_1 dx_2 \tag{4.29}
\]

Combining (4.28) and (4.29) results in

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_0(x_1^{\alpha_{ij}} x_2^{\beta_{ij}})|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_j} dx_1 dx_2 \tag{4.30}
\]

This in turn implies

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_2 x_2|^{-\frac{1}{2}}, |\lambda_0(x_1^{\alpha_{ij}} x_2^{\beta_{ij}})|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_j} dx_1 dx_2 \tag{4.31}
\]

Exactly as in the $\alpha_i > 0, \beta_i = 0$ case, when $k_i(x_1)$ is linear one can go from (4.31) to

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_2 x_2|^{-\frac{1}{2}}, |\lambda_0(x_1^{\alpha_{ij}} x_2^{\beta_{ij}})|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_j} dx_1 dx_2 \tag{4.32}
\]

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This is exactly (4.9a), the desired result for the linear situation. To complete the proof of Theorem 4.1 in the case at hand, we must show (4.9b) holds when \( k_i(x_1) \) is not linear. This will follow from (4.31) unless \( |\lambda_2 x_1^{\alpha_i} | > C|\lambda_0(x_1^{\alpha_i} x_2^{\beta_i})| \) for a constant \( C \) of our choosing. Since \( \partial^2_{x_1} P(x_1, x_2) = \lambda_0 \partial^2_{x_1} f_i(x_1, x_2) + \lambda_2 \partial^2_{x_1} (l_i x_1^{\alpha_i} + o(x_1^{\alpha_i})) \), by Corollary 3.3, if \( |\lambda_2 x_1^{\alpha_i} | > C_0 |\lambda_0(x_1^{\alpha_i} x_2^{\beta_i})| \) for a sufficiently large \( C_0 \), we will have

\[
|\partial^2_{x_1} P(x_1, x_2)| > C|\lambda_2 x_1^{\alpha_i - 2}|
\]  

(4.33)

Then the arguments of the \( \alpha_i > 0, \beta_i = 0 \) case, using the Van der Corput lemma for second derivatives to achieve (4.15a), give

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_2 x_1^{\alpha_i}|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2
\]  

(4.34)

Combining (4.31) and (4.34) gives (4.9b) as needed.

**Case 3.** \( \alpha_i = 0 \) and \( \beta_i > 0 \).

Since \( S(x_1, x_2) \) is assumed to have a zero of order at least two at the origin, \( \beta_i > 1 \) here. Since by (3.2) we have \( \frac{\partial^2 P}{\partial x_1^2}(x_1, x_2) = d_i \beta_i (\beta_i - 1) x_2^{\beta_i - 2} \) plus a smaller error term, on \( D'_i \cap I_{lm} \) we have

\[
\left| \frac{\partial^2 P}{\partial x_2^2}(x_1, x_2) \right| \geq C|\lambda_0|2^{-m(\beta_i - 2)}
\]  

(4.35)

We now use Lemma 2.1 for second derivatives in the \( x_2 \) variable and then integrate the result in \( x_1 \). Using (4.4) – (4.6) to bound the amplitude and its derivative, and (4.35) to provide a lower bound on the second \( x_2 \) derivative, we obtain

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_0|^{-\frac{1}{2}} (2^{m \beta_i - m})(2^{-1}) \prod_{j=1}^{M} (2^{-l_{ij}} - m \beta_{ij})^{\gamma_j}
\]  

(4.36)

This may be rewritten as

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_0|^{-\frac{1}{2}} \left( 2^{(l_{ij} - 1) + m(\beta_i - 1)} \right) \prod_{j=1}^{M} (2^{-l_{ij}} - m \beta_{ij})^{\gamma_j}
\]  

(4.37)

This is precisely (4.27). The exact argument of Case 2 starting with (4.27) now gives (4.9a) or (4.9b). This completes the Case 3 argument, and therefore the proof of Theorem 4.1.
5. Proofs of Theorems 1.3 and 1.4.

Proof Theorem 1.3 part a).

Here we will only need the following consequence of (4.9a) – (4.9b).

\[ |U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_0(x_1^\alpha x_2^\beta)|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_{ij}} \, dx_1 \, dx_2 \]  

(5.1)

Adding this over all \( l \) and \( m \) gives

\[ |U_l(\lambda_0, \lambda_1, \lambda_2)| \leq C \sum_{\{(l,m):D'_l \cap I_{lm} \neq \emptyset\}} \int_{I_{lm}} \min(1, |\lambda_0 x_1^{\alpha_i} x_2^{\beta_i}|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_{ij}} \, dx_1 \, dx_2 \]  

(5.2)

Since \( D'_l \) is of the form \( \{(x_1, x_2) : 0 < x_1 < a, \ h_i(x_1) < x_2 < H_i(x_1)\} \) with \( h_i(x_1) \) being identically zero or having a zero of greater order at \( x_1 = 0 \) than \( H_i(x_1) \), the shape of \( D'_l \) is such that equation (5.2) implies

\[ |U_l(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{D'_l} \min(1, |\lambda_0 x_1^{\alpha_i} x_2^{\beta_i}|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_{ij}} \, dx_1 \, dx_2 \]  

(5.3)

Since \( f_i(x_1, x_2) \sim x_1^{\alpha_i} x_2^{\beta_i} \) and \( |h_{ij}(x_1, x_2)| \sim x_1^{\alpha_{ij}} x_2^{\beta_{ij}} \) on \( D'_l \), the above implies that

\[ |U_l(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{D'_l} \min(1, |\lambda_0 f_i(x_1, x_2)|^{-\frac{1}{2}}) \prod_{j=1}^{M} |h_{ij}(x_1, x_2)|^{\gamma_{ij}} \, dx_1 \, dx_2 \]  

(5.4)

Converting back into the original coordinates on \( D_i \) (before the resolution of singularities), we see that

\[ |U_l(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{D_i} \min(1, |\lambda_0 f(x_1, x_2)|^{-\frac{1}{2}}) \prod_{j=1}^{M} |h_j(x_1, x_2)|^{\gamma_j} \, dx_1 \, dx_2 \]  

(5.5)

Adding this over all \( i \) gives

\[ |U(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{E \cap D} \min(1, |\lambda_0 f(x_1, x_2)|^{-\frac{1}{2}}) \prod_{j=1}^{M} |h_j(x_1, x_2)|^{\gamma_j} \, dx_1 \, dx_2 \]  

(5.6)

In terms of the measure \( d\mu_h \) of Theorem 1.3 this can be written as

\[ |U(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{E \cap D} \min(1, |\lambda_0 f(x_1, x_2)|^{-\frac{1}{2}}) \, d\mu_h \]  

(5.7)
We rewrite the integral in (5.7) as

\[ \mu_h \left( \{(x_1, x_2) \in E \cap D : |f(x_1, x_2)| < |\lambda_0|^{-1} \} \right) + |\lambda_0|^{-\frac{3}{2}} \int_{\{(x_1, x_2) \in E \cap D : |f(x_1, x_2)| > \frac{1}{|\lambda_0|} \}} |f(x_1, x_2)|^{-\frac{3}{2}} d\mu_h \]  

(5.8)

By the characterization of integrals of powers of functions in terms of their distribution functions, applied to \( |f(x_1, x_2)|^{-1} \), the integral in (5.8) is equal to

\[ \frac{1}{3} \int_{|\lambda_0|^{-1}}^{\infty} t^{-\frac{4}{3}} \mu_h \left( \{(x_1, x_2) \in E \cap D : |\lambda_0|^{-1} < |f(x_1, x_2)| < t \} \right) dt \]  

(5.9)

Thus if \( \epsilon \) and \( l \) are as in Theorem 1.3a), (5.9) is bounded by

\[ C \int_{|\lambda_0|^{-1}}^{\infty} t^{-\frac{4}{3}} \min(1, t^\epsilon \ln |t|) dt \]  

(5.10)

We can put the minimum with 1 here since \( \mu_h \) is a bounded measure on \( E \cap D \). Given (5.10) and the fact that first term in (5.8) is bounded by \( C|\lambda_0|^{-\epsilon}(\ln |\lambda_0|)^l \) by the assumptions of Theorem 1.3, we conclude that

\[ |U(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_0|^{-\epsilon}(\ln |\lambda_0|)^l + C|\lambda_0|^{-\frac{3}{2}} \int_{|\lambda_0|^{-1}}^{\infty} t^{-\frac{4}{3}} \min(1, t^\epsilon \ln |t|) dt \]  

(5.11)

When \( \epsilon \leq \frac{1}{3} \), we use \( \min(1, t^\epsilon \ln |t|) \leq t^\epsilon \ln |t| \) in (5.11). If \( \epsilon < \frac{1}{3} \), the result is that \( |U(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_0|^{-\epsilon}(\ln |\lambda_0|)^l \), the desired estimate (1.7a). (Because the integrands are all integrable, we can always replace \( |\lambda_0| \) by \( 1 + |\lambda_0| \) and \( \ln |\lambda_0| \) by \( \ln(2 + |\lambda_0|) \) in our estimates. Similar considerations will hold for the other estimates of our theorems.)

If \( \epsilon = \frac{1}{3} \) we obtain an additional logarithmic factor in the above. If \( \epsilon > \frac{1}{3} \), we use 1 in the minimum if \( t > 1 \), and \( t^\epsilon \ln |t| \) in the minimum if \( t < 1 \). In this case we get

\[ |U(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_0|^{-\epsilon} + C|\lambda_0|^{-\frac{3}{2}} \int_{|\lambda_0|^{-1}}^{1} t^{-\frac{4}{3}} \ln |t| + C|\lambda_0|^{-\frac{3}{2}} \int_{1}^{\infty} t^{-\frac{4}{3}} dt \]  

(5.12)

Since \( \epsilon > \frac{1}{3} \), this is bounded by a constant times \( |\lambda_0|^{-\frac{3}{2}} \), again the desired estimate for part a) of Theorem 1.3. This completes the proof of Theorem 1.3a).

**Proof of Theorem 1.3b).**

We first consider the case where \( k_i(x_1) \) is not linear. Here we use the following consequence of (4.9b).

\[ |U_{itm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{tm}} \min(1, |\lambda_2 x_1^{s_1}|^{-\frac{3}{2}}) \prod_{j=1}^{M} \left(x_1^{\alpha_{ij}} x_2^{\beta_{ij}}\right)^{\gamma_j} dx_1 dx_2 \]  

(5.13)
We add this over all $l$ and $m$ to obtain
\[
|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{\{(l, m): D'_i \cap I_{l, m} \neq \emptyset\}} \min(1, |\lambda_2 x_1^{s_i}|^{-\frac{j}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij} x_2^{\beta_{ij}}} x_2^{\gamma_j}) dx_1 dx_2 \tag{5.14}
\]

Due to the shape of the $D'_i$, as when going from (5.2) to (5.3) we get
\[
|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{D'_i} \min(1, |\lambda_2 x_1^{s_i}|^{-\frac{j}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij} x_2^{\beta_{ij}}} x_2^{\gamma_j}) dx_1 dx_2 \tag{5.15}
\]

Since $|h_{ij}(x_1, x_2)| \sim x_1^{\alpha_{ij} x_2^{\beta_{ij}}}$ on $D'_i$, we get
\[
|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{D'_i} \min(1, |\lambda_2 x_1^{s_i}|^{-\frac{j}{2}}) \prod_{j=1}^{M} |h_{ij}(x_1, x_2)|^{\gamma_j} dx_1 dx_2 \tag{5.16}
\]

Since the coordinate change map here is of the form $(x_1, x_2) \rightarrow (x_1, x_2+p_i x_1+l_i x_1^{s_i}+o(x_1^{s_i}))$, translating (5.16) back into the original coordinates gives
\[
|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{D_i} \min(1, |\lambda_2 (x_2 - p_i x_1)|^{-\frac{j}{2}}) \prod_{j=1}^{M} |h_{ij}(x_1, x_2)|^{\gamma_j} dx_1 dx_2 \tag{5.17}
\]

We can bound this from above by replacing $D_i$ by $E \cap D$, so we get
\[
|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{E \cap D} \min(1, |\lambda_2 (x_2 - p_i x_1)|^{-\frac{j}{2}}) \prod_{j=1}^{M} |h_{ij}(x_1, x_2)|^{\gamma_j} dx_1 dx_2 \tag{5.18}
\]

Let $v$ be the unit vector perpendicular to the line $x_2 = p_i x_1$ and let $(\delta_v, e_v)$ be as in Lemma 1.1. Then the argument from (5.6) – (5.12) gives that if $\delta_v < \frac{1}{3}$ one has
\[
|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_2|^{-\delta_v} \ln |\lambda_2|^{e_v} \tag{5.19}
\]

If $\delta_v = \frac{1}{3}$ one obtains an additional logarithm, and if $\delta_v > \frac{1}{3}$ we get
\[
|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_2|^{-\frac{j}{3}} \tag{5.20}
\]

Without loss of generality we may assume that one has rotated coordinates before applying Theorem 3.2 so that one has $|\lambda_2| > \frac{1}{2}|\lambda'|$, where $|\lambda'|$ is the magnitude of $(\lambda_1, \lambda_2)$ as in Theorem 1.3. (We must choose from a fixed finite family of rotations to ensure the constants are uniform.) Hence we may replace $|\lambda_2|$ by $|\lambda'|$ in (5.20). Taking the supremum over all directions allows us to then replace $(\delta_v, e_v)$ by $(\delta, e)$ in (5.19) – (5.20). Adding over all $i$ then gives the statement of Theorem 1.3b). This completes the proof in the case where $k_i(x_1)$ is not linear.

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If \( k_i(x_1) \) is linear, we use the following consequence of (4.9b) in place of (5.13).

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_2 x_2|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2
\]  

(5.21)

Since \( |h_{ij}(x_1, x_2)| \sim x_1^{\alpha_{ij}} x_2^{\beta_{ij}} \) on \( D'_i \), we have

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_2 x_2|^{-\frac{1}{2}}) \prod_{j=1}^{M} |h_{ij}(x_1, x_2)|^{\gamma_j} \, dx_1 \, dx_2
\]  

(5.22)

Doing the linear coordinate change back into the original coordinates, this becomes

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda_2 (x_2 - p_i x_1)|^{-\frac{1}{2}}) \prod_{j=1}^{M} |h_j(x_1, x_2)|^{\gamma_j} \, dx_1 \, dx_2
\]  

(5.23)

This is exactly (5.18), and now the argument proceeds as in the nonlinear case. This completes the proof of Theorem 1.3b).

**Proof of Theorem 1.3c).**

The sharpness statement for part b) of Theorem 1.3 actually follows from the sharpness statements in [G6], where it is shown that even when restricted to \( \lambda_0 = 0 \) one cannot get a better \( \delta \) than in (1.8a). So we focus our attention the sharpness statement for part a). Suppose one has the estimate (1.7a) for some \( \epsilon > 0 \); we will show that \( \epsilon \leq \epsilon_0 \).

Let \( B(x) \) be a bump function on \( \mathbb{R} \) whose Fourier transform is nonnegative, compactly supported, and equal to 1 on a neighborhood of the origin, and let \( N \) be a large positive number. If \( 0 < \epsilon' < \epsilon \), then (1.7a) implies that for some constant \( A \) independent of \( N \) one has

\[
\int_{\mathbb{R}} |U(\lambda_0, 0, 0)||\lambda_0|^{\epsilon'-1} B(N\lambda_0) \, d\lambda_0 \leq A
\]  

(5.24)

In view of the definition (1.1) for \( U(\lambda_0, \lambda_1, \lambda_2) \), this implies that

\[
\left| \int_{\mathbb{R}^3} e^{i\lambda_0 f(x_1, x_2)} \phi(x_1, x_2)|\lambda_0|^{\epsilon'-1} B(N\lambda_0) \, d\lambda_0 \, dx_1 \, dx_2 \right| \leq A
\]  

(5.25)

If we do the integral in \( \lambda_0 \) first in (5.25), for a constant \( A' \) independent of \( N \) we get

\[
\left| \int_{\mathbb{R}^2} \beta_N(f(x_1, x_2)) \phi(x_1, x_2) \, dx_1 \, dx_2 \right| \leq A'
\]  

(5.26)

Here \( \beta_N(y) \) is the convolution of \( |y|^{-\epsilon'} \) with \( \frac{1}{N} \hat{B}(\frac{y}{N}) \). Next, let \( \phi(x_1, x_2) \) be equal to \( \alpha(x_1, x_2) \chi_E(x_1, x_2) \prod_{i=1}^{M} |h_i(x_1, x_2)|^{\gamma_i} \), where \( \alpha(x_1, x_2) \) is nonnegative and bounded below by a positive constant near the origin. Then for a constant \( A'' \) independent of \( N \) we get

\[
\left| \int_{\mathbb{R}^2} \beta_N(f(x_1, x_2))\alpha(x_1, x_2) \chi_E(x_1, x_2) \prod_{i=1}^{M} |h_i(x_1, x_2)|^{\gamma_i} \, dx_1 \, dx_2 \right| \leq A''
\]  

(5.27)
Letting \( N \to \infty \) gives

\[
\int_{\mathbb{R}^2} |f(x_1, x_2)|^{-\epsilon} \alpha(x_1, x_2) \chi_E(x_1, x_2) \prod_{i=1}^M |h_i(x_1, x_2)|^{\gamma_i} \, dx_1 \, dx_2 < \infty \quad (5.28)
\]

Since \( \alpha(x_1, x_2) \) is bounded below by a positive constant on a neighborhood of the origin, for small enough \( D \) we have

\[
\int_D |f(x_1, x_2)|^{-\epsilon} \chi_E(x_1, x_2) \prod_{i=1}^M |h_i(x_1, x_2)|^{\gamma_i} \, dx_1 \, dx_2 < \infty \quad (5.29)
\]

In other words, \( |f(x_1, x_2)|^{-\epsilon} \) is in \( L^1(E \cap D) \) with respect to the measure \( d\mu_h \). Hence it is in weak \( L^1 \), and we have the existence of a constant \( B \) such that

\[
\mu_h((x_1, x_2) \in E \cap D : |f(x_1, x_2)|^{-\epsilon} > t) \leq B \frac{1}{t} \quad (5.30)
\]

Replacing \( t \) by \( t^{-\epsilon} \), we get

\[
\mu_h((x_1, x_2) \in E \cap D : |f(x_1, x_2)| < t) \leq B t^\epsilon \quad (5.31)
\]

In view of the definition of \( \epsilon_0 \), we have \( \epsilon' \leq \epsilon_0 \). Since this holds for each \( \epsilon' \) satisfying \( 0 < \epsilon' < \epsilon \), we conclude that \( \epsilon \leq \epsilon_0 \) as needed.

**Proof of Theorem 1.4.**

Suppose the \( h_i(x_1, x_2) \) and \( q_j(x_1, x_2) \) are not compatible with \( f(x_1, x_2) \). The reason the proof of Theorem 1.3 does not work here is because there may be \( i \) for which \( s_i = \alpha_i \) and \( \beta_i = 0 \), in which case the arguments of Theorem 4.1 estimating \( U_{ilm}(\lambda_0, \lambda_1, \lambda_2) \) are not valid. We instead proceed as follows. If we replace \( E \) with \( E - \bigcup_i \{D_i : s_i = \alpha_i, \beta_i = 0\} \), then we have removed the problematic portions of \( E \) and the arguments leading to Theorem 1.3 work just as before. Thus in order to prove Theorem 1.4, it suffices to restrict our attention to the portion of the surface above a \( D_i \) corresponding to a given \( i \) with \( s_i = \alpha_i \), and \( \beta_i = 0 \).

Let \( q \) be any nonzero number such that a portion of a curve of the form \( \{ (x_1, x_2) : x_2 = p_i x_1 + q_2 x_2^{s_i} + o(x_2^{s_i}) \} \) is contained in \( D_i \) (in the original coordinates). We do the coordinate change \( (x_1, x_2, x_3) \to (x_1, x_2, x_3 - \frac{d_i}{q}(x_2 - p_i x_1)) \) on the surface. In other words we replace \( f(x_1, x_2) \) by \( f(x_1, x_2) - \frac{d_i}{q}(x_2 - p_i x_1) \). Then instead of (4.1), the Fourier transform of the new surface measure, with the same density function \( \phi(x_1, x_2) \), is given by

\[
U_i(\lambda_0, \lambda_1, \lambda_2) = \int_{D_i} e^{i\lambda_0 f(x_1, x_2) - i\lambda_0 \frac{d_i}{q}(x_2 - p_i x_1) + i\lambda_1 x_1 + i\lambda_2 x_2} \phi(x_1, x_2) \, dx_1 \, dx_2 \quad (5.32)
\]
In (5.32), we do the coordinate change transforming \((x_1, x_2)\) to \((x_1, x_2 + p_i x_1 + qx_1^{s_i})\). Then (5.32) can be rewritten as

\[
U_i(\lambda_0, \lambda_1, \lambda_2) = \int_{\tilde{D}_i} e^{i\lambda_0(\tilde{f}(x_1, x_2) - d_s x_1^{s_i}) + i(\lambda_1 + p_i \lambda_2)x_1 + i(\lambda_2 - \lambda_0 \frac{d_s}{\pi})x_2 + i\lambda_2 q x_1^{s_i}} \tilde{\phi}(x_1, x_2) \, dx_1 \, dx_2
\]

(5.33)

By Lemma 3.4, \(\tilde{f}(x_1, x_2) = d_s x_1^{s_i} + o(x_1^{s_i})\) on \(\tilde{D}_i\) since this was true for \(f(x_1, x_2)\) on \(D_i\). Hence on \(D_i\), the function \(\tilde{f}(x_1, x_2) - d_s x_1^{s_i}\) is \(o(x_1^{s_i})\). This means that if we apply the resolution of singularities algorithm of Theorem 3.1 on \(\tilde{f}(x_1, x_2) - d_s x_1^{s_i}\) (assuming it is not identically zero) on a domain \(D_i\cap \{(x_1, x_2): 0 < x_1 < a, \, |x_2| < \eta x_1^{s_i}\}\) for sufficiently small \(\eta\), then the arguments leading to (4.9b) in the proof of Theorem 1.3 still work, for the following reason.

If \(\eta\) is sufficiently small, all the coordinate changes will be of the form \((x_1, x_2) \to (x_1, x_2 + O(x_1^{s_i}))\) for some \(s_i' > s_i\). Thus any adjustments due to the coordinate changes of the nonlinear portion of \(\lambda_2\) coefficient, initially given by \(q x_1^{s_i}\), will just add higher order terms, and the overall term will still vanish in \(x_1\) to order \(s_i\) at \(x_1 = 0\). On the other hand, the adjusted nonlinear portion of the \(\lambda_0\) coefficient, given by \(\tilde{f}(x_1, x_2) - d_s x_1^{s_i}\) plus some possible new terms in \(x_1\) of order greater than \(s_i\), vanishes to higher order. Since the analogues of \(x_1^{s_i}\) for the new coordinate changes will always dominate \(\tilde{f}(x_1, x_2) - d_s x_1^{s_i}\), any such analogues will dominate the \(\lambda_0\) coefficient while still vanishing to order greater than \(s_i\). Hence the issues that arise in the incompatible case no longer occur.

Stated in terms of the original coordinates, there is a set of the form \(D_i\cap \{(x_1, x_2): 0 < x_1 < a, \, |x_2 - p_i x_1 - qx_1^{s_i}| < \eta x_1^{s_i}\}\) for which the argument leading to (4.9b) will apply, where in the resulting restatement of part a), \(f(x_1, x_2)\) gets replaced by some \((f(x_1, x_2) - d_s x_1^{s_i})\) terms of order greater than \(s_i\). (If \(f(x_1, x_2) - d_s x_1^{s_i}\) is identically zero, we use the zero function in the restatement.)

Since the vertical thickness of each \(D_i\) is at most \(C x_1^{s_i}\), by compactness we may write the portion of the surface above \(D_i\) as a finite union of such sets, on each of which a linear transformation of the form \((x_1, x_2, x_3) \to (x_1, x_2, x_3 + a_1 x_1 + a_2 x_2)\) will turn the surface into one for which the argument leading to (4.9b) applies with a corresponding restatement of part a) of Theorem 1.3. Recall that before even performing the resolution of singularities after Corollary 3.3 we did a linear coordinate change in the \(x_1\) and \(x_2\) variables. Thus the proper restatement of Theorem 1.3a)-1.3b) is as given in Theorem 1.4b). This completes the proof of Theorem 1.4.

6. Proof of Theorem 1.5.

In this proof, we use the notation \(C\) denote a constant that may depend on \(f, \alpha, \mu_b, A,\) and \(E \cap D\) as allowed in Theorem 1.3. We will once again bound each \(U_{ilm}(\lambda_0, \lambda_1, \lambda_2)\) in (4.7) and add the result in \(l\) and \(m\), but this time we will use the fact that \(\partial_{x_2} S(x_1, x_2)\) has also had its singularities resolved instead of using that \(S(x_1, x_2)\)
has had its singularities resolved. Like in the previous section, we assume one has rotated coordinates before applying Theorem 3.2 so that \(|\lambda_2| > \frac{1}{2} |\lambda'|\), where \(|\lambda'|\) is the magnitude of \((\lambda_1, \lambda_2)\) as in Theorem 1.3. (As mentioned before, the rotation must be from a fixed finite family of rotations to ensure all constants are uniform.) Let \(\alpha'_i\) and \(\beta'_i\) be such that \(\partial_{x_2}S(x_1, x_2)\) is of the form \(d'_ix_1^{\alpha'_i}x_2^{\beta'_i}\) plus a smaller error term. Our key technical lemma is the following.

**Lemma 6.1.** For all \(i, l,\) and \(m\) we have the estimate

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda'x_2|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}}x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \quad (6.1)
\]

**Proof.** The argument splits into three cases. In the following, \(B\) denotes a large constant to be determined by our arguments.

**Case 1.** \(|\lambda_2| > B|\lambda_0|2^{-l\alpha'_i-m\beta'_i}\)

We use Lemma 2.2. If \(P(x_1, x_2)\) is the phase function in (4.8) as before, then we have

\[
\left| \frac{\partial P}{\partial x_2}(x_1, x_2) \right| = \left| \lambda_0 \frac{\partial f_i}{\partial x_2}(x_1, x_2) + \lambda_2 \right| > \frac{1}{2} |\lambda_2| \quad (6.2)
\]

The right-hand inequality follows from the fact that \(|\lambda_2| > B|\lambda_0|2^{-l\alpha'_i-m\beta'_i}\). By Corollary 3.3 we also have

\[
\left| \frac{\partial^2 P}{\partial x_2^2}(x_1, x_2) \right| = \left| \lambda_0 \frac{\partial^2 f_i}{\partial x_2^2}(x_1, x_2) \right| \leq C \frac{1}{x_2} \left| \lambda_0 \frac{\partial f_i}{\partial x_2}(x_1, x_2) \right| \leq C \frac{1}{x_2} |\lambda_2| \quad (6.3)
\]

Since we are in Case 1, this is bounded by

\[
\leq C \frac{1}{x_2} |\lambda_2| \quad (6.3)
\]

We now apply Lemma 2.2 in the \(x_2\) direction, and integrate the result in \(x_1\), using (4.4) – (4.6) to bound the amplitude and its first derivative. We obtain

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_2|^{-1}(2^{-l}) \prod_{j=1}^{M} (2^{-l\alpha_{ij}-m\beta_{ij}})^{\gamma_j} \quad (6.4)
\]

We write this in the following more convenient form, using that \(|\lambda_2| \sim |\lambda'|\).

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda'|^{-1} \int_{I_{lm}} x_2^{-1} \prod_{j=1}^{M} (x_1^{\alpha_{ij}}x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \quad (6.5a)
\]
Since by taking absolute values of the integrand and integrating one has \( |U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \), (6.5a) can be improved to

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda' x_2|^{-1}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \quad (6.5b)
\]

The integrand in (6.5b) is at most the integrand in (6.1), so we are done with the lemma in Case 1.

**Case 2.** \(|\lambda_2| \leq B|\lambda_0|2^{-\alpha'_{ij}-m\beta'_i}\) and \(\beta'_i > 0\).

In this case, we use Lemma 2.1 for second derivatives in the \(x_2\) direction and integrate the result in \(x_1\). By (3.2), \(\frac{\partial^2 f}{\partial x_2^2}(x_1, x_2) \sim x_1^{\alpha'_{ij}} x_2^{\beta'_i-1}\), so we have

\[
\left| \frac{\partial^2 P}{\partial x_2^2}(x_1, x_2) \right| = \left| \lambda_0 \frac{\partial^2 f}{\partial x_2^2}(x_1, x_2) \right| \geq C|\lambda_0|2^{-\alpha'_{ij}-m(\beta'_i-1)} \quad (6.6)
\]

Using Lemma 2.1 in the \(x_2\) variable in conjunction with (4.4) – (4.6) and (6.6), then integrating the result in \(x_1\), we obtain

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_0|^{-\frac{1}{2}} (2^{-t}) \left( \frac{2^{\alpha'_{ij}+m(\beta'_i-1)}}{2} \right) \prod_{j=1}^{M} (2^{-\alpha_{ij}-m\beta_{ij}})^{\gamma_j} \quad (6.7)
\]

By the condition that \(|\lambda_2| \leq B|\lambda_0|2^{-\alpha'_{ij}-m\beta'_i}\) we therefore have

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_2|^{-\frac{1}{2}} (2^{-t}) (2^{-\frac{m}{2}}) \prod_{j=1}^{M} (2^{-\alpha_{ij}-m\beta_{ij}})^{\gamma_j} \quad (6.8)
\]

Again using that \(|\lambda_2| \sim |\lambda'|\), we write this in the following more convenient form

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda'|^{-\frac{1}{2}} \int_{I_{lm}} x_2^{-\frac{1}{2}} \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \quad (6.9a)
\]

Like in (6.5b), simply by taking the absolute value of the integrand and integrating this can be improved to

\[
|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{I_{lm}} \min(1, |\lambda' x_2|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij}} x_2^{\beta_{ij}})^{\gamma_j} \, dx_1 \, dx_2 \quad (6.9b)
\]

This completes the proof of Lemma 6.1 in case 2.
Case 3. $|\lambda_2| \leq B|\lambda_0|2^{-l\alpha'_i-m\beta'_i}$ and $\beta'_i = 0$.

Here we use Lemma 2.3. By (3.1), we have $|\frac{\partial^2 f_i}{\partial x_1 \partial x_2}(x_1, x_2)| \sim x_1^{\alpha'_i-1}$, so we have

$$\left| \frac{\partial^2 P}{\partial x_1 \partial x_2}(x_1, x_2) \right| = \left| \lambda_0 \frac{\partial^2 f_i}{\partial x_1 \partial x_2}(x_1, x_2) \right| \geq C|\lambda_0|2^{-l(\alpha'_i-1)} \quad (6.10)$$

Using Lemma 2.3 in conjunction with (4.5) – (4.6) on the amplitude and (6.10) for the lower bound on the mixed derivative, we get

$$|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_0|^{-\frac{1}{2}}(2^{-\frac{1}{2}})(2^{-\frac{1}{2}}) \prod_{j=1}^{M} (2^{-l\alpha_{ij}-m\beta_{ij}}) \gamma_i \quad (6.11)$$

Using that $|\lambda_2| \leq B|\lambda_0|2^{-l\alpha'_i}$, equation (6.11) implies that

$$|U_{ilm}(\lambda_0, \lambda_1, \lambda_2)| \leq C|\lambda_2|^{-\frac{1}{2}}2^{-\frac{l\alpha'_i}{2}}(2^{-\frac{l\alpha'_i}{2}})(2^{-\frac{l(\alpha'_i-1)}{2}}) \prod_{j=1}^{M} (2^{-l\alpha_{ij}-m\beta_{ij}}) \gamma_i \quad (6.12)$$

This is the same as (6.8) and therefore (6.9a) – (6.9b) follows. This completes the proof for case 3 and therefore the proof of Lemma 6.1 is complete.

The proof of Theorem 1.5.

Adding (6.1) over all $l$ and $m$ gives

$$|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C \sum_{\{l,m: D'_i \cap I_{lm} \neq \emptyset\}} \int_{I_{lm}} \min(1, |\lambda' x_2|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij} \beta_{ij}}) \gamma_i \ dx_1 \ dx_2 \quad (6.13)$$

Since $D'_i$ is of the form $\{(x_1, x_2) : 0 < x_1 < a, h_i(x_1) < x_2 < H_i(x_1)\}$ with $h_i(x_1)$ being identically zero or having a zero of order at $x_1 = 0$ greater than that of $H_i(x_1)$, the shape of $D'_i$ is such that (6.13) implies

$$|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{D'_i} \min(1, |\lambda' x_2|^{-\frac{1}{2}}) \prod_{j=1}^{M} (x_1^{\alpha_{ij} \beta_{ij}}) \gamma_i \ dx_1 \ dx_2 \quad (6.14)$$

Since $|h_{ij}(x_1, x_2)| \sim x_1^{\alpha_{ij} \beta_{ij}}$ on $D'_i$, (6.14) implies

$$|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{D'_i} \min(1, |\lambda' x_2|^{-\frac{1}{2}}) \prod_{j=1}^{M} |h_{ij}(x_1, x_2)| \gamma_i \ dx_1 \ dx_2 \quad (6.15a)$$

Writing $H_i(x_1, x_2) = \prod_{j=1}^{M} |h_{ij}(x_1, x_2)| \gamma_i$, the above is the same as

$$|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C \int_{D'_i} \min(1, |\lambda' x_2|^{-\frac{1}{2}}) H_i(x_1, x_2) \ dx_1 \ dx_2 \quad (6.15b)$$

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Suppose $1 < p \leq \infty$ is such that $H_i \in L^p(D'_i)$. By Holder’s inequality, (6.15b) implies
\[
|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C||H_i||_{L^p(D'_i)} \left( \int_{D'_i} \left( \min(1, |\lambda'x_2|^{-\frac{1}{2}}) \right)^{p'} \right)^{\frac{1}{p'}} \tag{6.16}
\]
\[
\leq C_8||H_i||_{L^p(D'_i)} \left( \int_0^1 \left( \min(1, |\lambda'x_2|^{-\frac{1}{2}}) \right)^{p'} dx_2 \right)^{\frac{1}{p'}} \tag{6.17}
\]
Note that
\[
\int_0^1 \left( \min(1, |\lambda'x_2|^{-\frac{1}{2}}) \right)^{p'} dx_2 = |\lambda'|^{-1} + |\lambda'|^{-\frac{p'}{2}} \int_0^1 x_2^{-\frac{p'}{2}} dx_2 \tag{6.18}
\]
If $p' > 2$, this integral is bounded by $C|\lambda'|^{-1}$. If $p' = 2$, it is bounded by $C|\lambda'|^{-1} \ln |\lambda'|$, and if $p' < 2$ it is bounded by $C|\lambda'|^{-\frac{p'}{2}}$. Hence if $p' > 2$, (6.16) implies
\[
|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C||H_i||_{L^p(D'_i)} |\lambda'|^{-\frac{1}{p'}} \tag{6.19a}
\]
If $p' = 2$ we get an additional factor of $(\ln |\lambda'|)^{\frac{1}{2}}$, and if $p' < 2$ we get
\[
|U_i(\lambda_0, \lambda_1, \lambda_2)| \leq C||H_i||_{L^p(D'_i)} |\lambda'|^{-\frac{1}{2}} \tag{6.19b}
\]
Since $H_i(x_1, x_2)$ is the function $\prod_{j=1}^M |h_j(x_1, x_2)|^{\gamma_j}$ after the coordinate change of Theorem 3.2, if $p > 1$ is such that $\int_{E \cap D} (\prod_{j=1}^M |h_j(x_1, x_2)|^{\gamma_j})^p$ is finite, then $H_i(x_1, x_2) \in L^p(D'_i)$ for each $i$. Thus for such $p$, (6.19a) – (6.19b) hold for all $i$. Adding over all $i$ gives Theorem 1.5 when $p \neq 1$. (One can replace $|\lambda'|$ by $2 + |\lambda'|$ since the integral of the absolute value of the integrand is finite.) Since the $p = 1$ statements are immediate, we are done with the proof of Theorem 1.5.

References.


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