

Resolution of Singularities, Asymptotic Expansions of Oscillatory Integrals, and Related Phenomena

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1. Introduction

Suppose $f(x)$ is a real-analytic function defined in some neighborhood of the origin in \mathbf{R}^n . Consider the integral

$$I_\lambda = \int e^{i\lambda f(x)} \phi(x) dx \quad (1.1)$$

Here $\phi(x)$ is a smooth bump function defined on a neighborhood of the origin and λ is a real parameter whose absolute value we assume to be large. If $\nabla f(0) \neq 0$, then by repeated integrations by parts (see [S] Ch 8 for example), for any N one has an estimate $|I_\lambda| < C_{f,\phi,N} |\lambda|^{-N}$ for appropriate constants $C_{f,\phi,N}$. In the case where $\nabla f(0) = 0$, that is, when f has a critical point at the origin, it can be proven (see [M]) using Hironaka's resolution of singularities [H1]-[H2] that if the support of ϕ is contained in a sufficiently small neighborhood of the origin, then as $\lambda \rightarrow \infty$, I_λ has an asymptotic expansion of the form

$$e^{i\lambda f(0)} \sum_{\alpha} \sum_{i=0}^{n-1} a_{i,\alpha}(\phi) \lambda^{-\alpha} \ln(\lambda)^i \quad (1.2)$$

Here the sum in α goes over an increasing arithmetic progression of positive rational numbers, and the $a_{i,\alpha}$ are distributions with respect to the cutoff ϕ . We refer to the excellent resource [AGV] for further results along these lines. In Theorem 1.2, we will provide another proof of the existence of the expansion (1.2) using an elementary resolution of singularities theorem deriving from [Gr], thereby avoiding the use of Hironaka's theorem or other nonelementary techniques. Illustrating this elementary method of proving (1.2) and related results while giving the precise estimates of Theorems 1.1-1.3 can be viewed as the main purpose of this paper. In Theorem 1.4, we will also give an elementary proof of the well-known result of Atiyah [At] and Bernstein-Gelfand [BGe] concerning the meromorphy of integrals of f^z for nonnegative real-analytic f . It should be pointed out that there exist other elementary resolution of singularities algorithms which have been used for various purposes, notably [BiMi].

The analysis of (1.1) is closely related to the analysis of sublevel set integrals (see Ch 6-7 of [AGV] for example). Namely, let $f(x)$ be as above and assume $f(0) = 0$. We

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consider the integrals

$$J_t = \int_{\{x:0 < f(x) < t\}} \phi(x) dx \quad (1.3)$$

Theorem 1.1 below, also proved using an extension of [Gr], will show that for small $t > 0$ we have an asymptotic expansion

$$J_t \sim \sum_{\alpha} \sum_{i=0}^{n-1} b_{i,\alpha}(\phi) t^{\alpha} \ln(t)^i \quad (1.4)$$

Furthermore, each $\frac{\partial^k J_t}{\partial t^k}$ will be seen to have asymptotic expansion given by termwise differentiation of (1.4). This appears to only have been explicitly done in the case where f has an isolated zero at the origin [J] [Va], although one should note that if one is willing to assume Hironaka's results it can be proved without a huge amount of difficulty. [L] considers some related issues in this subject. Once one knows the expansion (1.4), one can obtain the expansion (1.2) using well-known techniques. In the integral (1.1), after factoring out a $e^{i\lambda f(0)}$ if necessary, one may work under the assumption that $f(0) = 0$. In this situation one first integrates over a given level set $f = t$ and then integrates with respect to t . Consequently, the integral may be rewritten as

$$\int_0^{\infty} \frac{\partial J_t}{\partial t} e^{i\lambda t} \gamma(t) dt + \int_0^{\infty} \frac{\partial \bar{J}_t}{\partial t} e^{-i\lambda t} \gamma(t) dt \quad (1.5)$$

Here \bar{J}_t denotes the analogue of J_t with f replaced by $-f$, and $\gamma(t)$ denotes a bump function such that $\gamma(f(x)) = 1$ for all $x \in \text{supp}(\phi)$. With a little care, one can then substitute (1.4) for f and $-f$ respectively into (1.5), integrate term by term, and obtain (1.2). We will do this rigorously in the proof of Theorem 1.2 below.

It turns out that there are some natural generalizations of (1.4) which are not any harder to prove using resolution of singularities than (1.4) itself. For example suppose $A = \{x \in \mathbf{R}^n : g_1(x) > 0, \dots, g_k(x) > 0\}$ where the g_i are real-analytic. Assume further that $0 \in \text{bd}(A)$. Then one can obtain an asymptotic expansion for $\int_{\{x \in A: 0 < f(x) < t\}} \phi(x) dx$ in the same fashion that one obtains (1.4). Furthermore, if one has several real-analytic functions $f_1(x), \dots, f_l(x)$ each satisfying $f_i(0) = 0$, then one can similarly obtain an asymptotic expansion for $\int_{\{x \in A: 0 < f_1(x), \dots, 0 < f_l(x) < t\}} \phi(x) dx$. However, the issue of exhibiting an asymptotic expansion for $\int_{\{x \in A: 0 < f_1(x) < t_1, \dots, 0 < f_l(x) < t_l\}} \phi(x) dx$ in t_1, \dots, t_l appears to be quite a bit harder, and will not be addressed here. Our first result is the following, to be proven in section 3 using the elementary resolution of singularities theorem of section 2.

Theorem 1.1: Suppose $f_1(x), \dots, f_l(x)$ and $g_1(x), \dots, g_k(x)$ are real-analytic functions defined on a neighborhood of the origin in \mathbf{R}^n , with $f_i(0) = 0$ for all i . Let $A = \{x \in \mathbf{R}^n : g_1(x) > 0, \dots, g_k(x) > 0\}$, and assume $0 \in \text{bd}(A)$. There is a neighborhood V of the origin such that if $\phi(x)$ is a C^∞ function supported in V , then $J_t =$

$\int_{\{x \in A: 0 < f_1(x) < t, \dots, 0 < f_i(x) < t\}} \phi(x) dx$ has an asymptotic expansion given by

$$J_t = \sum_{\alpha \leq a} \sum_{i=0}^{n-1} b_{i,\alpha}(\phi) t^\alpha \ln(t)^i + E_a(t) \quad (1.6)$$

Here the α range over an arithmetic progression of positive rational numbers depending on the f_i and the g_i . Let $Z = \{x \in cl(A) : f_i(x) = 0 \text{ for some } i.\}$ There are $M > 0$ and $A_\alpha > 0$ (depending on the f_i and g_i) such that each $b_{i,\alpha}$ is a distribution with respect to ϕ , supported on Z and satisfying

$$|b_{i,\alpha}(\phi)| \leq A_\alpha \sup_{|\beta| \leq M\alpha} \sup_{x \in Z} |\partial^\beta \phi(x)| \quad (1.7a)$$

The error term $E_a(t)$ is such that there is $\epsilon > 0$ and $C_a > 0$ such that if $0 \leq m \leq a$, then

$$\left| \frac{d^m}{dt^m} E_a(t) \right| < C_a \sup_{|\beta| \leq M\alpha} \sup_x |\partial^\beta \phi(x)| t^{a+\epsilon-m} \quad (1.7b)$$

The theorem of this paper regarding oscillatory integrals is as follows:

Theorem 1.2: Suppose $f(x)$ is a real-analytic function defined on a neighborhood of the origin in \mathbf{R}^n with $f(0) = 0$, and let A and V be as in Theorem 1.1. Then as $\lambda \rightarrow \infty$, $I_\lambda = \int_A e^{i\lambda f(x)} \phi(x) dx$ has an asymptotic expansion

$$\sum_{\alpha \leq a} \sum_{i=0}^{n-1} a_{i,\alpha}(\phi) \lambda^{-\alpha} \ln(\lambda)^i + E'_a(\lambda) \quad (1.8)$$

The α range over an arithmetic progression of positive rational numbers depending on f and the g_j . Let $Z = \{x \in cl(A) : f(x) = 0\}$. There are $M' > 0$ and $A'_\alpha > 0$ such that each $a_{i,\alpha}$ is a distribution with respect to ϕ , supported on Z and satisfying

$$|a_{i,\alpha}(\phi)| \leq A'_\alpha \sup_{|\beta| \leq M'\alpha} \sup_{x \in Z} |\partial^\beta \phi(x)| \quad (1.9a)$$

The error term $E'_a(\lambda)$ is such that there is $\epsilon' > 0$ and $C'_{m,a} > 0$ such that for any m one has

$$\left| \frac{d^m}{d\lambda^m} E'_a(\lambda) \right| < C'_{m,a} \sup_{|\beta| \leq M'(a+m)} \sup_x |\partial^\beta \phi(x)| \lambda^{-a-\epsilon'-m} \quad (1.9b)$$

Proof: As in (1.5) we write

$$I_\lambda = \int_0^\infty \frac{\partial J_t}{\partial t} e^{i\lambda t} \gamma(t) dt + \int_0^\infty \frac{\partial \bar{J}_t}{\partial t} e^{-i\lambda t} \gamma(t) dt \quad (1.10)$$

As before \bar{J}_t denotes the analogue of J_t with f replaced by $-f$ and $\gamma(t)$ has compact support and is equal to 1 on a neighborhood of the origin. The two terms of (1.10) are done in the same fashion, so we focus our attention on the first term. By Theorem 1.1, for a given k we can write

$$\frac{dJ_t}{dt} = \sum_{\alpha \leq k-1} \sum_{i=0}^{n-1} B_{i,\alpha}(\phi) t^\alpha \ln(t)^i + \frac{dE_k}{dt}(t)$$

Here the $B_{i,\alpha}$ are obtained from the $b_{i,\alpha}$ by performing the appropriate term-by-term differentiation. In order to be able to differentiate the expansion (1.8) up to λ^{-a} a total of m times, $m \geq 0$, we need to insert the above expansion for $k = a + m + 2$ into the term of (1.10). We get

$$I_\lambda = \sum_{\alpha \leq a+m+1} \sum_{i=0}^{n-1} B_{i,\alpha}(\phi) \int_0^\infty t^\alpha \ln(t)^i e^{i\lambda t} \gamma(t) dt + \int_0^\infty \frac{dE_{a+m+2}}{dt}(t) e^{i\lambda t} \gamma(t) dt \quad (1.11)$$

It is well-known (see [F]) that for any $l > 0$ one has

$$\int_0^\infty e^{i\lambda t} t^\alpha \ln(t)^m \gamma(t) dt = \frac{\partial^m}{\partial \alpha^m} \frac{\Gamma(\alpha + 1)}{(-i\lambda)^{\alpha+1}} + O(\lambda^{-l}) \quad (1.12)$$

As a result, for any l , (1.11) becomes

$$I_\lambda = \sum_{\alpha \leq a+m+1} \sum_{i=0}^{n-1} B_{i,\alpha}(\phi) \frac{\partial^i}{\partial \alpha^i} \frac{\Gamma(\alpha + 1)}{(-i\lambda)^{\alpha+1}} + \int_0^\infty \frac{dE_{a+m+2}}{dt}(t) e^{i\lambda t} \gamma(t) dt + O(\lambda^{-l}) \quad (1.13)$$

Equation (1.13) will give the desired expression for the m th derivative of I_λ 's expansion up to order λ^{-a} . For in (1.12) the $O(\lambda^{-l})$ behaves as needed under differentiation (again see [F]). As for the $\frac{dE_{a+m+2}}{dt}$ term in (1.13), one can differentiate $\int_0^\infty \frac{dE_{a+m+2}}{dt}(t) e^{i\lambda t} \gamma(t) dt$ under the integral sign, obtaining a power of it for each of the m λ -derivatives taken. Then one does $a + m + 1$ integrations by parts in t , integrating the $e^{i\lambda t}$ factor and differentiating the rest. Equation (1.7b) ensures the left endpoint terms disappear. We obtain

$$\begin{aligned} \left| \frac{d^m}{d\lambda^m} \int_0^\infty \frac{dE_{a+m+2}}{dt}(t) e^{i\lambda t} \gamma(t) dt \right| &\leq C_{a,m} \lambda^{-a-m-1} \sup_t \left| \frac{d^{a+m+2} E_{a+m+2}}{dt^{a+m+2}}(t) \right| \\ &\leq C'_{a,m} \sup_{|\beta| \leq M'(a+m)} \sup_x |\partial^\beta \phi(x)| \lambda^{-a-m-1} \end{aligned} \quad (1.14)$$

The last inequality follows from the $\frac{d^{a+m+2} E_{a+m+2}}{dt^{a+m+2}}(t)$ case of (1.7b). Inserting this back into (1.13) gives the desired result. This completes the proof of Theorem 1.2.

Next, we focus on the situation where $f \geq 0$ on A . Where τ is now a large positive parameter, we consider the Laplace Transform-like object defined by

$$L_\tau = \int_A e^{-\tau f(x)} \phi(x) dx \quad (1.15)$$

We have the following theorem regarding L_τ .

Theorem 1.3: Suppose $f(x)$ is a real-analytic function defined on a neighborhood of the origin in \mathbf{R}^n with $f(0) = 0$, and let A , V , and Z be as in Theorem 1.1. Suppose on a sufficiently small neighborhood U of the origin we have that $f(x) \geq 0$ on $U \cap A$. Then if the support of ϕ is contained in $V \cap U$, as $\tau \rightarrow \infty$, L_τ has an asymptotic expansion

$$L_\tau = \sum_{\alpha \leq a} \sum_{i=0}^{n-1} d_{i,\alpha}(\phi) \tau^{-\alpha} \ln(\tau)^i + E_a''(\tau) \quad (1.16)$$

The α range over an arithmetic progression of positive rational numbers depending on f and the g_j . There are $M'' > 0$ and $A''_\alpha > 0$ such that each $d_{i,\alpha}$ is a distribution with respect to ϕ , supported on Z and satisfying

$$|d_{i,\alpha}(\phi)| \leq A''_\alpha \sup_{|\beta| \leq M''_\alpha} \sup_{x \in Z} |\partial^\beta \phi(x)| \quad (1.17)$$

The error term $E_a''(\tau)$ is such that there is $\epsilon'' > 0$ and $C''_{m,a} > 0$ such that for any m one has

$$\left| \frac{d^m}{d\tau^m} E_a''(\tau) \right| < C''_{m,a} \sup_{|\beta| \leq M''(a+m)} \sup_x |\partial^\beta \phi(x)| \tau^{-a-\epsilon''-m} \quad (1.18)$$

Proof: The proof is essentially a repeat of that of Theorem 1.2. Namely, we have

$$L_\tau = \int_0^\infty \frac{\partial J_t}{\partial t} e^{-\tau t} \gamma(t) dt \quad (1.19)$$

Because of the $f \geq 0$ condition, there is only one term instead of two this time. Instead of using (1.12), here we use

$$\int_0^\infty e^{-\tau t} t^\alpha \ln(t)^i \gamma(t) dt = \frac{\partial^i}{\partial \alpha^i} \frac{\Gamma(\alpha+1)}{\tau^{\alpha+1}} + O(\tau^l) \quad (1.20)$$

Otherwise the proof is identical to that of Theorem 1.2 so we omit the details. Incidentally, equation (1.20) is somewhat easier to prove than (1.12). When $i = 0$ one can write (1.20) as as

$$\int_0^\infty e^{-\tau t} t^\alpha dt - \int_0^\infty e^{-\tau t} t^\alpha (1 - \gamma(t)) dt \quad (1.21)$$

The first term is exactly $\frac{\Gamma(\alpha+1)}{\tau^{\alpha+1}}$. On the other hand, via repeated integrations by parts, the second term and its τ derivatives are seen to be $O(\tau^l)$ for any l . The $i > 0$ case follows from differentiating (1.21) under the integral with respect to α .

Theorem 1.1 also gives as a relatively straightforward consequence the following theorem of Atiyah [At]; a similar result is due to Bernstein-Gelfand [BGE]:

Theorem 1.4: Let f , A , U , and V be as in Theorem 1.3. Define $F_\phi(z) = \int_A f(x)^z \phi(x) dx$. Then if the support of ϕ is contained in $V \cap U$, the function $F_\phi(z)$, initially defined as a holomorphic function of z on $Re(z) > 0$, extends to a meromorphic function on all of \mathbf{C} . The poles of this extension are located on an arithmetic progression of negative rational numbers depending on f and the g_j , and each pole is of order at most the dimension n .

Proof: Replacing f by cf for an appropriate $c > 0$ if necessary, we may assume that $|f| \leq 1$ on $supp(\phi)$. Analogous to (1.19), we have

$$F_\phi(z) = \int_0^1 t^z \frac{dJ_t}{dt} dt \quad (1.22)$$

Inserting (1.11) into (1.22), one obtains

$$F_\phi(z) = \sum_{\alpha \leq a} \sum_{i=0}^{n-1} B_{i,\alpha}(\phi) \int_0^1 t^{\alpha+z} \ln(t)^i dt + \int_0^1 \frac{dE_a}{dt}(t) t^z dt \quad (1.23)$$

When $Re(z) > -1$, each of the integrations of the sum in (1.23) can be performed directly, and we obtain

$$F_\phi(z) = \sum_{\alpha \leq a} \sum_{i=1}^n B'_{i,\alpha}(\phi) (z + \alpha + 1)^{-i} + \int_0^1 \frac{dE_a}{dt}(t) t^z dt \quad (1.24)$$

Note that the sum in (1.24) automatically extends to a meromorphic function on \mathbf{C} with poles of order at most n . As for the error term, one can rewrite it as $\int_0^1 F_a(t) t^{z+a-1+\epsilon} dt$, where in view of (1.7b) $F_a(t)$ is bounded. Hence on $Re(z) > -a + 1 - \epsilon$, the error term is an average of locally uniformly bounded analytic functions. Hence it is itself an analytic function on $Re(z) > -a + 1 - \epsilon$. Since a can be made arbitrarily large, the theorem follows.

The remainder of the paper is organized as follows. In section 2, a version of the elementary resolution of singularities algorithm of [Gr] will be developed that will be appropriate for proving the type of theorems of this paper. It may also be useful for other purposes as well. Section 3 will be devoted to proving Theorem 1.1 using the algorithm of section 2.

2. A resolution of singularities theorem

We now give some terminology from [Gr]:

Definition: We say that a function $g : A \subset \mathbf{R}^n \rightarrow \mathbf{R}^n$ a *quasitranlation* if there is a real analytic function $r(x)$ of $n-1$ variables such that $g(x) = (g_1(x), \dots, g_n(x))$, where for some j we have $g_j(x) = x_j - r(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ and where $g_i(x) = x_i$ for all $i \neq j$. In other words $g(x)$ is a translation in the x_j variable when the others are fixed.

Definition: We call a function $m : A \subset \mathbf{R}^n \rightarrow \mathbf{R}^n$ an *invertible monomial map* if there are nonnegative integers $\{\alpha_{ij}\}_{i,j=1}^n$ such that the matrix (α_{ij}) is invertible and $m(x) = (m_1(x), \dots, m_n(x))$ where $m_i(x) = x_1^{\alpha_{i1}} \dots x_n^{\alpha_{in}}$. The matrix (α_{ij}) being invertible ensures that h is a bijection on $\{x : x_l > 0 \text{ for all } l\}$.

Definition: Let $E = \{x : x_i > 0 \text{ for all } i\}$. If $h(x)$ is a bounded, nonnegative, compactly supported function on E , we say $h(x)$ is a *quasibump function* if $h(x)$ is of the following form:

$$h(x) = a(x) \prod_{l=1}^j b_l(c_l(x) \frac{p_l(x)}{q_l(x)}) \quad (2.1)$$

Here $p_l(x), q_l(x)$ are monomials, $a(x) \in C^\infty(\text{cl}(E))$, the $c_l(x)$ are nonvanishing real-analytic functions defined on a neighborhood of $\text{supp}(h)$, and $b_l(x)$ are nonnegative functions in $C^\infty(\mathbf{R})$ such that there are constants $c_1 > c_0 > 0$ with each $b_l(x) = 1$ for $x < c_0$ and $b_l(x) = 0$ for $x > c_1$.

The main theorem from [Gr] is as follows:

Main Theorem of [Gr]: Let $f(x)$ be a real-analytic function defined in a neighborhood of the origin in \mathbf{R}^n . Then there is a neighborhood U of the origin such that if $\phi(x) \in C_c^\infty(U)$ is nonnegative with $\phi(0) > 0$, then $\phi(x)$ can be written (up to a set of measure zero) as a finite sum $\sum_i \phi_i(x)$ of nonnegative functions such that for all i , $0 \in \text{supp}(\phi_i)$ and $\text{supp}(\phi_i)$ is a subset of one of the 2^n closed quadrants defined by the hyperplanes $\{x_m = 0\}$. The following properties hold.

- (1) For each i there are bounded open sets $D_i^0, \dots, D_i^{k_i}$, and maps $g_i^1, \dots, g_i^{k_i}$, each a reflection, translation, invertible monomial map, or quasitranslation, such that $D_i^0 = \{x : \phi_i(x) > 0\}$ and such that each g_i^j is a real-analytic diffeomorphism from D_i^j to D_i^{j-1} . The function g_i^j extends to a neighborhood N_i^j of $\text{cl}(D_i^j)$ with $g_i^j(N_i^j) \subset N_i^{j-1}$ for $j > 1$ and $g_i^1(N_i^1) \subset U$.
- (2) Let $E = \{x : x_i > 0 \text{ for all } i\}$ and $\Psi_i = g_i^1 \circ \dots \circ g_i^{k_i}$. Then $D_i^{k_i} \subset E$, and there is a quasibump function Φ_i such that $\chi_{D_i^{k_i}}(x)(\phi_i \circ \Psi_i(x)) = \Phi_i(x)$.
- (3) $0 \in N_i^{k_i}$ with $\Psi_i(0) = 0$.
- (4) On $N_i^{k_i}$, the functions $f \circ \Psi_i$, $\det(\Psi_i)$, and each j th component function $(\Psi_i)_j$ is of the form $c(x)m(x)$, where $m(x)$ is a monomial and $c(x)$ is nonvanishing.

The corollary to the main theorem of [Gr] says that one can resolve several functions simultaneously in such a way that the resolution satisfies the conclusions of the main theorem. However, these theorems are not precisely what is needed for the arguments of this paper, because here a quasibump function is not the appropriate form for the function $\Phi_i(x)$ of part (2). Instead we will need the following:

Theorem 2.1: If in the main theorem of [Gr] and its corollary, if one replaces the condition in part (2) that $\Phi_i(x) = \chi_{D_i^{k_i}}(x)(\phi_i \circ \Psi_i(x))$ is a quasibump function with the condition that

for some rectangle $R_i = (0, a_1^i) \times \dots \times (0, a_n^i)$ the function $\Phi_i(x)$ is of the form $\chi_{R_i}(x)\gamma(x)$, where $\gamma(x)$ is a C^∞ function on $cl(R_i)$, then the rest of the main theorem and its corollary respectively still holds.

One way of looking at Theorem 2.1 is that in the blown-up coordinates, one can replace the issues coming from the singularities of the quasibump function by the issues coming from the jumps in the characteristic function of R_i . These latter issues will turn out to cause no problems in the analysis of the integral quantities of this paper. Theorem 2.1 will be a consequence of the following, which we will prove later in this section.

Theorem 2.2: Let $\Phi(x)$ be a quasibump function, and let $p_l(x)$ and $q_l(x)$ be as in the definition (2.1) of quasibump function applied to $\Phi(x)$. Then there is a J such that for $1 \leq j \leq J$ and $1 \leq k \leq n$ there are invertible monomial maps $g_j(x)$, quasibump functions $Q_j(x)$ of the form $\prod_{l=1}^m \alpha(\frac{p_{jl}(x)}{q_{jl}(x)})$ and sets $F_j = \{x \in E : \frac{r_{jk}(x)}{s_{jk}(x)} < 1 \text{ for } 1 \leq k \leq n\}$ where $r_{jk}(x)$ and $s_{jk}(x)$ are monomials, such that the following hold.

(1) Up to a set of measure zero one has a decomposition

$$\Phi(x) = \sum_{j=1}^J \Phi(x)Q_j(x)\chi_{F_j}(x) \quad (2.2)$$

(2) Each $\frac{p_{jl}(g_j(x))}{q_{jl}(g_j(x))}$ and each $\frac{p_l(g_j(x))}{q_l(g_j(x))}$ is a monomial.

(3) For each j there is some rectangle $R_j = (0, a_1^j) \times \dots \times (0, a_n^j)$ such that $\chi_{F_j}(g_j(x)) = \chi_{R_j}(x)$.

Comment: Note that $\Phi(g_j(x)) = a(g_j(x)) \prod_{l=1}^L b_l(c_l(g_j(x))\frac{p_l(g_j(x))}{q_l(g_j(x))})$, a and c_l smooth, and that $Q_j(g_j(x)) = \prod_{l=1}^m \alpha(\frac{p_{jl}(g_j(x))}{q_{jl}(g_j(x))})$. Hence (2) implies that $\Phi(g_j(x))$ and each $Q_j(g_j(x))$ are smooth. Thus by (3), for each j there is and a smooth $\gamma_j(x)$ on R_j with

$$\Phi(g_j(x))Q_j(g_j(x))\chi_{F_j}(g_j(x)) = \chi_{R_j}(x)\gamma_j(x) \quad (2.3)$$

Proof that Theorem 2.2 implies Theorem 2.1:

Suppose Theorem 2.2 is known to hold, and let $f(x)$ and U be as in the main theorem (or its corollary). Suppose $\phi(x)$ is a bump function defined in U , and let $\phi = \sum_i \phi_i$ be the decomposition of the main theorem (or its corollary). Let $\Phi_i(x)$ be the quasibump function and Ψ_i the composition of coordinate changes associated to ϕ_i . Let $Q_{ij}(x)$, F_{ij} , and g_{ij} be as given by Theorem 2.2 applied to $\Phi_i(x)$. I claim that the decomposition $\phi(x) = \sum_{ij} \phi_i(x)Q_{ij}(\Psi_i^{-1}(x))\chi_{F_{ij}}(\Psi_i^{-1}(x))$ satisfies the conditions of the main theorem of [Gr], with coordinate changes $g_i^1, \dots, g_i^{k_i}, g_{ij}$, if one specifies the domains D_i and N_i as

follows. At the k_i th level, the domain $D_{ij}^{k_i}$ is defined to be $D_i^{k_i} \cap F_{ij} \cap \{x : Q_{ij}(x) > 0\}$. For $k < k_i$, the domains are successively defined by $D_{ij}^{k-1} = g_i^k(D_{ij}^k)$. At the final $k_i + 1$ th level, one puts $D_{ij}^{k_i+1} = g_{ij}^{-1}D_{ij}^{k_i}$. For $j \leq k_i$ one can define $N_{ij}^{k_i}$ to just be N_i^j , and then $N_{ij}^{k_i+1}$ to be $g_{ij}^{-1}N_{ij}^{k_i}$.

With the above definitions, part (1) of the main theorem is readily seen to hold. As for (2), since $\chi_{D_i^{k_i}}(x)(\phi_i \circ \Psi_i(x)) = \Phi_i(x)$, we have

$$\chi_{D_i^{k_i}}(x)(\phi_i \circ \Psi_i(x))Q_{ij}(x)\chi_{F_{ij}}(x) = \Phi_i(x)Q_{ij}(x)\chi_{F_{ij}}(x) \quad (2.4)$$

Therefore,

$$\chi_{D_i^{k_i} \cap \{x:Q_{ij}(x)>0\} \cap F_{ij}}(x)(\phi_i \circ \Psi_i(x))Q_{ij}(x)\chi_{F_{ij}}(x) = \Phi_i(x)Q_{ij}(x)\chi_{F_{ij}}(x)$$

Equivalently,

$$\chi_{D_{ij}^{k_i}}(x)(\phi_i \circ \Psi_i(x))Q_{ij}(x)\chi_{F_{ij}}(x) = \Phi_i(x)Q_{ij}(x)\chi_{F_{ij}}(x)$$

Composing with g_{ij} , we have

$$\chi_{D_{ij}^{k_i+1}}(x)(\phi_i \circ \Psi_i \circ g_{ij}(x))Q_{ij}(g_{ij}(x))\chi_{F_{ij}}(g_{ij}(x)) = \Phi_i(g_{ij}(x))Q_{ij}(g_{ij}(x))\chi_{F_{ij}}(g_{ij}(x)) \quad (2.5)$$

If one lets $\Psi_{ij} = \Psi_i \circ g_{ij}$ and $\phi_{ij}(x) = \phi_i(x)Q_{ij}(\Psi_i^{-1}(x))\chi_{F_{ij}}(\Psi_i^{-1}(x))$, from (2.5) and the assumption (2.3), for an appropriate rectangle R_{ij} one gets

$$\begin{aligned} \chi_{D_{ij}^{k_i+1}}(x)\phi_{ij}(\Psi_{ij}(x)) &= \chi_{D_{ij}^{k_i+1}}(x)(\phi_i \circ \Psi_i \circ g_{ij}(x))Q_{ij}(g_{ij}(x))\chi_{F_{ij}}(g_{ij}(x)) \\ &= \Phi_i(g_{ij}(x))Q_{ij}(g_{ij}(x))\chi_{F_{ij}}(g_{ij}(x)) \\ &= \chi_{R_{ij}}(x)\gamma_{ij}(x) \end{aligned} \quad (2.6)$$

This gives the version of part (2) of the main theorem that is needed in Theorem 2.1. Parts (3) and (4) are immediate, and we are done.

Proof of Theorem 2.2: The proof is by induction on the dimension n . When $n = 1$, since 1-dimensional quasibump functions are smooth already the proof is straightforward. Namely, one selects r_1 such that $\text{supp}(\Psi) \subset [0, r_1]$. We then let there be a single $F_j = \chi_{(0, r_1)}(x)$ and a single $Q_j(x) = \alpha(x)$ where $\alpha(x) = 1$ on $[0, r_1]$. The corresponding g_j is just the identity map, and the case $n = 1$ follows. Assume now we that know Theorem 2.2 in dimension $n - 1$ and the hypotheses of Theorem 2.2 hold for some n -dimensional situation. We break into two cases.

Case 1: The first case is when there are distinct monomials $t(x)$ and $u(x)$ and positive constants c_1 and c_2 such that whenever $\Phi(x) \neq 0$ we have

$$c_1 < \frac{t(x)}{u(x)} < c_2 \quad (2.7)$$

Write $t(x) = \prod_{i \in I} x_i^{l_i}$ and $u(x) = \prod_{i \in I'} x_i^{l_i}$, where each $l_i > 0$. We can assume that $I \cap I' = \emptyset$ and $I \cup I' \neq \emptyset$. We change variables as follows. For a given $i \in I \cup I'$, let $m_i = \prod_{j \in I \cup I', j \neq i} l_j$ and let $x_i = y_i^{m_i}$. If $i \notin I \cup I'$, let $x_i = y_i$. Let $x = g_1(y)$ be this coordinate change, and let l denote $\prod_{i \in I \cup I'} l_i$. Then we have

$$t(g_1(y)) = \prod_{i \in I} y_i^l, \quad u(g_1(y)) = \prod_{i \in I'} y_i^l \quad (2.8)$$

Consequently, if $t_1(y) = \prod_{i \in I} y_i = [t(g_1(y))]^{\frac{1}{l}}$, and $u_1(y) = \prod_{i \in I'} y_i = [u(g_1(y))]^{\frac{1}{l}}$, equation (2.7) says that whenever $\Phi(g_1(y)) \neq 0$ one has

$$c_1^{\frac{1}{l}} < \frac{t_1(y)}{u_1(y)} < c_2^{\frac{1}{l}} \quad (2.9)$$

Effectively, we have reduced to the case when each l_i is 1. We now prove Theorem 2.2 for case 1 by induction on $m = \min(|I|, |I'|)$. We start with the case where $m = 0$. In this case, either $u_1(y)$ or $t_1(y)$ is a nonconstant monomial and by (2.9) whenever $\Phi(g_1(y)) \neq 0$ that monomial is bounded below. Since the support of a quasibump function is also bounded, every y_i appearing in this monomial is therefore bounded below on the support $\Phi(g_1(y))$. Specifically, there is some y_i and some constant c such that $y_i > c$ whenever $\Phi(g_1(y))$ is nonzero.

Since $\Phi(x)$ is a quasibump function, so is $\Phi(g_1(y))$. As in (2.1), we write $\Phi(g_1(y))$ as

$$a(y) \prod_{l=1}^k b_l(c_l(y) \frac{p_l(y)}{q_l(y)}) \quad (2.10)$$

Because y_i is bounded below, one can incorporate any powers of y_i appearing in each $p_l(y)$ and $q_l(y)$ into the associated $c_l(y)$. Thus we may assume that the $p_l(y)$ and $q_l(y)$ do not depend on y_i . Let $c > 0$ be a constant such that $\prod_{l=1}^k b_l(c \frac{p_l(y)}{q_l(y)}) = 1$ on the support of $\Phi(g_1(y))$. So we have

$$\Phi(g_1(y)) = \Phi(g_1(y)) \prod_{l=1}^k b_l(c \frac{p_l(y)}{q_l(y)}) \quad (2.11)$$

We apply the induction hypothesis to $\prod_{l=1}^k b_l(c \frac{p_l(y)}{q_l(y)})$, a function of $n - 1$ variables. Let Q_j and F_j be as in Theorem 2.2 applied to this function, and let $g_j^2(z)$ be the associated invertible monomial map, called $g_j(x)$ in the statement of that theorem. Then

$$\prod_{l=1}^k b_l(c \frac{p_l(y)}{q_l(y)}) = \sum_j \left(\prod_{l=1}^k b_l(c \frac{p_l(y)}{q_l(y)}) Q_j(y) \chi_{F_j}(y) \right) \quad (2.12)$$

Combining the last two equations gives

$$\Phi(g_1(y)) = \sum_j \left([\Phi(g_1(y)) \prod_{l=1}^k b_l(c \frac{p_l(y)}{q_l(y)})] Q_j(y) \chi_{F_j}(y) \right) \quad (2.13)$$

Using this again on the bracketed expression, we have

$$\Phi(g_1(y)) = \sum_j \Phi(g_1(y))Q_j(y)\chi_{F_j}(y) \quad (2.14)$$

This will lead to our needed decomposition for $\Phi(g_1(y))$. Next, note that we have

$$\Phi(g_1 \circ g_2^j(z)) = \sum_j \Phi(g_1(g_2^j(z)))Q_j(g_2^j(z))\chi_{F_j}(g_2^j(z)) \quad (2.15)$$

Since Theorem 2.2 holds for $\prod_{l=1}^k b_l(c_{ql}(z))$ and $g_2^j(z)$, each $\frac{p_l(g_2^j(z))}{q_l(g_2^j(z))}$ in $\Phi(g_1(g_2^j(z)))$ is a monomial. Similarly, if as in Theorem 2.2 we write $Q_j(y) = \prod_l b_{jl}(c_{jl}(y)\frac{p_{jl}(y)}{q_{jl}(y)})$, we also have that each $\frac{p_{jl}(g_2^j(z))}{q_{jl}(g_2^j(z))}$ appearing in $Q_j(g_2^j(z))$ is a monomial. So since $\chi_{F_j}(g_2(z)) = \chi_{R_j}(z)$, equation (2.15) shows that Theorem 2.2 holds for Φ , with the associated invertible monomial maps given by $g_1 \circ g_2^j$. Hence we are done with the case where $m = \min(|I|, |I'|) = 0$.

Now we assume $\min(|I|, |I'|) = m > 0$, and that we know the theorem for $\min(|I|, |I'|) = m - 1$. For this fixed m , we induct on $\max(|I|, |I'|)$ which we denote by M . The initial step $M = m$ is done the same way as the inductive step, so we assume either $M = m$ or that $M > m$ and that we know the result for $M - 1$. Let $i_1 \in I$ and let $i_2 \in I'$. Let $\alpha(t)$ be a C^∞ function on $[0, \infty)$ that is equal to 1 for small enough t and which satisfies $\alpha(t) + \alpha(\frac{1}{t}) = 1$. In particular,

$$\alpha\left(\frac{y_{i_1}}{y_{i_2}}\right) + \alpha\left(\frac{y_{i_2}}{y_{i_1}}\right) = 1 \quad (2.16)$$

Correspondingly, we decompose $\Phi(g_1(y)) = \Phi_1(y) + \Phi_2(y)$, where

$$\Phi_1(y) = \Phi(g_1(y))\alpha\left(\frac{y_{i_1}}{y_{i_2}}\right), \quad \Phi_2(y) = \Phi(g_1(y))\alpha\left(\frac{y_{i_2}}{y_{i_1}}\right) \quad (2.17)$$

Let $G_1(y)$ be the invertible monomial map whose i_1 th component is $y_{i_1}y_{i_2}$, and whose i th component is y_i for all $i \neq i_1$. Similarly, let $G_2(y)$ be the invertible monomial map whose i_2 th component is $y_{i_1}y_{i_2}$, and whose i th component is y_i for all $i \neq i_2$. We have

$$\Phi_1(G_1(y)) = \Phi(g_1 \circ G_1(y))\alpha(y_{i_1}), \quad \Phi_2(G_2(y)) = \Phi(g_1 \circ G_2(y))\alpha(y_{i_2}) \quad (2.18)$$

Both $\Phi_1(G_1(y))$ and $\Phi_2(G_2(y))$ are quasibump functions for the following reasons. First, since $\Phi(g_1(y))$ has compact support and for $k = 1, 2$ the coordinate change G_k is in the y_{i_k} variable only, the $\Phi_k(G_k(y))$ have compact support in the y_i variable for $i \neq i_k$. When $i = i_k$, the $\alpha(y_{i_k})$ factor ensures that $\Phi_k(G_k(y))$ has compact support in the y_{i_k} variable as well. We conclude that both $\Phi_k(G_k(y))$ are compactly supported. Furthermore, since

the G_k are invertible monomial maps and Φ is a quasibump function, the $\Phi_k(G_k(y))$ are automatically of the form (2.1). We conclude they are quasibump functions.

Next, we translate the condition (2.9) in the new variables. Since y_{i_1} appears in $t_1(y)$ and y_{i_2} appears in $u_1(y)$, both of degree 1, the expression for $\frac{t_1(G_1(y))}{u_1(G_1(y))}$ is obtained from the expression for $\frac{t_1(y)}{u_1(y)}$ by removing y_{i_2} from the denominator, while $\frac{t_1(G_2(y))}{u_1(G_2(y))}$ is obtained from the expression for $\frac{t_1(y)}{u_1(y)}$ by removing y_{i_1} from the numerator. In either case, either $m = \min(|I|, |I'|)$ or $M = \max(|I|, |I'|)$ is reduced by 1. So by the inductive hypothesis Theorem 2.2 applies to both $\Phi_1(G_1(y))$ and $\Phi_2(G_2(y))$. Let the decompositions from Theorem 2.2 applied to these functions be given by

$$\Phi_1(G_1(y)) = \sum_j \Phi_1(G_1(y))Q_{ij}^1(y)\chi_{F_{ij}^1}(y), \quad \Phi_2(G_2(y)) = \sum_j \Phi_2(G_2(y))Q_{ij}^2(y)\chi_{F_{ij}^2}(y) \quad (2.19)$$

Pulling back by G_1 and G_2 respectively, we have

$$\Phi_1(y) = \sum_j \Phi_1(y)Q_{ij}^1(G_1^{-1}y)\chi_{F_{ij}^1}(G_1^{-1}y), \quad \Phi_2(y) = \sum_j \Phi_2(y)Q_{ij}^2(G_2^{-1}y)\chi_{F_{ij}^2}(G_2^{-1}y) \quad (2.20)$$

Since $\Phi(g_1(y)) = \Phi_1(y) + \Phi_2(y)$, we may add these, obtaining

$$\Phi(g_1(y)) = \sum_j \Phi_1(y)Q_{ij}^1(G_1^{-1}y)\chi_{F_{ij}^1}(G_1^{-1}y) + \sum_j \Phi_2(y)Q_{ij}^2(G_2^{-1}y)\chi_{F_{ij}^2}(G_2^{-1}y) \quad (2.21)$$

Going back to the original x coordinates

$$\begin{aligned} \Phi(x) &= \sum_j \Phi_1(g_1^{-1}(x))Q_{ij}^1((g_1 \circ G_1)^{-1}x)\chi_{F_{ij}^1}((g_1 \circ G_1)^{-1}x) \\ &\quad + \sum_j \Phi_2(g_1^{-1}(x))Q_{ij}^2((g_1 \circ G_2)^{-1}x)\chi_{F_{ij}^2}((g_1 \circ G_2)^{-1}x) \end{aligned} \quad (2.22)$$

Equation (2.22) gives the required decomposition of the form (2.2) for $\Phi(x)$. If g_{ij}^1, g_{ij}^2 denote the invertible monomial maps coming from Theorem 2.2 in (2.19), then the invertible monomial maps corresponding to the decomposition (2.22) are given by $g_1 \circ G_1 \circ g_{ij}^1$ or $g_1 \circ G_2 \circ g_{ij}^2$. Since $\Phi_1(g_1^{-1}(x)) = \alpha\left(\frac{(g_1^{-1}(x))_{i_1}}{(g_1^{-1}(x))_{i_2}}\right)\Phi(x)$ and $\Phi_2(g_1^{-1}(x)) = \alpha\left(\frac{(g_1^{-1}(x))_{i_2}}{(g_1^{-1}(x))_{i_1}}\right)\Phi(x)$, the quasibump functions for (2.22) (corresponding to the $Q_j(x)$ in (2.2)) are given by $\alpha\left(\frac{(g_1^{-1}(x))_{i_1}}{(g_1^{-1}(x))_{i_2}}\right)Q_{ij}^1((g_1 \circ G_1)^{-1}x)$ and $\alpha\left(\frac{(g_1^{-1}(x))_{i_2}}{(g_1^{-1}(x))_{i_1}}\right)Q_{ij}^2((g_1 \circ G_2)^{-1}x)$, while the sets corresponding to F_j in (2.2) are given by $(g_1 \circ G_1)^{-1}F_{ij}^1$ or $(g_1 \circ G_2)^{-1}F_{ij}^2$. That (2.22) satisfies the conclusions of Theorem 2.2 is a direct consequence of the fact that the decomposition (2.19) does. This completes the proof in Case 1.

Case 2: The second case is when no expression of the form (2.7) holds. Since $\Phi(x)$ is a quasibump function, as in (2.1) we can write

$$\Phi(x) = a(x) \prod_{l=1}^j b_l(c_l(x) \frac{p_l(x)}{q_l(x)}) \quad (2.23)$$

Since $\Phi(x)$ is compactly supported, we can multiply (2.23) through by $\prod_{m=1}^n \alpha(x_m)$ for an appropriate function $\alpha(x)$ and not change the result. Hence we can assume the x_m are amongst the $\frac{p_l(x)}{q_l(x)}$. Write $\frac{p_l(x)}{q_l(x)} = \prod_{i=1}^n x_i^{a_{il}}$. Here a_{il} can be positive, negative, or zero. By definition of quasibump function, there is a constant c_1 such that $\Phi(x)$ is supported on $\{x \in E : \frac{p_l(x)}{q_l(x)} < c_1 \text{ for all } l\}$. If we define (y_1, \dots, y_n) coordinates by $y_i = \ln(x_i)$, and write the associated coordinate change as $x = e(y)$, then $\Phi(e(y))$ is supported on the set A given by

$$A = \cap_{l=1}^j \{y \in \mathbf{R}^n : \sum_{i=1}^n a_{il} y_i < \ln(c_1)\} \quad (2.24)$$

Define the set A' by

$$A' = \cap_{l=1}^j \{y \in \mathbf{R}^n : \sum_{i=1}^n a_{il} y_i < 0\} \quad (2.25)$$

Lemma: A' is nonempty.

Proof: We will show that if $A' = \emptyset$, then one must be in case 1 of this proof. Suppose $A' = \emptyset$. Note that we may assume $\ln(c_1) > 0$; otherwise $A' \supset A \neq \emptyset$. Since $A' = \emptyset$, we have $A = A - A'$ and therefore

$$A \subset \cup_{l=1}^j \{y \in \mathbf{R}^n : 0 < \sum_{i=1}^n a_{il} y_i < \ln(c_1)\} \quad (2.26)$$

I claim that there is some $M > 0$ such that

$$A \subset \{y \in \mathbf{R}^n : -M < \sum_{i=1}^n a_{il} y_i < \ln(c_1)\} \quad (2.27)$$

For suppose not. Then any $M > 0$ we can find a $y^l \in A$ such that $\sum_{i=1}^n a_{il} y_i^l < -M$. If y denotes the average $\frac{1}{j} \sum_{l=1}^j y^l$, then by the convexity of A , $y \in A$. Furthermore, since for all l_1 and l_2 we have $\sum_{i=1}^n a_{il_1} y_i^{l_2} < \ln(c_1)$, taking this average leads to

$$\sum_{i=1}^n a_{il} y_i = \frac{1}{j} \sum_{l=1}^j \sum_{i=1}^n a_{il} y_i^l < \frac{j-1}{j} \ln(c_1) - \frac{M}{j}$$

Hence if M is large enough, one has

$$\sum_{i=1}^n a_{il} y_i < 0 \quad (2.28)$$

Hence $y \in A'$, contradicting that $A' = \emptyset$. Therefore (2.27) must hold. This automatically implies that we are in case 1; for the y coordinates, (2.7) translates to the existence of numbers $e_1, e_2, d_1, \dots, d_n$ such that $e_1 < \sum_{i=1}^n d_i y_i < e_2$ whenever $(y_1, \dots, y_n) \in A$. Thus (2.27) implies we are in case 1 and the lemma is proved.

Lemma: There is a vector v such that $A \subset A' + v$

Proof: let w be any vector in A' . Then we have $\sum_{i=1}^n a_{il} w_i < 0$ for every l . If $y \in A$, for any $t > 0$, the vector $y + tw$ satisfies

$$\sum_{i=1}^n a_{il}(y + tw)_i = \sum_{i=1}^n a_{il} y_i + t \left(\sum_{i=1}^n a_{il} w_i \right) < \ln(c_1) + t \left(\sum_{i=1}^n a_{il} w_i \right)$$

Thus if t is large enough, one has $\sum_{i=1}^n a_{il}(y + tw)_i < 0$ for all $y \in A$. In other words, $y + tw \in A'$. Hence $A \subset A' - tw$ and we are done.

Since the coordinate functions x_m are amongst the $\frac{p_l(x)}{q_l(x)}$ (see below (2.23)), the n inequalities $\{y_m < 0\}$ are amongst the defining inequalities for A' and therefore

$$A' \subset \{y : y_m < 0 \text{ for all } m\} \quad (2.29)$$

We now "triangulate" A' . To be precise, by (2.29) we see that $A' \cap \{y : \sum_{m=1}^n y_m = -1\}$ is a bounded convex polyhedron and therefore up to a set of measure zero can be written as a finite union $\cup_j S_j$ of $n - 1$ -dimensional simplices. For a given j , we let T_j be the union of all lines containing the origin and passing through S_j . We then have

$$A' = \cup_j T_j$$

Since $A \subset A' + v$ for an appropriate vector v , we have

$$A \subset \cup_j (T_j + v) \quad (2.30)$$

Furthermore, each T_j can be written in the form

$$T_j = \cap_{l=1}^n \{y \in \mathbf{R}^n : \sum_{i=1}^n h_{ilj} y_i < 0\} \quad (2.31)$$

Equivalently,

$$T_j + v = \cap_{l=1}^n \{y \in \mathbf{R}^n : \sum_{i=1}^n h_{ilj} y_i < \sum_{i=1}^n h_{ilj} v_i\}$$

We write $\eta_j = \sum_{i=1}^n h_{ilj} v_i$, so that the above becomes

$$T_j + v = \cap_{l=1}^n \{y \in \mathbf{R}^n : \sum_{i=1}^n h_{ilj} y_i < \eta_j\} \quad (2.32)$$

Next, we move everything back to the original x coordinates. (Recall $y_i = \ln(x_i)$ for each i). Denote the associated coordinate change by $x = e(y)$. Writing $E = \{x : x_i > 0 \text{ for all } i\}$, we have

$$e(T_j + v) = \cap_{i=1}^n \{x \in E : \prod_{i=1}^n x_i^{h_{ilj}} < \exp(\eta_j)\} \quad (2.33)$$

By (2.30), $e(A) \subset \cup_j e(T_j + v)$, while in (2.24) we defined A so that the original quasibump function $\Phi(x)$ is nonzero only on $e(A)$. Hence if we denote $e(T_j + v)$ by F_j , we have

$$\Phi(x) = \sum_j \Phi(x) \chi_{F_j}(x)$$

For each j let $Q_j(x)$ be some bump function of the form $\prod_{i=1}^n \alpha(x_i)$ that is equal to 1 on the support of $\Phi(x)$. One then has

$$\Phi(x) = \sum_j \Phi(x) Q_j(x) \chi_{F_j}(x) \quad (2.34)$$

This will be the decomposition needed for Theorem 2.2. For a given j , the associated coordinate changes $g_j(z)$ are defined as follows. Let H_j be the matrix whose il entry is h_{ilj} , and let $M_j = \{\epsilon_{dij}\}_{d,i=1}^n$ be a matrix such that integer coordinates such that $M_j H_j$ is N times the identity matrix for a large integer N . If one does the substitution $x_i = \prod_{d=1}^n z_d^{\epsilon_{dij}}$, then $\prod_{i=1}^n x_i^{h_{ilj}}$ becomes z_i^N . We define $g_j(z)$ to be the map

$$g_j(z) = \left(\prod_{d=1}^n z_d^{\epsilon_{d1j}}, \dots, \prod_{d=1}^n z_d^{\epsilon_{dnj}} \right)$$

Note that by (2.33) we have

$$g_j^{-1} F_j = \left(0, \exp\left(\frac{\eta_{1j}}{N}\right)\right) \times \dots \times \left(0, \exp\left(\frac{\eta_{nj}}{N}\right)\right) \quad (2.35)$$

Claim: The F_j and $g_j(z)$ defined this way satisfy the conclusions of Theorem 2.2.

Proof: First, we check the $g_j(z)$ are invertible monomial maps; we have not shown that each ϵ_{ijk} is nonnegative. Since $F_j \subset \text{supp}(\Phi)$, F_j has compact support. In particular each x_i is bounded on F_j . Pulling back to the z coordinates, this means each $\prod_{d=1}^n z_d^{\epsilon_{dij}}$ is bounded on $(0, \exp(\frac{\eta_{1j}}{N})) \times \dots \times (0, \exp(\frac{\eta_{nj}}{N}))$. This fact forces each ϵ_{dij} to be nonnegative.

For on the curve $z = (t^{\alpha_1}, \dots, t^{\alpha_n})$, $\alpha_d > 0$, the i th component of $g_j(z)$ is equal to $t^{\sum_d \alpha_d \epsilon_{dij}}$. If some ϵ_{dij} were negative, by choosing α_d to be far larger than the other α_k , the image of the curve under g_j would go off to infinity. We conclude that each ϵ_{dij} is nonnegative and therefore that $g_j(z)$ is an invertible monomial map.

Next, observe that by (2.34), part (1) of Theorem 2.2 holds. As for (2), we must show that if p_k and q_k are as in (2.23), then each $\frac{p_k(g_j(z))}{q_k(g_j(z))}$ is a monomial. Recall that by

(2.31), we have $T_j = \cap_{l=1}^n \{y \in \mathbf{R}^n : \sum_{i=1}^n h_{ilj} y_i < 0\}$, and that each equation $\frac{p_k(e(y))}{q_k(e(y))} < 1$ is one of the defining equations for A . Translating the statement that $T_j \subset A$ into the x coordinates gives: For fixed j , if $\prod_{i=1}^n x_i^{h_{ilj}} < 1$ for all l then $\frac{p_k(x)}{q_k(x)} < 1$ for all k . Pulling back by g_2 , we have that if $z \in (0, \exp(\frac{\eta_{1j}}{N})) \times \dots \times (0, \exp(\frac{\eta_{nj}}{N}))$, then $\frac{p_k(g_j(z))}{q_k(g_j(z))} < 1$ for all k . Note that $\frac{p_k(g_j(z))}{q_k(g_j(z))}$ is of the form $\prod_{d=1}^n z_d^{\delta_{dkj}}$ for integers δ_{dkj} . So exactly as in the last paragraph, we must have that $\delta_{dkj} \geq 0$ for each d and k . Hence for each k , $\frac{p_k(g_j(z))}{q_k(g_j(z))}$ is a monomial. Since j was arbitrary, we have part (2) of Theorem 2.2. Equation (2.35) gives part (3) and we are done with the proof of Theorem 2.2.

3. Proof of Theorem 1.1

Most of this section will be devoted to proving Theorem 1.1. We will use the resolution of singularities algorithm as described in Theorem 2.1 to reduce consideration to the simpler situation where there is only one $f_i(x)$, and where that $f_i(x)$ is of the form $c(x)m(x)$ where $m(x)$ is a monomial and $c(x)$ is nonvanishing. Namely, we will use Theorem 2.1 to reduce things to proving the following:

Theorem 3.1: Suppose $c(x)$ is a positive real-analytic function defined on a neighborhood of $[0, 1]^n$, and $m(x) = \prod_{i=1}^n x_i^{m_i}$ is a nonconstant monomial. Let Z denote $\{x \in [0, 1]^n : m(x) = 0\}$. Define $K_\phi(t)$ by

$$K_\phi(t) = \int_{\{x \in (0,1)^n : c(x)m(x) < t\}} \phi(x) dx \quad (3.1)$$

Then if $\phi(x)$ is a smooth function on $[0, 1]^n$ supported in a sufficiently small neighborhood of Z , $K_\phi(t)$ has an asymptotic expansion of the following form, where the α range over an arithmetic progression of positive numbers:

$$K_\phi(t) = \sum_{\alpha \leq a} \sum_{i=0}^{n-1} k_{i,\alpha}(\phi) t^\alpha \ln(t)^i + E_a(t) \quad (3.2)$$

There are $M > 0$ and $D_\alpha > 0$ depending on $c(x)$ and $m(x)$ such that each $k_{i,\alpha}$ is a distribution with respect to ϕ , supported on Z , satisfying

$$|k_{i,\alpha}(\phi)| < D_\alpha \sup_{|\beta| \leq M\alpha} \sup_{x \in Z} |\partial^\beta \phi(x)| \quad (3.3)$$

Also, there are $\epsilon > 0$ and $C_a > 0$ depending on $c(x)$ and $m(x)$ such that for any any a and any l satisfying $0 \leq l \leq a$ one has

$$\left| \frac{d^l}{dt^l} E_a(t) \right| < C_a \sup_{|\beta| \leq M(a+l)} \sup_{x \in (0,1)^n} |\partial^\beta \phi(x)| t^{a+\epsilon-l} \quad (3.4)$$

Proof that Theorem 3.1 implies Theorem 1.1: Suppose we are in the setting of Theorem 1.1. We apply the resolution of singularities algorithm simultaneously to each $f_l(x)$, each $g_l(x)$, and each difference $f_l(x) - f_m(x)$. Let U be as in the version of the main theorem of [Gr] from Theorem 2.1, and let $A = \{x \in \mathbf{R}^n : g_1(x) > 0, \dots, g_k(x) > 0\}$. Let $\eta \in C_c(U)$ such that $\eta = 1$ on a neighborhood V of the origin, and let $\eta = \sum_i \eta_i$ be the decomposition given by the version of the main theorem given by Theorem 2.1.

For any $\phi \in C_c(V)$, observe that $\phi = \sum_i \phi \eta_i$. Thus we have

$$J_t = \int_{\{x \in A : 0 < f_1(x) < t, \dots, 0 < f_l(x) < t\}} \phi(x) dx = \sum_i \int_{\{x \in A : 0 < f_1(x) < t, \dots, 0 < f_l(x) < t\}} \phi(x) \eta_i(x) dx$$

Let Ψ_i be the composition of coordinate changes corresponding to η_i , let $B_i = \Psi_i^{-1}A$, and let R_i be the rectangle $(0, a_1^i), \dots, (0, a_n^i)$ given by Theorem 2.1. Let $F_i = f_i \circ \Psi_i$. Then the i th term of the above expression becomes

$$\int_{\{x \in B_i \cap R_i : 0 < F_1(x) < t, \dots, 0 < F_l(x) < t\}} \phi \circ \Psi_i(x) (\eta_i \circ \Psi_i(x)) J_i(x) dx \quad (3.5)$$

Here $J_i(x)$ denotes the Jacobian Ψ_i , and by Theorem 2.1 each $\eta_i \circ \Psi_i(x)$ is smooth. Since each g_j is being resolved by $\Psi_i(x)$, in (3.5) each $g_l \circ \Psi_i(x)$ is of the form $c_l(x)m_l(x)$, where $m_l(x)$ is a monomial and $c_l(x)$ is nonvanishing. In particular each $g_l \circ \Psi_i(x)$ is either everywhere positive or everywhere negative on R_i . Thus we have that either some $B_i \cap R_i = \emptyset$, whereupon (3.5) is zero, or that each $B_i \cap R_i = R_i$, whereupon (3.5) is equal to

$$\int_{\{x \in R_i : 0 < F_1(x) < t, \dots, 0 < F_l(x) < t\}} \phi \circ \Psi_i(x) (\eta_i \circ \Psi_i(x)) J_i(x) dx \quad (3.6)$$

Clearly we only have to consider terms of the form (3.6). Next, notice that since we resolved all differences $f_l(x) - f_m(x)$, each $F_l(x) - F_m(x)$ is also of the form $c_{lm}(x)m_{lm}(x)$, where $m_{lm}(x)$ is a monomial and $c_{lm}(x)$ is nonvanishing. As a result, each $F_l(x) - F_m(x)$ is either everywhere positive or everywhere negative on R_i . Hence there is some $F_l(x)$ which is strictly larger than every other $F_m(x)$ everywhere on R_i , and (3.6) is

$$\int_{\{x \in R_i : 0 < F_l(x) < t\}} \phi \circ \Psi_i(x) (\eta_i \circ \Psi_i(x)) J_i(x) dx$$

Since the singularities of $g_l(x)$ are also being resolved, we can write $F_l(x) = c(x)m(x)$ and the above becomes

$$\int_{\{x \in R_i : 0 < c(x)m(x) < t\}} \phi \circ \Psi_i(x) (\eta_i \circ \Psi_i(x)) J_i(x) dx \quad (3.7)$$

To get (3.7) into the form (3.1), we do a scaling in (3.7) to convert R_i into $(0, 1)^n$. Denoting this scaling coordinate change by $(x_1, \dots, x_n) \rightarrow sx = (s_1 x_1, \dots, s_n x_n)$, (3.7) becomes

$$\prod_{l=1}^n s_l \int_{\{x \in (0, 1)^n : 0 < c(sx)m(sx) < t\}} \phi \circ \Psi_i(sx) (\eta_i \circ \Psi_i(sx)) J_i(sx) dx$$

This integral is of the form (3.1), with ϕ_i replaced by $\phi \circ \Psi_i(sx)(\eta_i \circ \Psi_i(sx))J_i(sx)$. Hence applying Theorem 3.1 will give an asymptotic expansion for (3.7) satisfying (3.2) – (3.4) for $\phi \circ \Psi_i(sx)(\eta_i \circ \Psi_i(sx))J_i(sx)$ in place of $\phi_i(x)$. By the chain and product rules, (3.2) – (3.4) then also hold for ϕ itself. Adding over all i gives Theorem 1.1 and we are done.

Proof of Theorem 3.1: We induct on the number of positive m_i in $m(x) = \prod_i x_i^{m_i}$. We start with the case where exactly one m_i is positive, and without generality we may assume $i = 1$, so that $c(x)m(x) = c(x)x_1^k$ for some k . Since $c(x)$ is a positive real analytic function, we can write $c(x) = c_1(x)^k$ for some positive real analytic c_1 , and we have $c(x)m(x) = (c_1(x)x_1)^k$. We now do a coordinate change as follows. For $i > 1$, we let $y_i = x_i$. For $i = 1$, we let $y_1 = c_1(x)x_1$. This is a smooth coordinate change on $x_1 < \delta$ for an appropriate δ . Denote the coordinate change map by $x = \Psi(y)$. Assuming $\phi(x)$ is supported on $x_1 < \delta$, we have

$$K_\phi(t) = \int_{\{x \in (0,1)^n : y_1 < t^{\frac{1}{k}}\}} \phi(\Psi(y))J(y) dy \quad (3.8)$$

Here $J(y)$ denotes the Jacobian of this coordinate change. Thus $K_\phi(t)$ is the indefinite integral of $\Phi(y) = \phi(\Psi(y))J(y)$ in the y_1 variable from 0 to $t^{\frac{1}{k}}$. Hence if we substitute $\phi(\Psi(y))J(y) = \sum_{l=0}^m \frac{\partial^l \Phi}{\partial y_1^l}(0, y_2, \dots, y_n) \frac{y_1^l}{l!} + O(y_1^{m+1})$ into (3.8), one obtains

$$K_\phi(t) = \sum_{l=0}^m \left(\int_{(0,1)^{n-1}} \frac{\partial^l \Phi}{\partial y_1^l}(0, y_2, \dots, y_n) dy_2 \dots dy_n \right) \frac{t^{\frac{l+1}{k}}}{(l+1)!} + O(t^{\frac{m+2}{k}})$$

This gives the desired asymptotic expression (3.2) for $K_\phi(t)$ in powers of $t^{\frac{1}{k}}$. The expressions (3.3) and (3.4) follow from the chain rule applied to $\Phi(y) = \phi(\Psi(y))J(y)$. This concludes the proof of Theorem 3.1 for the case that only one m_i is positive.

Next, we assume that some $l > 1$ of the m_i are positive, and we have shown the result for $l - 1$. Without loss of generality, once again we assume that $m_1 = k > 0$. Like before we let $c_1(x)$ be such that $c_1(x)^k = c(x)$, so that $c(x)m(x) = (c_1(x)x_1)^k \prod_{i>1} x_i^{m_i}$. Like above, for x_1 smaller than some δ we can do the coordinate change to variables y_i , where $y_1 = c_1(x)x_1$ and where $y_i = x_i$ for $i > 1$. Thus if $x = \Psi(y)$, we have

$$c(\Psi(y))m(\Psi(y)) = \prod_{i=1}^n y_i^{m_i} \quad (3.9)$$

Let $\xi(t)$ be a smooth function on $[0, \infty)$ that is supported in $[0, \delta)$, and which is equal to 1 on a neighborhood of 0. Correspondingly we write $\phi(x) = \phi_1(x) + \phi_2(x)$, where

$$\phi_1(x) = \phi(x)\xi(x_1), \quad \phi_2(x) = \phi(x)(1 - \xi(x_1)) \quad (3.10)$$

We correspondingly write $K_\phi(t) = K_\phi^1(t) + K_\phi^2(t)$, where

$$K_\phi^1(t) = \int_{\{x \in (0,1)^n : c(x)m(x) < t\}} \phi_1(x) dx, \quad K_\phi^2(t) = \int_{\{x \in (0,1)^n : c(x)m(x) < t\}} \phi_2(x) dx \quad (3.11)$$

Note that $K_\phi^2(t)$ reduces to the case when $l - 1$ of the m_i are positive, for the integrand is supported on $x_1 > \delta'$ and thus the x_1^k factor can be incorporated into the $c(x)$: One does a linear variable change in the x_1 variable to turn $[\delta', 1] \times [0, 1]^{n-1}$ into $[0, 1]^n$ and then applies the $l - 1$ case. Hence it suffices to restrict our attention to $K_\phi^1(t)$. We change to the y variables, obtaining

$$K_\phi^1(t) = \int_{\{x \in (0,1)^n: \prod_{i=1}^n y_i^{m_i} < t\}} \phi_1(\Psi(y))J(y) dy \quad (3.12)$$

$J(y)$ is the Jacobian of the coordinate change. In (3.12), the $c(x)$ factor has been removed from the domain of integration, a fact that will make our arguments simpler. Write $\Phi(x) = \phi_1(\Psi(y))J(y)$, and (3.12) becomes

$$K_\phi^1(t) = \int_{\{x \in (0,1)^n: \prod_{i=1}^n y_i^{m_i} < t\}} \Phi(y) dy \quad (3.12')$$

Lemma 3.2: To prove $K_\phi^1(t)$ has an asymptotic expansion satisfying the conditions of Theorem (3.2)–(3.4), thereby proving Theorem 3.1, it suffices to find an asymptotic expansion for $K_\phi^1(t)$ satisfying the analogues of (3.2)–(3.4) with ϕ replaced by Φ . Furthermore, it suffices to consider only the case where $m_i > 0$ for all i .

Proof: Since $\Phi(y) = \phi_1(\Psi(y))J(y) = \xi(\Psi_1(y))\phi(\Psi(y))J(y)$, by the chain and product rules, $|\partial^\alpha \Phi(x)|$ is bounded by $C_\alpha \sum_{|\beta| \leq |\alpha|} |\partial^\beta \phi(x)|$. Hence if the versions of (3.2)–(3.4) hold for $K_\phi^1(t)$ with Φ in place of ϕ , they will also hold for $K_\phi^1(t)$ with ϕ itself. This gives the first statement of Lemma 3.2. As for the second statement, suppose we have proved the asymptotics in the case where each $m_i > 0$. Then in the general situation, we can fix those y_i variables for which $m_i = 0$. Then the asymptotics hold for the integral in the remaining variables. By integrating the asymptotics with respect to the y_i variables for which $m_i = 0$, one sees that the asymptotics hold for the original integral. This completes the proof of Lemma 3.2.

Lemma 3.3: Suppose $g(y)$ is a C^∞ function on $[0, 1]^n$. Let $\sum_\alpha g_\alpha y^\alpha$ denote the Taylor expansion for $g(y)$ about the origin. Then for each N we can write

$$g(y) = \sum_{\alpha_1, \dots, \alpha_n < N} g_\alpha y^\alpha + \sum_\beta \frac{y^\beta}{\beta_1! \dots \beta_l!} [h_\beta(y_{l_\beta+1}, \dots, y_n) - \sum_{i=0}^{N-1} \frac{\partial^i h_\beta}{\partial y_{l_\beta+1}^i}(0, y_{l_\beta+2}, \dots, y_n) \frac{y_{l_\beta+1}^i}{i!}] \quad (3.13)$$

Here the sum in β ranges over all $(\beta_1, \dots, \beta_l)$ with $0 \leq l < n$ and $0 \leq \beta_i < N$ for each i , l_β denotes the number of entries β has, and $h_\beta(y_{l_\beta+1}, \dots, y_n)$ denotes $\partial^\beta g(0, \dots, 0, y_{l_\beta+1}, \dots, y_n)$.

Proof: Taylor expanding in y_1 we have

$$g(y) = \sum_{i=0}^{N-1} \frac{\partial^i g}{\partial y_1^i}(0, y_2, \dots, y_n) \frac{y_1^i}{i!} + (g(y) - \sum_{i=0}^{N-1} \frac{\partial^i g}{\partial y_1^i}(0, y_2, \dots, y_n) \frac{y_1^i}{i!}) \quad (3.14)$$

This gives the result for $n = 1$. When $n > 1$, one substitutes the $n - 1$ dimensional case for $\frac{\partial^i g}{\partial y_1^i}(0, y_2, \dots, y_n)$ into each term of the left series of (3.14) and the lemma follows.

Lemma 3.4: Let $p(y)$ denote one of the terms of the second sum of (3.13); that is, let $p(y)$ be of the form

$$p(y) = \frac{y^\beta}{\beta_1! \dots \beta_l!} [h_\beta(y_{l_\beta+1}, \dots, y_n) - \sum_{i=0}^{N-1} \frac{\partial^i h_\beta}{\partial y_{l_\beta+1}^i}(0, y_{l_\beta+2}, \dots, y_n) \frac{y_{l_\beta+1}^i}{i!}]$$

Let j denote the index called $l_\beta + 1$ in this term. Then if γ is a multiindex such that $\gamma_j = 0$, then

$$|\partial^\gamma p(y)| < C'_{|\gamma|, N} y_j^N \|g\|_{C^{nN+|\gamma|}}$$

Proof: If one takes the γ derivative of $p(y)$, one obtains some terms of the form

$$cy^m [q(y_j, \dots, y_n) - \sum_{i=0}^{N-1} \frac{\partial^i q}{\partial y_j^i}(0, y_{j+1}, \dots, y_n) \frac{y_j^i}{i!}] \quad (3.15)$$

Here y^m is a monomial, and q is some partial derivative of the appropriate h_β of order at most $|\gamma|$. Note that the bracketed expression is equal to $q(y_j, \dots, y_n)$ minus the first N terms of its Taylor expansion in the y_j direction. As a result, by Taylor's theorem, the bracketed expression is equal to $\frac{y_j^N}{N!} \frac{\partial^N q}{\partial y_j^N}(Y_j, y_{j+1}, \dots, y_n)$ for some Y_j between 0 and y_j . Hence (3.15) is bounded by $c'y_j^N \|q\|_{C^N}$. Since q is a derivative of order at most $|\gamma|$ of some h_β , which is itself a derivative of order at most $(n-1)N$ of g , we conclude that this term is bounded by $c''y_j^N \|g\|_{C^{nN+|\gamma|}}$. We add over all terms of $\partial^\gamma p(y)$ and we are done.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1: By lemma 3.2, it suffices to find an asymptotic expansion for $K_\phi^1(t)$ satisfying the analogue of (3.2) – (3.4) with ϕ replaced by Φ , and in $m(x) = \prod_i x_i^{m_i}$ we may assume $m_i > 0$ for all i . Also, we can assume we know the result inductively for dimensions $< n$. We fix some $a > 0$. With the goal of finding the asymptotic expansion (3.2) up to the power t^a , we apply Lemma 3.3 to $\Phi(y)$, setting N equal $\lfloor (a+\epsilon) \max_i m_i \rfloor + 1$. We then insert the result termwise into (3.12'). We obtain two types of terms. The first are terms of the form

$$\Phi_\alpha \int_{\{y \in (0,1)^n : \prod_{i=1}^n y_i^{m_i} < t\}} y^\alpha dy \quad (3.16)$$

Here Φ_α denotes the coefficient of y^α in Φ 's Taylor expansion about the origin. One can evaluate (3.16) directly using calculus, by induction on the dimension for example, and prove that (3.16) is of the form

$$\Phi_\alpha \sum_{\beta \leq M\alpha} \sum_{i=0}^{n-1} c_{i,\beta} t^\beta \ln(t)^i \quad (3.17)$$

This is the form that we need for our asymptotics. The second type of term we obtain comes from the second sum of (3.13), a term denoted by $p(y)$ in Lemma 3.4. We write $p(y) = p_{y_j}(\bar{y})$, where y_j is as in the previous lemma and where \bar{y} denotes the remaining $n - 1$ variables. We integrate the term first with respect to the \bar{y} variables and then with respect to the y_j variable. The term becomes the following, where $\bar{m}(\bar{y})$ denotes the monomial $\frac{m(y)}{y_j^{m_j}}$:

$$\int_0^1 \int_{\{y \in (0,1)^{n-1} : \bar{m}(\bar{y}) < \frac{t}{y_j^{m_j}}\}} p_{y_j}(\bar{y}) d\bar{y} dy_j \quad (3.18)$$

We next use the inductive hypothesis on the inner integral, and (3.18) becomes

$$\int_0^1 \left[\sum_{\alpha \leq a} \sum_{k=0}^{n-1} k_{k,\alpha} (p_{y_j}) \left(\frac{t}{y_j^{m_j}}\right)^\alpha \ln\left(\frac{t}{y_j^{m_j}}\right)^k + E_{a,y_j}\left(\frac{t}{y_j^{m_j}}\right) \right] dy_j \quad (3.19)$$

We expand $\ln\left(\frac{t}{y_j^{m_j}}\right)^k = (\ln(t) - m_j \ln(y_j))^k$ and (3.19) becomes

$$\int_0^1 \left[\sum_{\alpha \leq a} \sum_{k_1, k_2=0}^{n-1} \kappa_{k_1, k_2, \alpha} (p_{y_j}) \left(\frac{t}{y_j^{m_j}}\right)^\alpha \ln(t)^{k_1} \ln(y_j)^{k_2} + E_{a,y_j}\left(\frac{t}{y_j^{m_j}}\right) \right] dy_j \quad (3.20)$$

Here each $\kappa_{k_1, k_2, \alpha}$ is a constant multiple of some $k_{k,\alpha}$. In particular, each $\kappa_{k_1, k_2, \alpha}$ satisfies (3.3). We rewrite (3.20) as

$$\sum_{\alpha \leq a} \sum_{k_1, k_2=0}^{n-1} \left(\int_0^1 \frac{\kappa_{k_1, k_2, \alpha} (p_{y_j}) \ln(y_j)^{k_2}}{y_j^{m_j \alpha}} dy_j \right) t^\alpha \ln(t)^{k_1} dy_j + \int_0^1 E_{a,y_j}\left(\frac{t}{y_j^{m_j}}\right) dy_j \quad (3.21)$$

We will see that (3.21) gives the desired asymptotic expansion. For by (3.3) and then Lemma 3.4 we have

$$\begin{aligned} |\kappa_{k_1, k_2, \alpha} (p_{y_j})| &\leq D_\alpha \|p_{y_j}\|_{C^{M\alpha}(Z)} \\ &\leq D'_\alpha y_j^N \|\Phi\|_{C^{M\alpha+nN}(Z)} \end{aligned} \quad (3.22)$$

Hence, the absolute value of the $t^\alpha \ln(t)^{k_1}$ coefficient of (3.21) satisfies

$$\left| \sum_{k_2} \int_0^1 \frac{\kappa_{k_1, k_2, \alpha} (p_{y_j}) \ln(y_j)^{k_2}}{y_j^{m_j \alpha}} dy_j \right| \leq D'_\alpha \|\Phi\|_{C^{M\alpha+nN}(Z)} \int_0^1 y_j^N \frac{|\ln(y_j)|^{n-1}}{y_j^{m_j \alpha}} dy_j \quad (3.23)$$

Since $N \geq \max_k m_k(a + \epsilon) \geq m_j a \geq m_j \alpha$, the right-hand integral above is bounded, so (3.23) is at most $C'_\alpha \|\Phi\|_{C^{M\alpha+nN}(Z)}$. Since $\alpha \leq a$ and $N < M'a$ for some M' , we have $M\alpha + nN \leq (M + M'n)a$, and we conclude that (3.23) is at most

$$C'_\alpha \|\Phi\|_{C^{(M+M'n)a}(Z)}$$

$$\leq C_a'' \|\Phi\|_{C^{(M+M'n)_a}(Z)}$$

Here $C_a'' = \sup_{\alpha \leq a} C'_\alpha$. This gives us the desired estimate (3.3) for the $t^\alpha \ln(t)^{k_1}$ coefficient.

We also have to analyze the error term $\int_0^1 E_{a,y_j}(\frac{t}{y_j}) dy_j$. If $0 \leq l \leq a$, the l th derivative of this with respect to t is equal to

$$\int_0^1 y_j^{-lm_j} \frac{\partial^m E_{a,y_j}}{\partial t^m}(\frac{t}{y_j}) dy_j \quad (3.24)$$

Substituting (3.4) into this, this is bounded in absolute value by

$$C_a \int_0^1 y_j^{-lm_j} \|p_{y_j}\|_{C^{Ma}((0,1)^{n-1})} (\frac{t}{y_j})^{a+\epsilon-l} \quad (3.25)$$

By Lemma 3.4 this in turn is bounded by

$$\begin{aligned} & C_a \int_0^1 y_j^{-lm_j+N} \|\Phi\|_{C^{Ma+nN}((0,1)^n)} (\frac{t}{y_j})^{a+\epsilon-l} dy_j \\ &= C_a \|\Phi\|_{C^{Ma+nN}((0,1)^n)} (\int_0^1 y_j^{N-am_j-\epsilon m_j} dy_j) t^{a+\epsilon-l} \end{aligned} \quad (3.26)$$

Since $N \geq am_j + \epsilon m_j$, (3.26) is at most

$$C_a \|\Phi\|_{C^{Ma+nN}((0,1)^n)} t^{a+\epsilon-l} \quad (3.27)$$

And because $N \leq M'a$ for some M' , (3.27) is bounded by

$$C_a \|\Phi\|_{C^{(M+M'n)_a}((0,1)^n)} t^{a+\epsilon-l} \quad (3.28)$$

This gives the desired estimate (3.3) with Φ and we are done. To be clear, what we showed is that each $t^\alpha \ln(t)^j$'s coefficient is bounded in terms of $C^{M_0 a}$ norms of Φ for $M_0 = M + M'n$, with corresponding estimates for the error terms. Note that for (3.2)-(3.4) to hold we need $C^{M_0 \alpha}$ norms, which makes a difference if α is a lot smaller than a . However, if for a given α we consider the estimate obtained from the expansion to degree $a = 2\alpha$ for example, we get the desired estimates.

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