Sharp Estimates for One-dimensional Oscillatory Integral Operators with C^∞ Phase

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1. Introduction

The objects under consideration in this paper are oscillatory integral operators on $L^2(\mathbf{R})$ of the form

$$Tf(x) = \int e^{i\lambda S(x,y)}\phi(x,y)f(y)\,dy \tag{1.1}$$

Here $\phi(x,y)$ is smooth function defined on an appropriately small neighborhood of the origin with $\phi(0,0) \neq 0$, λ is a real parameter, and $S(x,y)$ is a smooth function. The question being addressed in this paper is the determination of the best constant $\epsilon > 0$ for which for some constant $C > 0$ we have the estimate

$$||T||_{L^2 \rightarrow L^2} < C|\lambda|^{-\epsilon} \tag{1.2}$$

It turns out that the answer to this question depends on the behavior of the second derivative $\frac{\partial^2 S}{\partial x \partial y}(x,y)$. If $\frac{\partial^2 S}{\partial x \partial y}(0,0) \neq 0$, then by a straightforward TT^* argument, one has that the best ϵ in (1.2) is exactly $\frac{1}{2}$. On the other hand, if each multiindex (α, β) with $\alpha, \beta \geq 1$ satisfies $\frac{\partial^{\alpha+\beta} S}{\partial x^\alpha \partial y^\beta}S(0,0) = 0$, then it is not too hard to show that there is no $\epsilon > 0$ for which (1.2) holds. Hence in this paper we will assume we are in the degenerate, finite-type case; in other words we will assume that for some $(\alpha, \beta) \neq (1,1)$ with $\alpha, \beta \geq 1$ we have the following:

$$\frac{\partial^2 S}{\partial x \partial y}(0,0) = 0, \;\; \frac{\partial^{\alpha+\beta} S}{\partial x^\alpha \partial y^\beta}(0,0) \neq 0 \tag{1.3}$$

In view of prior work on this subject, such as [PS2] and [R], one expects that in general the optimal ϵ in (1.2) is expressible in terms of the *reduced Newton polygon* of $S(x,y)$. Namely, let $\sum s_{a,b}x^a y^b$ be the (possibly nonconvergent) Taylor expansion of $S(x,y)$ at the origin. For each $(a,b) \in \mathbf{R}^2$, let $Q_{a,b}$ be the set $\{(x,y) \in \mathbf{R}^2 : x \geq a,\ y \geq b\}$.

Definition: The *reduced Newton polygon* $N_r(S)$ is defined to be the convex hull of the $Q_{a,b}$ for which $s_{a,b} \neq 0$, $a > 0$, and $b > 0$.

It is a well-known fact that the boundary of $N_r(S)$ consists of a vertical ray going out to infinity in the y direction, followed by finitely many (possibly zero) bounded segments of negative slope, followed by a horizontal ray going out to infinity in the x-direction.

Definition: The *reduced Newton distance* δ_r of S is defined to be the least $\delta > 0$ for which $(\delta, \delta) \in N_r(S)$.

One often thinks of the reduced Newton distance geometrically as the x and y coordinates of the intersection of the line $y = x$ with $N_r(S)$.

The main theorem of this paper is that the best ϵ in (1.2) can be expressed in terms of the reduced Newton distance in a canonical way:

Theorem 1.1: For any $S(x, y)$ satisfying (1.3), if $\phi(x, y)$ is supported in a sufficiently small neighborhood of the origin then for an appropriate C (1.2) holds for $\epsilon = -\frac{1}{2\delta_r}$ This value of ϵ is sharp; there exists a constant C' such that for any λ we can find an $f(x)$ and $g(x)$ such that $|\int Tf(x)g(x)\,dx| > C'|\lambda|^{-\frac{1}{2\delta_r}}||f||_2||g||_2$

A brief history of the work done on this problem is as follows. In the 70's, inspired by a conjecture of Arnold, Varchenko [V] proved an analogue to Theorem 1.1 for *scalar* oscillatory integrals (i.e integrals $\int e^{i\lambda S(x,y)}\phi(x, y)dx\,dy$) in a number of cases, using a method involving resolutions of singularities. In [PS2], Phong and Stein proved Theorem 1.1 for a class of operators, and many of the analytic tools used in subsequent papers were developed. They later conjectured that Theorem 1.1 holds for any real-analytic $S(x, y)$, and proved this conjecture in [PS1]. A rather different argument, in the context of multilinear operators, was later given in [PSSt]. Furthermore, in [Se], Seeger proved non-endpoint estimates for the operators in [PS1] as well as more general averaging operators. Endpoint estimates in the fold case, which include higher-dimensional operators, are given in [Co]. In his thesis [R], Rytchkov simplified the arguments in [PS1] and was able to generalize their work to most C^∞ situations. However, some exceptional cases remained. The author later found a proof of Theorem 1.1 in the real-analytic situation [G1] that combined the analytic methods of [PS2] with a resolution of singularities procedure more in the spirit of [V]. This proof avoided quoting any theorems from algebraic geometry, but still did not cover the general C^∞ situation since the resolution scheme only applied to real-analytic functions. At around the same time, the author proved an analogous result for objects related to scalar oscillatory integrals [G2] which used a substantially simpler resolution scheme that applies to general C^∞ functions; the question arose if there was some way of proving an operator version that used the same simplified resolution scheme in conjuction with extensions of the analytic tools used in the various papers on this subject, giving a theorem that covers the full C^∞ situation. This paper does exactly that.

2. The Set-up

In most of the work that has been done on this type of problem, starting with [PS1], one proceeds by writing the operator T as the sum of operators T_i where the cutoff $\phi(x, y)$ is replaced by a cutoff $\phi_i(x, y)$ with smaller support. In general each $\phi_i(x, y)$ will be supported on a rectangle on which it satisfies appropriate derivative estimates. Determining the precise set of rectangles to be used is a delicate matter. Typically, on the support of a given $\phi_i(x,y)$, the function $\frac{\partial^2 S}{\partial x \partial y}$ is within a factor of C of a fixed value on the support of $\phi_i(x,y)$. As a result, one may use "operator Van der Corput lemmas" such as those in [PS1] [PS2] to find sharp estimates for T_i. Carefully adding the estimates obtained for each T_i, using almost-orthogonality when appropriate, one obtains sharp estimates for the whole operator T. To figure out how to divide T into the T_i's, one typically uses the resolution of the zero set of $\frac{\partial^2 S}{\partial x \partial y}$ in some way to give information about the behavior of $\frac{\partial^2 S}{\partial x \partial y}$ and its x and y derivatives.

One can view the operator T to fall into three cases, depending on whether the line $y = x$ intersects the reduced Newton polygon $N_r(S)$ on one of the infinite rays, at a vertex, or in the interior of a bounded segment. The first two cases are far easier than the third, and the decomposition into rectangles is straightforward. As a result, we focus our attention on the third case. In the following arguments up to the proof of Lemma 2.3, we focus on the part of the support of $\phi(x,y)$ where $x > 0$; the portion where $x < 0$ is done in the analogous fashion.

Let $-\frac{1}{M}$ denote the slope of the bounded segment intersecting the line $y = x$. We write the Taylor expansion of $\frac{\partial^2 S}{\partial x \partial y}$ as follows, keeping in mind the series may not converge.

$$\frac{\partial^2 S}{\partial x \partial y}(x,y) \sim \sum_{a,b} S_{a,b} x^a y^b \tag{2.1}$$

Define the quantity d by

$$d = \min\{e : S_{a,b} \neq 0 \text{ for some } (a,b) \text{ with } a + Mb = e\} \tag{2.2}$$

Let the polynomial $p(y)$ be given by

$$p(y) = \sum_{a+Mb=d} S_{a,b} y^b \tag{2.3}$$

It turns out that the zeroes of the polynomial $p(y)$ give a lot of information on the location of the zero set of $\frac{\partial^2 S}{\partial x \partial y}$. To see why this is case, for a fixed c we consider the behavior of $\frac{\partial^2 S}{\partial x \partial y}$ on the curve $x \rightarrow (x, cx^M)$, which we denote by σ_c. We have

$$\frac{\partial^2 S}{\partial x \partial y}(x, cx^M) = p(c)x^d + O(x^{d+1}) \tag{2.4}$$

Hence on a curves σ_c with $p(c) = 0$, the function $\frac{\partial^2 S}{\partial x \partial y}$ decays at least as fast as $O(x^{d+1})$ as $x \rightarrow 0$, while on a curve with $p(c) \neq 0$, it decays only as fast as x^d. Hence we would expect the decomposition into rectangles to be most difficult near curves σ_c with $p(c) = 0$. However, it turns out that if k denotes the order of the zero of p at c, then near σ_c the functions $\frac{\partial^k}{\partial x^k}\frac{\partial^2 S}{\partial x \partial y}$ and $\frac{\partial^k}{\partial y^k}\frac{\partial^2 S}{\partial x \partial y}$ are nonzero and can be bounded below. This will make it possible to divide the support of $\phi(x,y)$ near σ_c into rectangles appropriate for the use of the various Van der Corput-like lemmas that have been developed for this subject. Needless to say however, doing so will be a delicate matter. The next lemma makes some of these considerations precise.

Lemma 2.1: Suppose c is a zero of $p(y)$ of order k. If ϵ and $x > 0$ are sufficiently small, then if $|y - cx^M| < \epsilon x^M$ there exists a constant $\delta > 0$ such that

$$|\frac{\partial^{k+2} S}{\partial x^{k+1}\partial y}| > \delta |y|^{\frac{d-k}{M}},\ |\frac{\partial^{k+2} S}{\partial x \partial y^{k+1}}| > \delta x^{d-kM} \tag{2.4}$$

Proof: By symmetry we only have to prove the second estimate. We do the variable change $(x,y) = (x, x^M y')$, and we obtain that

$$\frac{\partial^2 S}{\partial x \partial y}(x,y) = p(y')x^d + O(x^{d+1}) \tag{2.5}$$

One should keep in mind here that the $O(x^{d+1})$ term may depend continuously on x and y, but for fixed y has a zero of order $d+1$ at $x=0$. Taking a y derivative of $\frac{\partial^2 S}{\partial x \partial y}$ corresponds to taking a y' derivative of the right hand side of (2.5) and multiplying the result by x^{-M}. Thus we have

$$\frac{\partial^{k+2} S}{\partial x \partial^{k+1} y}(x,y) = \frac{\partial^k p}{\partial y'^k}(y')x^{d-kM} + O(x^{d+1-kM}) \tag{2.6}$$

The set $|y - cx^M| < \epsilon x^M$ corresponds to the set $|y' - c| < \epsilon$. So if we take ϵ sufficiently small that the function $\frac{\partial^k p}{\partial y'^k}(y')$ has no zero on $|y' - c| \leq \epsilon$, and if we then let x be sufficiently small, the right hand side of (2.6) is of absolute value at least δx^{d-kM} for some $\delta > 0$. Thus the lemma is proven.

In order to use the operator Van der Corput lemmas of section 3, we also need upper bounds on the functions $\frac{\partial^{l+2} S}{\partial x^{l+1}\partial y}$ and $\frac{\partial^{l+2} S}{\partial x \partial y^{l+1}}$ for $l = 0, 1$, and 2.

Lemma 2.2: Suppose $N > 0$ is fixed. For $l = 0, 1$, and 2, on the set $|y| < Nx^M$, if x is sufficiently small we have the estimate

$$|\frac{\partial^{l+2} S}{\partial x \partial y^{l+1}}(x,y)| < C_N x^{d-Ml}$$

On the set $|y| > \frac{1}{N}x^M$, if $|y|$ is sufficiently small we have the estimate

$$|\frac{\partial^{l+2} S}{\partial x^{l+1}\partial y}(x,y)| < C'_N |y|^{\frac{d-l}{M}}$$

Proof: Again, by symmetry we need only consider the y derivative estimate. Again we do the coordinate change $(x,y) = (x, x^M y')$. The lemma follows directly from (2.6); since we are on the set $|y'| < N$, the $O(x^{d+1-lM})$ term is bounded in absolute value by $A_N x^{d+1-lM}$, A_N a constant depending only on N.

We also have lower bounds for $\frac{\partial^2 S}{\partial x \partial y}$ away from the curves $y = cx^M$ where $p(c) = 0$.

Lemma 2.3: Let $\epsilon, N > 0$ be fixed. Then if $|y| < Nx^M$ and $|y - cx^M| > \epsilon x^M$ for any c satisfying $p(c) = 0$, then if $x > 0$ is sufficiently small we have the estimate

$$|\frac{\partial^2 S}{\partial x \partial y}(x,y)| > Cx^d \tag{2.7a}$$

If the condition $|y| < Nx^M$ is replaced by $|y| > \frac{1}{N}x^M$, then the conclusion is replaced by the following holding for $|y|$ sufficiently small

$$|\frac{\partial^2 S}{\partial x \partial y}(x,y)| > C|y|^{\frac{d}{M}} \tag{2.7b}$$

Proof: As before by symmetry it suffices to prove (2.7*a*). We again do the coordinate change $(x,y) = (x, x^M y')$ and examine (2.5). We are on the set of (x,y') satisfying $|y'| < N$ and $|y' - c| > \epsilon$ for any zero c of $p(y')$. Since $|y'| < N$, the remainder term is bounded by $A_N x^{d+1}$. Since $|y' - c|$ is at least ϵ for each zero c of $p(y')$, the first term is of magnitude at least δx^d for some $\delta > 0$. Thus if x is sufficiently small, (2.7*a*) holds and we are done.

We are now in a position to describe the first stages of the decomposition of T into a sum of simpler operators. Let $\psi(x)$ be a nonnegative C^∞ function that is supported on $[-1,1]$ and which is equal to 1 on $[-\frac{1}{2}, \frac{1}{2}]$. Let $\chi(x,y)$ be the characteristic function of $\{(x,y) : x \geq 0\}$. We enumerate the zeroes c_i of $p(y)$, *except* $y = 0$ if it is a zero of p. For each of the c_i, for an appropriately small $\epsilon > 0$ we define the operator T_i by

$$T_i f(x) = \int e^{i\lambda S(x,y)} \phi(x,y) \psi(\frac{y - c_i x^M}{\epsilon x^M}) \chi(x,y) f(y)\, dy \tag{2.8}$$

We also define the analogous collection of operators T_i', corresponding to $x < 0$. Next, we define operators V and V' by

$$Vf(x) = \int e^{i\lambda S(x,y)} \phi(x,y) \psi(\frac{|y|}{\epsilon |x|^M}) f(y)\, dy \tag{2.9a}$$

$$V'f(x) = \int e^{i\lambda S(x,y)} \phi(x,y) \psi(\frac{|x|^M}{\epsilon |y|}) f(y)\, dy \tag{2.9b}$$

Observe that if ϵ is sufficiently small, then the integrand in the definition of $T - \sum_i T_i - \sum_i T_i' - V - V'$ is supported on several disjoint curved triangles; for fixed x the support consists of several disjoint intervals of measure $\sim |x|^M$. We write $T - \sum_i T_i - \sum_i T_i' - V - V' = \sum_i U_i + \sum_i U_i'$, where each U_i corresponds to a curved triangle with $x > 0$ and each

U_i' corresponds to a curved triangle with $x < 0$. We thus have created a decomposition of T given by

$$T = \sum_i T_i + \sum_i T_i' + \sum_i U_i + \sum_i U_i' + V + V' \tag{2.10}$$

The next stage of the decomposition is to divide each of the terms in (2.10) dyadically in x, except V' which must be decomposed dyadically in y. To this end, we let $\psi_j(x) = 2^j\psi(2^jx) - 2^{j+1}\psi(2^{j+1}x)$, and we write $T_i = \sum T_{ij}$, $U_i = \sum U_{ij}$, etc. Each T_{ij} is obtained by multiplying the integrand in T_i by $\psi_j(x)$, with the analogous definition for the U_{ij} and so on, except the V_j' which are defined by the analogous decomposition in y instead of x. In this way we get the next level of the decomposition

$$T = \sum T_{ij} + \sum T_{ij}' + \sum U_{ij} + \sum U_{ij}' + V_j + V_j' \tag{2.11}$$

As a direct consequence of Lemmas 2.1 - 2.3 we have the following:

Lemma 2.4: If ϵ is chosen sufficiently small in the definitions of the terms of (2.11), then we have the following.

(i) Write $T_{ij}f(x)$ as

$$\int e^{i\lambda S(x,y)}\phi_{ij}(x,y)f(y)\,dy$$

Let k be the order of the zero of $p(y)$ at $y = c_i$. On the support of ϕ_{ij}, for some constant C we have

$$\left|\frac{\partial^{k+2}S}{\partial x\partial y^{k+1}}\right| > C2^{-jd+jkM},\ \left|\frac{\partial^{k+2}S}{\partial x^{k+1}\partial y}\right| > C2^{-jd+jk} \tag{2.12a}$$

For $l = 0, 1, 2$ we also have

$$\left|\frac{\partial^{l+2}S}{\partial x\partial y^{l+1}}\right| < C2^{-jd+jlM},\ \left|\frac{\partial^{l+2}S}{\partial x^{l+1}\partial y}\right| < C2^{-jd+jl} \tag{2.12b}$$

The analogous statements hold for the operators T_{ij}'.

(ii) Write $U_{ij}f(x)$ as

$$\int e^{i\lambda S(x,y)}\phi_{ij}'(x,y)f(y)\,dy$$

Then on the support of $\phi_{ij}'(x,y)$ we have

$$\left|\frac{\partial^2 S}{\partial x\partial y}\right| > C2^{-jd} \tag{2.13}$$

Again for $l = 0, 1, 2$ equation (2.12b) holds. The analogous statements hold for the operators U_{ij}'.

(iii) Write $V_jf(x)$ as

$$\int e^{i\lambda S(x,y)}\phi_j''(x,y)f(y)\,dy$$

If $p(y)$ has a zero at $x=0$, let k be the order of this zero. Otherwise, let $k=0$. Then if $x>0$, on the support of $\phi_{ij}''(x,y)$ we have

$$|\frac{\partial^{k+2}S}{\partial x\partial^{k+1}y}| > C2^{-jd+jkM} \tag{2.14}$$

And again (2.12*b*) holds. If $x<0$ the analogous statement holds with $p(y)$ replaced by the corresponding polynomial. For the operators V_j', the analogous statements hold with the roles of the x and y axes reversed.

3. Analytic Lemmas

In this section we give some analytic lemmas that are necessary for the subsequent arguments. The first is a variant of Schur's test taken verbatim from [G1], where a proof may be found.

Lemma 3.1: Suppose R is an operator defined by $Rf(x)=\int f(y)K(x,y)dy$, where $K(x,y)$ is supported in an a by b rectangle and $|K(x,y)|\leq 1$. Then $||R||_{L^2\rightarrow L^2}\leq (ab)^{\frac{1}{2}}$.

The next lemma is what is often referred to as an "operator Van der Corput lemma". This is also proved in detail in [G1]. Very similar lemmas have appeared in papers dating back to [PS1].

Lemma 3.2: Suppose $Q(x,y)$ and $\chi(x,y)$ are smooth functions on an open set U, with $\chi(x,y)$ supported on an a by b rectangle. Assume for a constant A that $\chi(x,y)$ satisfies

$$0\leq|\chi(x,y)|\leq 1,\;\; |\partial_y\chi(x,y)|<\frac{A}{b},\;\; |\partial_y^2\chi(x,y)|<\frac{A}{b^2}$$

Also assume that for some $\mu>0$, $Q(x,y)$ satisfies

$$|\frac{\partial^2 Q}{\partial x\partial y}(x,y)|\geq\mu,\;\; |\frac{\partial^3 Q}{\partial x\partial^2 y}(x,y)|<\frac{A\mu}{b},\;\; |\frac{\partial^4 Q}{\partial x\partial^3 y}(x,y)|<\frac{A\mu}{b^2}$$

For a real parameter λ, define the operator R by

$$Rf(x)=\int e^{i\lambda Q(x,y)}\chi(x,y)f(y)dy$$

Then for a constant C depending only on A, we have

$$||R||_{L^2\rightarrow L^2}<C(\lambda\mu)^{-\frac{1}{2}}$$

The third lemma is a direct consequence of Cotlar-Stein almost-orthogonality. It is also used in [R] for a similar purpose.

Lemma 3.3: Suppose $R = \sum_l R_l$ is an operator, such that each R_l can be written as $R_l f(x) = \int f(y) K_l(x,y)\, dy$, where $K_l(x,y)$ is supported on a product of intervals $I_l \times J_l$. Suppose there is a positive integer N such that for a fixed l, the number of m with $I_l \cap I_m \neq \emptyset$ is bounded by N, and the number of m with $J_l \cap J_m \neq \emptyset$ is also bounded by N. Then $||R||_{L^2 \to L^2} \leq N \sup_l ||R_l||_{L^2 \to L^2}$.

The following variant of Van der Corput's Lemma is a consequence of Lemma 3.3 in [C].

Lemma 3.4: Suppose f is a C^k function on an interval I satisfying $|f^{(k)}(x)| > a$ for all $x \in I$. Then for a constant $C_k > 0$ we have

$$\sup_{x \in I} |f(x)| \geq C_k a |I|^k$$

The next lemma contains the Bernstein-type inequalities needed for this paper. It is conceivable that it has appeared somewhere before.

Lemma 3.5: Let $f(x,y)$ be a smooth function defined in a neighborhood of the origin and (a,b) a multiindex such that

$$\frac{\partial^{a+b} f}{\partial x^a \partial y^b}(0,0) = \delta \neq 0 \tag{3.1}$$

Suppose that $\epsilon > 0$. There is a neighborhood U of the origin such that if $R \subset U$ is an r_1 by r_2 rectangle with $r_1 \leq r_2^\epsilon$, $r_2 \leq r_1^\epsilon$, then for any multiindex (α, β) we have the following.

$$\sup_{(x,y) \in R} |\frac{\partial^{\alpha+\beta} f}{\partial x^\alpha \partial y^\beta}(x,y)| < C r_1^{-\alpha} r_2^{-\beta} \sup_{(x,y) \in R} |f(x,y)| \tag{3.2}$$

Furthermore, if S is a subrectangle of R of dimensions $\frac{r_1}{2}$ by $\frac{r_2}{2}$, then we also have

$$\sup_{(x,y) \in R} |f(x,y)| < C \sup_{(x,y) \in S} |f(x,y)| \tag{3.3}$$

The constants in (3.2) and (3.3) and the neighborhood U depend on ϵ, a, b, α, β, δ, and the suprema of the absolute value of finitely many derivatives of $f(x,y)$ on U.

Proof: Let R be a rectangle satisfying the conditions of the lemma. Let $d = \frac{a+b}{\epsilon}$, and let $r = \max(r_1, r_2)$. We define the function $q_N(x,y)$ by

$$q_N(x,y) = \sum_{0 \leq\, i,j\, \leq d} |\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x,y)| r_1^i r_2^j N^{i+j} \tag{3.4}$$

Observe that by (3.1), in a sufficiently small neighborhood of the origin we have

$$|q_N(x,y)| > \frac{\delta}{2} r^d \tag{3.5}$$

Sublemma: If N is sufficiently large, then for any (x_1,y_1), $(x_2,y_2) \in R$, we have

$$|q_N(x_1,y_1) - q_N(x_2,y_2)| < \frac{1}{2} q_N(x_1,y_1) \tag{3.6}$$

In particular, the function $q_N(x,y)$ remains within a factor of 2 of any given $q_N(x_1,y_1)$ on R.

Proof of Sublemma: We have

$$|q_N(x_1,y_1) - q_N(x_2,y_2)| \leq \sum_{0\leq i,j\leq d} |\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x_2,y_2) - \sum_{0\leq i,j\leq d} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x_1,y_1)| r_1^i r_2^j N^{i+j} \tag{3.7}$$

We Taylor expand a given term in (3.7) at (x_1,y_1), obtaining an expression bounded by

$$C \sum_{\{k,l:\, d\geq k\geq i,\, d\geq l\geq j,\, (k,l)\neq(i,j)\}} |\frac{\partial^{k+l} f}{\partial x^k \partial y^k}(x_1,y_1)| r_1^k r_2^l N^{i+j} + O(r^{d+1}) N^{i+j} \tag{3.8}$$

Observe that each term in the sum in (3.8) is bounded by $\frac{1}{N} q_N(x_1,y_1)$, and that by (3.5) the remainder term is at most $CrN^{i+j} q_N(x_1,y_1)$. Hence if we make N sufficiently large, and then assume we are in a small enough neighborhood of the origin to ensure that each CrN^{i+j} is small, (3.6) follows and the sublemma is proven.

We now let $F(x,y) = f(r_1 x, r_2 y)$ and R' be the rectangle $\{(r_1 x, r_2 y) : (x,y) \in R\}$. Correspondingly, we define

$$q_N(x,y) = \sum_{0\leq i,j\leq d} |\frac{\partial^{i+j} F}{\partial x^i \partial y^j}(x,y)| N^{i+j} \tag{3.9}$$

The conclusion of the sublemma is that whenever (x_1,y_1) and (x_2,y_2) are in R' and N is sufficiently large, we have $|Q_N(x_1,y_1) - Q_N(x_2,y_2)| < \frac{1}{2} Q_N(x_1,y_1)$. Let (a',b') be a multiindex such that

$$|\frac{\partial^{a'+b'} F}{\partial x^{a'} \partial y^{b'}}(x_1,y_1) N^{a'+b'}| \geq \frac{1}{(d+1)^2} Q_N(x_1,y_1) \tag{3.10}$$

Such a selection is possible because there are $(d+1)^2$ terms in the series defining $Q_N(x,y)$. If $a' = b' = 0$, the conclusions of the lemma are easy: equation (3.2) follows directly from (3.10) and the fact that $Q_N(x,y)$ dominates the various $r_1^\alpha r_2^\beta |\frac{\partial^{\alpha+\beta} F}{\partial x^\alpha \partial y^\beta}(x,y)|$, and (3.3) will automatically hold if N is chosen appropriately large since $F(x,y)$ remains within a factor of 2 of a fixed value on R'. Hence we assume that a' or b' are nonzero. We next let v be a direction such that for some constant $c > 0$ we have

$$\left|\frac{\partial^{a'+b'}F}{\partial v^{a'+b'}}(x_1, y_1)\right| \geq c\,Q_N(x_1, y_1) \tag{3.11}$$

The existence of such a v follows from the general fact that the directional derivatives of a given order n span the whole space of operators generated by partial derivatives of order n (See [S] for details.) In analogy with the sublemma, the following holds for all $(x, y) \in R'$, if N is sufficiently large.

$$\left|\frac{\partial^{a'+b'}F}{\partial v^{a'+b'}}(x, y) - \frac{\partial^{a'+b'}F}{\partial v^{a'+b'}}(x_1, y_1)\right| < \frac{c}{2}Q_N(x_1, y_1) \tag{3.12}$$

This is proved by going back to the original coordinates and then arguing exactly as in the sublemma. Hence if N is sufficiently large, the following holds for any $(x, y) \in R'$:

$$\left|\frac{\partial^{a'+b'}F}{\partial v^{a'+b'}}(x, y)\right| > \frac{c}{2}Q_N(x_1, y_1) \tag{3.13}$$

As a result, by Lemma 3.4 and the fact that the cross sections of the square R' are between 1 and 2, we have

$$\sup_{(x,y)\in R'} |F(x, y)| > c'Q_N(x_1, y_1) \tag{3.14}$$

However, by the sublemma the following holds.

$$Q_N(x_1, y_1) > \frac{1}{2} \sup_{(x,y)\in R'} Q_N(x, y) \tag{3.15}$$

By definition of Q_N, for any $0 \leq i, j \leq d$, we have

$$Q_N(x, y) \geq N^{i+j}\left|\frac{\partial^{i+j}F}{\partial x^i \partial y^j}(x, y)\right| \tag{3.16}$$

Combining (3.14) – (3.16), we have

$$\sup_{(x,y)\in R'} |F(x, y)| > C' \sup_{(x,y)\in R'} \left|\frac{\partial^{i+j}F}{\partial x^i \partial y^j}(x, y)\right| \tag{3.17}$$

Translating (3.17) back from R''s coordinates to R's coordinates gives (3.2) for $\alpha, \beta \leq d$. If α or β is greater than d, then d was defined large enough so that we have the following, where again we use the notation $r = \max(r_1, r_2)$ and where (a, b) is as in (3.1).

$$r_1^\alpha r_2^\beta \left|\frac{\partial^{\alpha+\beta}f}{\partial x^\alpha \partial y^\beta}(x, y)\right| \leq r^{d+1}\left|\frac{\partial^{\alpha+\beta}f}{\partial x^\alpha \partial y^\beta}(x, y)\right| < Cr \times r_1^a r_2^b \tag{3.18}$$

On the other hand, by (3.5) the right-hand side of (3.18) is at most

$$C'r \times r_1^a r_2^b \left|\frac{\partial^{a+b} f}{\partial x^a \partial y^b}(x,y)\right| \tag{3.19}$$

By virtue of the fact that we already have shown (3.2) for the multiindex (a,b), (3.19) is at most $C''r$ times the supremum of f on R. Hence in view of (3.18), equation (3.2) holds for the multiindex (α,β) as well. Thus we have shown (3.2) in all cases.

Moving on to proving (3.3), recall by (3.13) that for $(x,y) \in R'$ we have $|\frac{\partial^{a'+b'}F}{\partial v^{a'+b'}}(x,y)| > \frac{c}{2}Q_N(x_1,y_1)$ for any fixed $(x_1,y_1) \in R'$. Thus by Lemma 3.4, if I is any cross-section of R' in the v direction of length $\frac{1}{2}$, for some constant $c' > 0$ we have

$$\sup_{(x,y)\in I} |F(x,y)| > c'Q_N(x_1,y_1)$$

So in particular, if S' is a subsquare of R' of diameter $\frac{1}{2}$, we have

$$\sup_{(x,y)\in S'} |F(x,y)| > c'Q_N(x_1,y_1) \tag{3.20}$$

By definition of Q_N, we have that $|F(x,y)| \leq Q_N(x,y)$, and by the sublemma we also have that $Q_N(x,y) < 2Q_N(x_1,y_1)$. Thus we have

$$\sup_{(x,y)\in R'} |F(x,y)| < 2Q_N(x_1,y_1) \tag{3.21}$$

Combining (3.20) and (3.21) gives

$$\sup_{(x,y)\in R'} |F(x,y)| < \frac{2}{c'} \sup_{(x,y)\in S'} |F(x,y)|$$

Changing back into the coordinates of R and letting $C = \frac{2}{c'}$ gives (3.3). This completes the proof of Lemma 3.5.

4. The decomposition of the T_{ij}

As in Lemma 2.4, we write $T_{ij}f(x)$ as

$$\int e^{i\lambda S(x,y)}\phi_{ij}(x,y)f(y)\,dy \tag{4.1}$$

The goal is to write $\phi_{ij}(x,y)$ as a sum of cutoff functions with smaller support in such a way as to most effectively apply the lemmas of section 3. In general the new cutoff functions will have supports that are comparable to rectangles; if we used any other shape the almost-orthogonality lemma can generally be used to reduce to the rectangular case.

Furthermore, in general one wishes to make the supports as large as possible while keeping the mixed partial $\frac{\partial^2 S}{\partial x \partial y}$ comparable to a fixed value. For suppose R_1 is an a by b rectangle which we write as $R_1 = R_2 \cup R_3$, where R_2 is an a by b' rectangle and R_3 is an a by b'' rectangle. Let $T_1 = T_2 + T_3$ be a corresponding sum of operators of the form (4.1) except with cutoffs that correspond to the associated R_i's. Assume that there is a number A such that $\frac{\partial^2 S}{\partial x \partial y}$ is between A and $2A$ on R_1. Then if Lemma 3.1 is applied directly to T_1, one gets an estimate $||T_1|| < (ab)^{\frac{1}{2}}$, while if one applies the same lemma to T_2 and T_3 and adds the results, one gets the worse estimate $||T_1|| < (ab_1)^{\frac{1}{2}} + (ab_2)^{\frac{1}{2}}$. Hence in using Lemma 3.1 it is better to make the rectangles as large as possible. The same is true for Lemma 3.2; applying it to T_1 directly gives $||T_1|| < C(|\lambda|A)^{-\frac{1}{2}}$, while adding the estimates for T_2 and T_3 gives $||T_1|| < 2C(|\lambda|A)^{-\frac{1}{2}}$. In addition, Lemma 3.3 is generally easier to use the fewer rectangles one has.

This suggests that if one is doing the most efficient decomposition, a support rectangle containing a given point (x, y) should be, roughly speaking, as large as one can get while keeping $H = \frac{\partial^2 S}{\partial x \partial y}$ within a factor of two of a fixed value. One way this would happen would be if we fixed a ratio r and specified that the dimensions $a \times ar$ of a support rectangle R were maximal such that the following holds:

$$\sup_{(x,y)\in R} \left(a\left|\frac{\partial H}{\partial x}(x,y)\right| + ar\left|\frac{\partial H}{\partial y}(x,y)\right|\right) < \frac{1}{4} \sup_{(x,y)\in R} |H(x,y)| \tag{4.2}$$

In the real-analytic case, such a decomposition into rectangles is possible with and the lemmas of section 3 can be used on the resulting decomposition of T_{ij} to get sharp estimates. The ratio r is given by the ratio corresponding to the function ϕ_{ij} in (4.1), namely $\frac{2^{-jM}}{2^{-j}} = 2^{j-jM}$, where M is as in section 2. To show that this all works one uses detailed information concerning the zero set of the mixed partial $H(x, y)$, as in [PS1] or [G1].

In the general C^∞ case things can be more complicated. Regardless of which ratio r one chooses, it can happen that a decomposition into rectangles satisfying (4.2) does not satisfy the orthogonality lemma. The reason for this is that $\sup_{(x,y)\in R} a|\frac{\partial H}{\partial x}(x, y)|$ can be far larger or far smaller than $\sup_{(x,y)\in R} ar|\frac{\partial H}{\partial y}(x, y)|$. If it is far smaller, for example, there will be too many rectangles in the x direction for a given y. Hence one needs to extend such rectangles in the x direction until one obtains an a' by ar rectangle such that

$$\sup_{(x,y)\in R} a'\left|\frac{\partial H}{\partial x}(x,y)\right| \sim \sup_{(x,y)\in R} ar\left|\frac{\partial H}{\partial y}(x,y)\right| \tag{4.3}$$

One has to do this in a way to make sure that the mixed partial $H(x, y)$ stays within a factor of two of a fixed value on the extended rectangle.

So in the general situation one does a two-stage process, which we will describe in detail shortly. In the first stage, one decomposes the support of ϕ_{ij} into squares where (4.2) is satisfied with $r = 1$. One could use other values of r for this stage, we choose

$r = 1$ for simplicity of exposition. In the second stage, one extends the squares of the first stage in the vertical or horizontal directions until (4.3) is satisfied (with $r = 1$), making sure that $H(x, y)$ doesn't vary by more than a factor of two on a given enlarged rectangle. Sometimes, multiple rectangles from the first stage will correspond to a single rectangle in the second stage. It will turn out that the resulting collection of rectangles have bounded overlap at any given (x, y), and that any two intersecting rectangles in this collection have comparable dimensions. Hence the rectangles will be suitable for defining a partition of unity and thus an appropriate decomposition of T_{ij}. The relevant properties of this decomposition for the future arguments will be given in Lemma 4.8. In section 5, the lemmas of section 3 will be used together with Lemma 4.8 on each term of the decomposition, and the results will be added, giving sharp estimates for T_{ij}.

We now describe the first stage of the decomposition. We will start by covering the support of ϕ_{ij} by boundedly many squares which are as "large as possible"; by this the following is meant: Since the support of ϕ_{ij} is comparable to a 2^{-j} by 2^{-Mj} rectangle, if $M > 1$ we should make the squares have side 2^{-Mj}, while if $M \leq 1$ we should make them have side length 2^{-j}. Thus letting $M_0 = max(M, 1)$, we define B_0 be a set of boundedly many $2^{-M_0 j} \times 2^{-M_0 j}$ squares that covers the support of ϕ_{ij}.

At the end of the first stage of the decomposition, we will have a collection B of squares that we will obtain by dyadically subdividing the squares in B_0 using a stopping time argument; the squares in B will not overlap and their union will be the points in the union of the squares in B_0 for which $H \neq 0$. It should be pointed out that other stopping-time arguments have also been used in this subject such as in [R] and [PS3].

The stopping time procedure begins as follows. Let S be any square in B_0. Suppose the following holds.

$$2^{-M_0 j} \sup_{(x,y)\in S} |\nabla H(x, y)| < \frac{1}{4} \sup_{(x,y)\in S} |H(x, y)| \tag{4.4}$$

In other words, assume (4.2) is already satisfied. In this case we add S to the collection B. Next, for each square S' for which (4.4) does not hold, we dyadically divide S' into four subsquares. We denote the collection of all the resulting subsquares of the various S' by B_1. For a given S'' in B_1, we add S'' to B if and only if we have

$$2^{-M_0 j-1} \sup_{(x,y)\in S''} |\nabla H(x, y)| < \frac{1}{4} \sup_{(x,y)\in S''} |H(x, y)| \tag{4.5}$$

The members of B_1 that don't satisfy (4.5) are then divided into 4 subsquares, each of which is added to the set B_2 of squares, and the above procedure is repeated. Inductively, in the $k + 1$th step of this argument, a square S in B_k is added to B if and only if we have

$$2^{-M_0 j-k} \sup_{(x,y)\in S} |\nabla H(x, y)| < \frac{1}{4} \sup_{(x,y)\in S} |H(x, y)| \tag{4.6}$$

Performing this stopping time procedure for all positive integers k results in a set B of disjoint squares whose union contains the portion of the support of ϕ_{ij} where $H \neq 0$. A square S comes from the $k+1$th step of the stopping time argument if (4.6) holds for S, but (4.6) does not hold for any of the larger dyadic supersquares containing S.

A way of saying that the squares in B are as large as possible for $H(x,y)$ to stay within a factor of 2 of a fixed value on each square is the following lemma.

Lemma 4.1: There exists a uniform constant δ_1 such that if S is one of the squares in B, coming from B_k in the $k+1$th stage of the stopping time procedure described above, then we have

$$\delta_1 \sup_{(x,y)\in S} |H(x,y)| < 2^{-M_0 j - k} \sup_{(x,y)\in S} |\nabla H(x,y)| < \frac{1}{4} \sup_{(x,y)\in S} |H(x,y)| \tag{4.7}$$

Furthermore, $H(x,y)$ stays within a factor of 2 of a fixed value on S.

Proof: The final statement of Lemma 4.1 follows from the right-hand inequality of (4.7), so it suffices to show (4.7). The stopping time procedure was defined so that the right-hand inequality holds, so it suffices to show the left-hand inequality. We fist consider the case where S comes from a B_k for some $k > 0$. In this case, S is contained in a square S' in B_{k-1}, and since S' was not selected in the kth step of the stopping time procedure we must have

$$2^{-M_0 j - k + 1} \sup_{(x,y)\in S'} |\nabla H(x,y)| \geq \frac{1}{4} \sup_{(x,y)\in S'} |H(x,y)| \tag{4.8}$$

By (3.3) of Lemma 3.5, for some $\delta_0 > 0$ we therefore have

$$2^{-M_0 j - k + 1} \sup_{(x,y)\in S} |\nabla H(x,y)| \geq \delta_0 \sup_{(x,y)\in S'} |H(x,y)| \tag{4.9}$$

$$\geq \delta_0 \sup_{(x,y)\in S} |H(x,y)| \tag{4.10}$$

Hence the left hand inequality of (3.7) holds for $\delta_1 = \frac{\delta_0}{2}$. This completes the proof of Lemma 4.1 for squares coming from B_k for $k > 0$.

So suppose S is a square in B_0. Then by Lemma 2.2 we have

$$\sup_{(x,y)\in S} |H(x,y)| < C2^{-dj} \tag{4.11}$$

(Recall that d was defined by the Newton polygon of H having an edge with equation $x + My = d$.) Since S intersects the curves $y = cx^M$ for a range $c_1 < c < c_2$ of values of c, by applying Lemma 2.3 to $\frac{\partial H}{\partial x}$ and $\frac{\partial H}{\partial y}$ respectively we have

$$\sup_{(x,y)\in S} |\frac{\partial H}{\partial x}(x,y)| > C' 2^{-dj+j} \tag{4.12a}$$

$$\sup_{(x,y)\in S} |\frac{\partial H}{\partial y}(x,y)| > C' 2^{-dj+Mj} \tag{4.12b}$$

Recalling that $M_0 = max(M,1)$, we thus have

$$\sup_{(x,y)\in S} |\nabla H(x,y)| > C' 2^{-dj+M_0 j} \tag{4.13}$$

Combining (4.11) and (4.13) gives

$$2^{-M_0 j} \sup_{(x,y)\in S} |\nabla H(x,y)| > \frac{C'}{C} \sup_{(x,y)\in S} |H(x,y)| \tag{4.14}$$

So long as $\delta_1 \leq C'/C$ the left-hand inequality of (4.7) holds and we are done.

We now move to the second stage of the stopping-time procedure. We will define a collection D of rectangles. Each member of D will either be a member of B or an elongation in the horizontal or vertical direction of a member of B. We create D as follows. By Lemma 4.1, for each S in B of side length $2^{-M_0 j-k}$ at least one of the following two inequalities must hold:

$$2^{-M_0 j-k} \sup_{(x,y)\in S} |\frac{\partial H}{\partial x}(x,y)| > \frac{\delta_1}{2} \sup_{(x,y)\in S} |H(x,y)| \tag{4.15a}$$

$$2^{-M_0 j-k} \sup_{(x,y)\in S} |\frac{\partial H}{\partial y}(x,y)| > \frac{\delta_1}{2} \sup_{(x,y)\in S} |H(x,y)| \tag{4.15b}$$

In the proof of Lemma 4.6 we will define a constant $\delta_2 < \frac{\delta_1}{2}$. By (4.15$a$) and (4.15$b$) at least one of the following holds.

$$2^{-M_0 j-k} \sup_{(x,y)\in S} |\frac{\partial H}{\partial x}(x,y)| > \delta_2 \sup_{(x,y)\in S} |H(x,y)| \tag{4.16a}$$

$$2^{-M_0 j-k} \sup_{(x,y)\in S} |\frac{\partial H}{\partial y}(x,y)| > \delta_2 \sup_{(x,y)\in S} |H(x,y)| \tag{4.16b}$$

If both (4.16a) and (4.16b) hold, then we add S unchanged to D. Suppose now that one of (4.16a) and (4.16b) holds and not the other. Say (4.16b) holds, the case where (4.16a) holds is dealt with in the analogous fashion. Write S as a product of intervals as $I \times J$. Then we let I' be the largest dyadic interval containing I of side length at most 2^{-j} such that we have

$$|I'| \sup_{(x,y)\in I'\times J} |\frac{\partial H}{\partial x}(x,y)| \leq \delta_2 \sup_{(x,y)\in I'\times J} |H(x,y)| \tag{4.17}$$

We include the rectangle $I' \times J$ to the collection D. We do this for all squares S in B such that (4.16b) holds but not (4.16a), and we then add to D all the analogous vertical elongations of squares in B for which (4.16a) holds but not (4.16b). Once this is done we have defined D.

The next lemma ensures that $H(x,y)$ stays within a factor of 8 of a fixed value on each of the rectangles in D.

Lemma 4.2: Suppose that R is a rectangle in D, and let $A_R = \sup_{(x,y)\in R} |H(x,y)|$. Then for each $(x,y) \in R$ we have

$$\frac{A_R}{8} \leq |H(x,y)| \leq A_R \tag{4.18}$$

Proof: Assume R was obtained by extending in the horizontal direction; the case where R was obtained by extending vertical direction is done similarly and the case where R was not extended at all follows from Lemma 4.1. Let (x_0, y_0) denote a point in R where $|H(x_0, y_0)| = A_R$, and let (x, y) denote any other point in R. We may let x_1 be such that (x_1, y_0) is in the square S of B which was extended to create R. By (4.17) and the mean-value theorem, we have

$$|H(x_1, y_0) - H(x_0, y_0)| \leq \delta_2 A_R$$

Since $|H(x_0, y_0)| = A_R$, we thus have

$$|H(x_1, y_0)| > \frac{A_R}{2}$$

Furthermore, by Lemma 4.1, on this square S the function $H(x,y)$ stays within a factor of two of a fixed value. Since (x_1, y) and (x_1, y_0) are both in this square, we therefore have

$$|H(x_1, y)| > \frac{A_R}{4}$$

Applying (4.17) with the mean-value theorem again we have

$$|H(x,y) - H(x_1, y)| \leq \delta_2 A_R$$

The last two equations imply that $|H(x,y)| > \frac{A_R}{8}$ and we are done with the proof.

The next lemma allows us to apply finite-type "doubling lemmas", in particular Lemma 3.5, to the rectangles in D:

Lemma 4.3: There exists an $\epsilon > 0$ such that for each rectangle R in D of dimensions $r_1 \times r_2$ we have

$$r_2 < Cr_1^{\epsilon},\ r_1 < Cr_2^{\epsilon} \tag{4.19}$$

Proof: Without loss of generality we assume R was obtained by extending in the x direction, so that $r_2 < r_1$. Thus we must find an $\epsilon > 0$ such that for some C we have

$$r_1 < Cr_2^{\epsilon} \tag{4.20}$$

Like above we let $A_R = \sup_{(x,y)\in R} |H(x,y)|$. By equation (2.12$a$) of Lemma 2.4(i), there is a $k > 0$ such that the following holds on R:

$$|\frac{\partial^k H}{\partial x^k}| > C2^{-dj+kj}$$

As a result, by Lemma 3.4 we have

$$2^{-dj+kj} r_1^k \leq CA_R \tag{4.21}$$

Since we are assuming R was created by horizontally extending a square S in B, by (4.15b) we have

$$\frac{\delta_1}{2} \sup_{(x,y)\in S} |H(x,y)| < r_2 \sup_{(x,y)\in S} |\frac{\partial H}{\partial y}(x,y)|$$

By Lemma 4.2, we have

$$\frac{\delta_1 A_R}{16} \leq \frac{\delta_1}{2} \sup_{(x,y)\in S} |H(x,y)|$$

Since $S \subset R$, the last two equations imply

$$\frac{\delta_1 A_R}{16} < r_2 \sup_{(x,y)\in R} |\frac{\partial H}{\partial y}(x,y)|$$

By (2.12b) of Lemma 2.4(i) we have

$$\sup_{(x,y)\in R} |\frac{\partial H}{\partial y}(x,y)| < C2^{-dj+Mj}$$

Combining the last two equations, we have

$$A_R < C2^{-dj+Mj} r_2 \tag{4.22}$$

The idea now will be to combine (4.21) and (4.22) to get the result we seek. We raise both sides of (4.21) to a power $a \geq 1$ we will specify shortly, obtaining

$$2^{-adj+akj} r_1^{ak} \leq CA_R^{a-1} A_R \tag{4.23}$$

We use the inequality $A_R < C2^{-dj}$ from (2.12b) of Lemma 2.4(i) on the A_R^{a-1} factor, obtaining

$$2^{-adj+akj} r_1^{ak} \leq C2^{-adj+dj} A_R \tag{4.24}$$

Note that we require $a \geq 1$ here. Combining (4.24) and (4.22) gives

$$2^{-adj+akj} r_1^{ak} \leq C2^{-adj+Mj} r_2$$

It is preferable to write this as

$$r_1^{ak} \leq C2^{-akj+Mj} r_2$$

As a result, so long as a is at least $\frac{M}{k}$ we have

$$r_1^{ak} \leq Cr_2$$

So for example we can set $a = \max(1, \frac{M}{k})$. In this case (4.20) holds for $\epsilon = \frac{1}{ak}$ and we are done.

The following lemma says that the rectangles in D have been extended as much as possible; it makes the heuristics of (4.3) precise:

Lemma 4.4: Suppose R is one of the rectangles in D, of dimensions r_1 by r_2. Then for some constants δ and C the following hold on R:

$$\delta \sup_{(x,y)\in R} |H(x,y)| < r_1 \sup_{(x,y)\in R} |\frac{\partial H}{\partial x}(x,y)| < C \sup_{(x,y)\in R} |H(x,y)| \tag{4.25a}$$

$$\delta \sup_{(x,y)\in R} |H(x,y)| < r_2 \sup_{(x,y)\in R} |\frac{\partial H}{\partial y}(x,y)| < C \sup_{(x,y)\in R} |H(x,y)| \tag{4.25b}$$

Proof: First, by the Bernstein-type inequalities of Lemma 3.5 the right-hand sides of (4.25a) and (4.25b) hold. Thus we focus our attention on the left-hand sides. If R is a square, then R is one of the squares in B and by Lemma 4.1 R must satisfy (4.25a) and (4.25b) with $\delta = \delta_1$ and $C = \frac{1}{4}$. Hence we assume that R is not a square. The cases where $r_2 < r_1$ and $r_2 > r_1$ are done the same way, so we assume that $r_2 < r_1$. In this case, R was created by extending a square in B in the x-direction, and we have that (4.15b) holds. Therefore the left side of (4.25b) holds with $\delta = \frac{\delta_1}{2}$.

It remains the prove the left side of (4.25a). By the definition of the stopping-time procedure, R is a product of intervals $I' \times J$, where either I' is of length 2^{-j} or I' is the maximal dyadic interval of length less than 2^{-j} where (4.17) holds. First consider the case where $|I'| = 2^{-j}$. In this situation, R intersects the curve $y = cx^M$ for a range $c_1 < c < c_2$ of c. As a result, by Lemma 2.3 applied to $\frac{\partial H}{\partial x}$ we have

$$\sup_{(x,y)\in R} |\frac{\partial H}{\partial x}(x,y)| > C_1 2^{-dj+j} \tag{4.26}$$

On the other hand, by Lemma 2.2, we also have

$$\sup_{(x,y)\in R} |H(x,y)| < C_2 2^{-dj} \tag{4.27}$$

Combining (4.26) and (4.27) gives the left side of (4.25a), where we take $\delta = \frac{C_1}{C_2}$.

Hence it remains to prove the left hand side of (4.25a) when I' is the maximal dyadic interval of length less than 2^{-j} for which (4.17) holds. Let I'' be the dyadic interval containing I' with twice its length. By assumption, (4.17) does not hold when I' is replaced with I''. Since $|I''| = 2r_1$, this means that

$$2r_1 \sup_{(x,y)\in I''\times J} |\frac{\partial H}{\partial x}(x,y)| \geq \delta_2 \sup_{(x,y)\in I''\times J} |H(x,y)|$$

By Lemma 3.5, this implies that

$$2r_1 \sup_{(x,y)\in I'\times J} |\frac{\partial H}{\partial x}(x,y)| \geq C\delta_2 \sup_{(x,y)\in I''\times J} |H(x,y)|$$

$$\geq C\delta_2 \sup_{(x,y)\in I'\times J} |H(x,y)|$$

Hence the left side of $(4.25a)$ holds with $\delta = \frac{C\delta_2}{2}$ and we are done.

Since we are using the rectangles in D for a partition of unity, it is necessary to also understand the behavior of $H(x,y)$ on dilations of the rectangles in D. For example, the following lemma is useful.

Lemma 4.5: Suppose R is a rectangle in D, of dimensions $r_1 \times r_2$. Let R_ϵ denote the rectangle obtained by dilating R by a factor of $1+\epsilon$, keeping the same center. Like before let $A_R = \sup_{(x,y)\in R}|H(x,y)|$. Then if ϵ is sufficiently small, for each (x,y) in R_ϵ we have

$$\frac{A_R}{16} < |H(x,y)| < 2A_R \tag{4.26}$$

Proof: Let (x,y) be any point in R_ϵ. Let (x',y') be the point in R nearest to (x,y). Then we have

$$|x'-x| \leq \epsilon r_1,\ |y'-y| \leq \epsilon r_2 \tag{4.27}$$

By the mean value theorem, we have

$$|H(x,y) - H(x',y')| < |x'-x| \sup_{R_\epsilon} \frac{\partial H}{\partial x} + |y'-y| \sup_{R_\epsilon} \frac{\partial H}{\partial y}$$

$$\leq \epsilon r_1 \sup_{R_\epsilon} \frac{\partial H}{\partial x} + \epsilon r_2 \sup_{R_\epsilon} \frac{\partial H}{\partial y}$$

By the Bernstein-type inequalities of Theorem 3.5, there is a constant C such that

$$\sup_{R_\epsilon} \frac{\partial H}{\partial x} < C \sup_{R} \frac{\partial H}{\partial x}, \quad \sup_{R_\epsilon} \frac{\partial H}{\partial y} < C \sup_{R} \frac{\partial H}{\partial y}$$

As a result,

$$|H(x,y) - H(x',y')| < C\epsilon r_1 \sup_{R} \frac{\partial H}{\partial x} + C\epsilon r_2 \sup_{R} \frac{\partial H}{\partial y} \tag{4.28}$$

Again using the Bernstein-type inequalities of Lemma 3.5 we thus have

$$|H(x,y) - H(x',y')| < C'\epsilon \sup_R |H(x,y)| = C'\epsilon A_R \tag{4.29}$$

Since (x',y') is in R, by Lemma 4.2 $|H(x',y')|$ is between $\frac{A_R}{8}$ and A_R. As a result, if ϵ is such that $C'\epsilon < \frac{A_R}{16}$, then (4.29) implies that (4.26) holds and we are done.

The next lemma will help us prove that the rectangles in D have bounded overlap, and that intersecting rectangles in D have comparable dimensions.

Lemma 4.6: Let $R = I \times J$ be one of the rectangles of D, of dimensions r_1 by r_2, and let A_R be as above. Let R_ϵ be as in Lemma 4.5, and let I_ϵ and J_ϵ be intervals such that $R_\epsilon = I_\epsilon \times J_\epsilon$. If the δ_2 used in defining D was sufficiently small, the following hold, where δ_1 is as in Lemma 4.1:

a) If $r_2 < r_1$, then there is a $y_0 \in J$ such that for each $x \in I_\epsilon$ we have the following:

$$r_2|\frac{\partial H}{\partial y}(x,y_0)| > \frac{\delta_1}{4}A_R \tag{4.30}$$

b) If $r_2 > r_1$, then there is an $x_0 \in I$ such that for each $y \in J_\epsilon$ we have the following:

$$r_1|\frac{\partial H}{\partial x}(x_0,y)| > \frac{\delta_1}{4}A_R \tag{4.31}$$

Proof: The proofs of a) and b) are identical, so we consider a) only. Since $r_2 < r_1$, equation (4.15b) had to have held, so we may let (x_0,y_0) in R be such that

$$r_2|\frac{\partial H}{\partial y}(x_0,y_0)| > \frac{\delta_1}{2}A_R \tag{4.32}$$

By the mean-value theorem, for any x in I_ϵ we have

$$r_2|\frac{\partial H}{\partial y}(x,y_0) - \frac{\partial H}{\partial y}(x_0,y_0)| \leq (1+\epsilon)r_1r_2 \sup_{R_\epsilon}|\frac{\partial^2 H}{\partial x \partial y}| \tag{4.33}$$

However, by equation (3.2) of Lemma 3.5 we have

$$r_1r_2 \sup_{R_\epsilon}|\frac{\partial^2 H}{\partial x \partial y}| < Cr_1 \sup_{R_\epsilon}|\frac{\partial H}{\partial x}| \tag{4.34}$$

Furthermore, by equation (4.17) we have

$$Cr_1 \sup_{R_\epsilon}|\frac{\partial H}{\partial x}| < C\delta_2 \sup_R |H| = C\delta_2 A_R \tag{4.35}$$

As a result, so long as δ_2 was chosen such that $2C\delta_2 < \frac{\delta_1}{4}$, we have that (4.32) and the string of inequalities (4.33) – (4.35) imply that (4.30) holds and we are done.

Observe that each rectangle R in D is of the form $[a2^{-k}, (a+1)2^{-k}] \times [b2^{-l}, (b+1)2^{-l}]$ for some integers a, b, k, and l. As a result, for a given pair (k, l), there are at most four rectangles R_ϵ containing any given point. Consquently, if we can show that for each point x, there exists a (k_0, l_0) such that $|k - k_0|$, $|l - l_0| < C$ for any $2^{-k} \times 2^{-l}$ rectangle R for which $x \in R_\epsilon$, then we will have that the rectangles R_ϵ have bounded overlap and have comparable dimensions. This is given by the following lemma.

Lemma 4.7: There are constants δ, C such that the following holds. Suppose R and Q are rectangles in D for which the point x is in both R_ϵ and Q_ϵ, of dimensions $(1+\epsilon)r_1 \times (1+\epsilon)r_2$ and $(1+\epsilon)q_1 \times (1+\epsilon)q_2$ respectively. Then

$$\delta < \frac{r_1}{q_1}, \ \frac{r_2}{q_2} < C \tag{4.36}$$

Proof: We break into two cases. Case 1 is where $r_1 \leq q_1$ and $r_2 \leq q_2$, or $r_1 \geq q_1$ and $r_2 \geq q_2$. Case 2 is where $r_1 \geq q_1$ and $r_2 \leq q_2$, or $r_1 \leq q_1$ and $r_2 \geq q_2$.

We consider Case 1 first. Switching names if necessary, we assume that $r_1 \leq q_1$ and $r_2 \leq q_2$. By Lemma 4.4, for some $\delta_0 > 0$ we have

$$\delta_0 A_R < r_1 \sup_R \left|\frac{\partial H}{\partial x}\right|, \ r_2 \sup_R \left|\frac{\partial H}{\partial y}\right| \tag{4.37}$$

Since $r_1 \leq q_1$ and $r_2 \leq q_2$ and $R_\epsilon \cap Q_\epsilon$ is nonempty, the rectangle R must be contained in a C-fold dilate of Q. (In fact C can be taken to be 3 here.) As a result, by Lemma 3.5 we have

$$r_1 \sup_R \left|\frac{\partial H}{\partial x}\right| < C_0 r_1 \sup_Q \left|\frac{\partial H}{\partial x}\right| \tag{4.38a}$$

$$r_2 \sup_R \left|\frac{\partial H}{\partial y}\right| < C_0 r_2 \sup_Q \left|\frac{\partial H}{\partial y}\right| \tag{4.38b}$$

By Lemma 4.4, we have

$$r_1 \sup_Q \left|\frac{\partial H}{\partial x}\right| < C_1 \frac{r_1}{q_1} A_Q \tag{4.39a}$$

$$r_2 \sup_Q \left|\frac{\partial H}{\partial y}\right| < C_1 \frac{r_2}{q_2} A_Q \tag{4.39b}$$

By Lemma 4.5, $|H(x, y)|$ stays within factor of 32 of a fixed value on each of R_ϵ and Q_ϵ. Since R_ϵ and Q_ϵ intersect, this means that A_Q and A_R are within a factor of 1024 of each other. So we have

$$C_1 \frac{r_1}{q_1} A_Q < C_2 \frac{r_1}{q_1} A_R \tag{4.40a}$$

$$C_1\frac{r_2}{q_2}A_Q < C_2\frac{r_2}{q_2}A_R \tag{4.40b}$$

Combining the string of inequalities (4.37) – (4.40) gives

$$\frac{r_1}{q_1}, \frac{r_2}{q_2} > \frac{\delta_0}{C_2} \tag{4.41}$$

Since we are working in case 1, we also have $\frac{r_1}{q_1}, \frac{r_2}{q_2} \leq 1$, and (4.36) follows. Thus we are done with case 1.

We now move to case 2. Assume that r_2 is the smallest of r_1, r_2, q_1, and q_2; the other three cases are done exactly the same way. Since we are in case 2, the minimality of r_2 implies that $r_1 > r_2$. We write $R = I \times J$ and $R_\epsilon = I_\epsilon \times J_\epsilon$ as before. By Lemma 4.6, we may let $y_0 \in J$ be such that for each x in I_ϵ we have

$$\frac{\delta_1}{4}A_R < r_2\left|\frac{\partial H}{\partial y}(x, y_0)\right| \tag{4.42}$$

Since we are in case 2 and $r_1 > r_2$, we have that $q_2 \geq q_1$. Because of this and the fact that $R_\epsilon \cap Q_\epsilon$ is nonempty, the dilation of Q by a factor of 3, which we denote by Q', will contain a vertical cross section of R. In particular, Q' contains a point (x, y_0) in $I_\epsilon \times \{y_0\}$. By Lemma 3.5 applied to Q', we have that

$$(3q_2)\left|\frac{\partial H}{\partial y}(x, y_0)\right| < C_3 \sup_{Q'} |H|$$

We rewrite this as

$$r_2\left|\frac{\partial H}{\partial y}(x, y_0)\right| < 3C_3\frac{r_2}{q_2}\sup_{Q'}|H| \tag{4.43}$$

Using Lemma 3.5 again, this is bounded by

$$C_4\frac{r_2}{q_2}\sup_{Q}|H| = C_4\frac{r_2}{q_2}A_Q \tag{4.44}$$

As in case 1, since R_ϵ and Q_ϵ intersect, by Lemma 4.5 A_R and A_Q are within a factor of 1024 of each other. Hence combining (4.42) – (4.44) gives

$$\frac{r_2}{q_2} > C_5 \tag{4.45}$$

Since we are assuming r_2 is the smallest of r_1, r_2, q_1, and q_2, (4.45) implies

$$1 \geq \frac{r_2}{q_2} > C_5 \tag{4.46}$$

This is half of (4.36). Next, note that depending on whether $r_1 \leq q_1$ or vice-versa, (4.46) implies that either $r_1 \leq q_1$ and $r_2 \leq q_2$, or $q_1 \leq r_1$ and $q_2 \leq \frac{1}{C_5}r_2$. The first possibility is just case 1, while the second possibility is case 1 modulo the $\frac{1}{C_5}$ factor. However, the argument of case 1 covers this situation; all one needed there was that one of R and Q is contained in the C-fold dilation of the other for a constant C. This completes the proof of Lemma 4.7.

We are now in a position to decompose the T_{ij}. Let $\rho(x)$ be a nonnegative function that is equal to 1 on $[-\frac{1}{2}, \frac{1}{2}]$ which is supported on $[-\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2} + \frac{\epsilon}{2}]$ and which is nonzero on the interior of its support. Here ϵ is as in Lemma 4.7. Suppose R is a rectangle in D with dimensions r_1 by r_2. Denote the center of R by (x_0, y_0). Let $\psi_R(x,y)$ be defined by

$$\psi_R(x,y) = \rho(\frac{x - x_0}{r_1})\rho(\frac{y - y_0}{r_2})$$

Let $\xi_R(x,y)$ be defined by

$$\xi_R(x,y) = \frac{\psi_R(x,y)}{\sum_{Q \in D} \psi_Q(x,y)}$$

By Lemma 4.7, the functions $\xi_R(x,y)$ form a partition of unity such that each $\xi_R(x,y)$ satisfies derivative estimates appropriate to the rectangle R. Correspondingly, we let $\{\phi_{ijk}(x,y)\}$ be the collection of functions $\{\phi_{ij}(x,y)\xi_R(x,y) : R \in D\}$. Here $\phi_{ij}(x,y)$ is as in (4.1). We define the operators T_{ijk} by

$$T_{ijk}f(x) = \int e^{i\lambda S(x,y)}\phi_{ijk}(x,y)f(y)\,dy \tag{4.47}$$

Observe that $\sum_k T_{ijk} = T_{ij}$. Also observe that ϕ_{ijk} is supported in the rectangle in D from which it was derived.

The next lemma summarizes what we will need in section 5 from this section.

Lemma 4.8: Suppose $\phi_{ijk} = \phi_{ij}(x,y)\xi_R(x,y)$ for some r_1 by r_2 rectangle R. For $\alpha = 0, 1, 2$ we have

$$|\frac{\partial^\alpha \phi_{ijk}}{\partial x^\alpha}| < C r_1^{-\alpha} \tag{4.48a}$$

$$|\frac{\partial^\alpha \phi_{ijk}}{\partial y^\alpha}| < C r_2^{-\alpha} \tag{4.48b}$$

In addition, there is a constant M_{ijk} such that on R we have

$$\frac{M_{ijk}}{32} < |H(x,y)| < M_{ijk} \tag{4.49}$$

Also, there exist $\delta, C > 0$ such that we have

$$\delta M_{ijk} \;<\; r_1 \sup_{(x,y)\in R} |\frac{\partial H}{\partial x}(x,y)|,\; r_2 \sup_{(x,y)\in R} |\frac{\partial H}{\partial y}(x,y)| \tag{4.50}$$

$$r_1 \sup_{(x,y)\in R} |\frac{\partial H}{\partial x}(x,y)|,\ r_1^2 \sup_{(x,y)\in R} |\frac{\partial^2 H}{\partial x^2}(x,y)| \ < \ CM_{ijk} \tag{4.51a}$$

$$r_2 \sup_{(x,y)\in R} |\frac{\partial H}{\partial y}(x,y)|,\ r_2^2 \sup_{(x,y)\in R} |\frac{\partial^2 H}{\partial y^2}(x,y)| \ < \ CM_{ijk} \tag{4.51b}$$

Proof: (4.48a) and (4.48b) follow from examining the definiton of the ϕ_{ijk} and Lemma 4.7. Equation (4.49) follows from Lemma 4.5. Equation (4.50) follows from Lemma 4.4, and (4.51a) $-$ (4.51b) are a consequence of the Bernstein-type inequalties of Lemma 3.5, or alternatively Lemma 4.4.

5. Estimating $||T_{ij}||$; proof of sharp estimates for T_i

We now fix i and j and we will bound $||T_{ij}||_{L^2\rightarrow L^2}$ using the decomposition $T_{ij} = \sum_k T_{ijk}$ of (4.47) and Lemmas 3.1-3.5. We first observe from Lemma 2.4 (i) that $H(x,y)$ satisfies

$$|H(x,y)| < C2^{-jd} \tag{5.1}$$

Correspondingly, we divide the operators $\{T_{ijk}\}$ into the following classes D_l, where A_{ijk} denotes the rectangle called R in Lemma 4.8.

$$D_l = \{T_{ijk} : 2^{l-dj} < \sup_{(x,y)\in A_{ijk}} |H(x,y)| \leq 2^{l+1-dj}\} \tag{5.2}$$

Our constructions were such that any A_{ijk} has dimensions $(1+\epsilon)2^{-m-j} \times (1+\epsilon)2^{-n-Mj}$ for some nonnegative integers m, n. Thus it is reasonable to subdivide the classes D_l into subclasses D_{lmn}, where

$$D_{lmn} = \{T_{ijk} \in D_l : A_{ijk} \text{ has dimensions } (1+\epsilon)2^{-m-j} \times (1+\epsilon)2^{-n-Mj}\} \tag{5.3}$$

Lemma 5.1: Suppose T_{ijk} is an element of D_l whose corresponding A_{ijk} has dimensions $a \times b$. Then there exists an integer $p > 0$ depending only on i such that for some constants C_1 and C_2 we have

$$C_1 2^{-j-l} < a < C_2 2^{-j-\frac{l}{p}},\ C_1 2^{-Mj-l} < b < C_2 2^{-Mj-\frac{l}{p}} \tag{5.4}$$

As a result, for a given l, there are at most $C(l+1)^2$ values of (m,n) for which D_{lmn} is nonempty.

Proof: We consider the inequalities for b. By Lemma 2.4 i), for all $(x,y) \in A_{ijk}$ we have

$$|\frac{\partial H}{\partial y}(x,y)| < C2^{-jd+jM} \tag{5.5}$$

By Lemma 4.8 (or Lemma 4.4), there is a $(x',y') \in A_{ijk}$ for which we have

$$|\frac{\partial H}{\partial y}(x',y')| > \delta b^{-1} 2^{-l-jd} \tag{5.6}$$

Combining (5.5) and (5.6), we have that for some C_1

$$b > C_1 2^{-l-jM} \tag{5.7}$$

On the other hand, if p is the number denoted by k in Lemma 2.4 (i), on A_{ijk} we have

$$|\frac{\partial^p H}{\partial y^p}| > C2^{-jd+jMp} \tag{5.8}$$

Thus by applying Lemma 3.4 in the y direction, we have

$$\sup_{A_{ijk}} |H(x,y)| > C' 2^{-jd+jMp}\ b^p \tag{5.9}$$

By (5.2) we also have

$$\sup_{A_{ijk}} |H(x,y)| < C'' 2^{-l-jd} \tag{5.10}$$

Combining (5.9) and (5.10) we conclude that for some constant C_2 we have

$$b < C_2 2^{-jM-\frac{l}{p}}$$

Thus we have proven the inequalities for b. The inequalities for a are proved in exactly the same way, switching the roles of the x and y axes and using (2.12a) and (2.12b) in the x direction.

Lemma 5.2: For a fixed (l,m,n), the number of operators $T_{ijk} \in D_{lmn}$ whose corresponding A_{ijk} intersects any given vertical line is uniformly bounded. The number of $T_{ijk} \in D_{lmn}$ whose corresponding A_{ijk} intersects a given horizontal line is also uniformly bounded. Consequently, by Lemma 3.3 there exists a constant C such that

$$||\sum_{T_{ijk}\in D_{lmn}} T_{ijk}||_{L^2\to L^2} < C \sup_{T_{ijk}\in D_{lmn}} ||T_{ijk}||_{L^2\to L^2} \tag{5.11}$$

Proof: We prove the statement for vertical lines, the horizontal case being entirely analogous. We fix an interval I of length $(1+\epsilon)2^{-m-j}$, and the goal is to show that the number of A_{ijk} whose x-projection is I is uniformly bounded. Denote this number by N.

For each such A_{ijk}, by Lemma 4.8 there is a point $(x',y') \in A_{ijk}$ such that $|\frac{\partial H}{\partial y}(x',y')| > \delta 2^{n+Mj-l-dj}$. In fact, it is also true that for a sufficiently small $\delta' > 0$, whenever $|x'-x| < \delta' 2^{-m-j}$ and $|y'-y| < \delta' 2^{-n-Mj}$, we still have

$$|\frac{\partial H}{\partial y}(x,y)| > \frac{\delta}{2} 2^{n+Mj-l-dj} \tag{5.12}$$

This follows from the bounds on $|\nabla \frac{\partial H}{\partial y}|$ obtained from the Bernstein inequalities of lemma 3.5. Hence if we let B be the set

$$B = \{(x,y) : x \in I,\ |H(x,y)| > 2^{-l-jd},\ |\frac{\partial H}{\partial y}(x,y)| > \frac{\delta}{2} 2^{n+Mj-l-dj}\} \tag{5.13}$$

then B contains all N of these subrectangles of dimensions $\delta' 2^{-m-j} \times \delta' 2^{-n-Mj}$. As a result, we have

$$|B| \geq N 2^{-m-j-n-Mj} \delta'^2 \tag{5.14}$$

On the other hand, since $H(x,y)$ has nonvanishing pth derivative in the y direction for some $p \geq 1$, the y-cross section of B for a fixed x' consists of boundedly many intervals. By (5.13) each such interval has measure at most $2^{-l-jd} / \frac{\delta}{2} 2^{n+Mj-l-dj} = C 2^{-n-Mj}$. Hence we have the following upper bound on $|B|$:

$$|B| < C 2^{-m-j-n-Mj} \tag{5.15}$$

Combining (5.14) and (5.15) gives uniform bounds on N, and we are done.

Combining Lemmas 5.1 and 5.2, we have

Corollary 5.3: There exist constants c_1 and c_2 such that

$$||T_{ij}|| < C \sum_l \sum_{\{(m,n): c_1 + \frac{l}{p} < \ m,n \ < c_2 + l\}} \sup_{T_{ijk} \in D_{lmn}} ||T_{ijk}|| \tag{5.16}$$

To bound a given $||T_{ijk}||$, we will use Lemmas 3.1 and 3.2; the conditions of the latter lemma are satisfied by Lemma 4.7. Lemma 3.1 tells us that $||T_{ijk}|| < C 2^{-\frac{m+j+n+Mj}{2}}$, while Lemma 3.2 tells us that $||T_{ijk}|| < C |\lambda|^{-\frac{1}{2}} 2^{\frac{l+dj}{2}}$. Hence (5.16) implies

$$||T_{ij}|| < C \sum_l \sum_{\{(m,n): c_1 + \frac{l}{p} < \ m,n \ < c_2 + l\}} \min(2^{-\frac{m+j+n+Mj}{2}}, |\lambda|^{-\frac{1}{2}} 2^{\frac{l+dj}{2}}) \tag{5.17}$$

Since in any term of (5.17) we have $m, n > c_1 + \frac{l}{p}$, in a given summand in (5.17), the left term in the minimum is used if we have

$$2^{-\frac{l}{p} - \frac{(M+1)j}{2}} < |\lambda|^{-\frac{1}{2}} 2^{\frac{l+dj}{2}} \tag{5.18}$$

Equivalently,

$$\frac{p+2}{p} l > \log_2 |\lambda| - (M+1+d)j \tag{5.19}$$

Let l_0 denote the minimum possible value of l, and let j_0 be such that

$$\frac{p+2}{p} l_0 = \log_2 |\lambda| - (M+1+d)j_0 \tag{5.20}$$

If $j > j_0$, then the left term in the minimum will be used in every summand of (5.17), and we have

$$||T_{ij}|| < C \sum_{l \geq l_0} \sum_{m,n > \frac{l}{p}} 2^{-\frac{m+j+n+Mj}{2}}$$

$$< C2^{-\frac{(M+1)j}{2}} \tag{5.21}$$

On the other hand, if $j \leq j_0$, we use both sides of the minimum. For each $j \leq j_0$, let l_j denote the solution to

$$\frac{p+2}{p} l_j = \log_2 |\lambda| - (M+1+d)j \tag{5.22}$$

We now fix an l and add up the terms in (5.17) for this value of l. If $l \geq l_j$, we always use the left side of the minimum and the portion of the sum (5.17) corresponding to l is

$$C \sum_{m,\, n > \frac{l}{p}} 2^{-\frac{m+j+n+Mj}{2}}$$

This is at most

$$C 2^{-\frac{l}{p} - \frac{(M+1)j}{2}}$$

Therefore, the sum of all terms of (5.17) corresponding to $l \geq l_j$ is bounded by

$$C 2^{-\frac{l_j}{p} - \frac{(M+1)j}{2}} \tag{5.23}$$

If $l < l_j$ one has to use both sides of the minimum in (5.17), depending on m and n. The number N_l of terms where one uses the right side is bounded by the number of (m,n) with $m,\, n > \frac{l}{p}$ satisfying

$$2^{-\frac{m+j+n+Mj}{2}} > |\lambda|^{-\frac{1}{2}} 2^{\frac{l+dj}{2}}$$

Equivalently,

$$m + n < \log_2 |\lambda| - (1+M+d)j - l$$

Thus N_l is bounded by C times the area of the set Q_l defined by

$$Q_l = \{(x,y) : x, y > \frac{l}{p},\ x + y < \log_2 |\lambda| - (1+M+d)j - l\} \tag{5.24}$$

The set Q_l is a 45-90-45 triangle. Denote its height by h_l, and its area by A_l, so that $A_l = \frac{h_l^2}{2}$. Observe that for some constant c we have

$$\frac{\partial A_l}{\partial l} = c h_l \tag{5.25}$$

In other words, $h_l \frac{dh_l}{dl} = c h_l$, and thus h_l is of the form $cl + c'$. Since l_j was chosen exactly so $h_{l_j} = 0$, we in fact have $h_l = c(l - l_j)$, and thus

$$A_l = \frac{1}{2}(l - l_j)^2 \tag{5.26}$$

Hence N_l, the number of terms of the sum (5.17) corresponding to l such that the right hand side of the minimum is used, is bounded by $C(l - l_j)^2$, and the sum of all these terms, which we denote here by Σ_l^2, satisfies

$$\Sigma_l^2 < C(l - l_j)^2 |\lambda|^{-\frac{1}{2}} 2^{\frac{l+dj}{2}} \tag{5.27}$$

On the other hand, for our fixed l, the terms in the sum (5.17) corresponding to taking the left hand side of the minimum form a geometric series in m and n, whose sum is thus bounded by C times the largest term. This largest term corresponds to where the two sides of the minimum are equal. Hence if Σ_l^1 denotes the sum of all these terms where the left hand side is used, we have

$$\Sigma_l^1 < C|\lambda|^{-\frac{1}{2}}2^{\frac{l+dj}{2}} \tag{5.28}$$

Adding (5.27) and (5.28), the sum of all of the terms of (5.17) corresponding to our given $l \leq l_0$ is bounded by

$$C'((l-l_j)^2+1)|\lambda|^{-\frac{1}{2}}2^{\frac{l+dj}{2}}$$

Summing this over all $l \leq l_0$, we conclude that the sum of the terms of (5.17) corresponding to $l \leq l_0$ is bounded by

$$C''|\lambda|^{-\frac{1}{2}}2^{\frac{l_j+dj}{2}} \tag{5.29}$$

In (5.23) we had an expression for the sum of all the terms of (5.17) corresponding to $l > l_0$. Observe that l_j was defined so that (5.23) and (5.29) are equal. Hence for our given value of $j < j_0$, the sum (5.17) is bounded by C times the expression in (5.23). So we have, for a given $j < j_0$

$$||T_{ij}|| < C2^{-\frac{l_j}{p}-\frac{(M+1)j}{2}} \tag{5.30}$$

Substituting back for l_j, we have

$$||T_{ij}|| < C|\lambda|^{-\frac{1}{p+2}}2^{(\frac{M+1+d}{p+2}-\frac{M+1}{2})j} \tag{5.31}$$

In view of (5.21) and (5.31), we have bounds for every $||T_{ij}||$. We are now in a position to bound the operator T_i. First observe that we may use almost-orthogonality on the sum $T_i = \sum_j T_{ij}$; there is an N such that if we take every Nth term in the sum, the x and y projections of the supports of their kernels are disjoint and the orthogonality lemma Lemma 3.3 applies. Hence we have

$$||T_i|| < C\sup_j ||T_{ij}|| \tag{5.32}$$

I claim now that the expression (5.31) is nondecreasing in j. For this do be true, one needs that $p \leq \frac{2d}{M+1}$. Recall that p is the order of a zero of the polynomial $p(y)$ from section 2, and each term cy^b in $p(y)$ derived from a term cx^ay^b in the Taylor expansion of $\frac{\partial^2 S}{\partial x \partial y}$ with $a + Mb = d$. There are at most $\min(d, \frac{d}{M}) + 1$ such terms, with the degree of $p(y)$ being at most $\min(d, \frac{d}{M}) \leq \frac{2d}{M+1}$. So $p \leq \frac{2d}{M+1}$, and (5.31) is an nondecreasing function of j. Clearly (5.21) is a decreasing funcction of j. Since (5.31) is used for $j < j_0$, and (5.21) is used for $j \geq j_0$, the supremum in (5.32) is realized for $j = j_0$ where we can use either (5.21) or (5.31). If we use (5.21) for $j = j_0$ then we get

$$||T_i|| < C2^{-\frac{(M+1)j_0}{2}}$$

Inserting in (5.20), we get

$$||T_i|| < C|\lambda|^{-\frac{M+1}{2(M+1+d)}} \tag{5.33}$$

Since the edge of the Newton polygon of $\frac{\partial^2 S}{\partial x \partial y}$ intersecting the line $y = x$ has equation $x + My = d$, the edge of the reduced Newton polygon of S intersecting the line $y = x$ will have equation $x + My = d + 1 + M$, and (5.33) is exactly the decay required by Theorem 1.1. Thus we have found the sharp estimates for T_i.

6. Estimating the $||U_i||$, $||V||$; completion of the proof.

Since $U_i = \sum_j U_{ij}$, we have that $||U_i|| \leq \sum_j ||U_{ij}||$. Using Lemma 3.1 on U_{ij}, we have

$$||U_{ij}|| < C2^{-\frac{M+1}{2}j} \tag{6.1}$$

By virtue of part (ii) of Lemma 2.4, each U_{ij} also satisfies the conditions of Lemma 3.2. Thus we also have the estimate

$$||U_{ij}|| < C|\lambda|^{-\frac{1}{2}}2^{\frac{jd}{2}} \tag{6.2}$$

Combining (6.1) and (6.2) we have

$$||U_i|| \leq C\sum_j \min(2^{-\frac{M+1}{2}j}, |\lambda|^{-\frac{1}{2}}2^{\frac{jd}{2}}) \tag{6.3}$$

The estimate in (6.1) decreases exponentially with j, and the estimate of (6.2) increases exponentially with j, so the sum in (6.3) is bounded by C times the term where the two estimates are equal. Denoting the j for which they are equal by j', we thus have

$$||U_i|| \leq C2^{-\frac{M+1}{2}j'} \tag{6.4}$$

One may calculate that $j' = \frac{\log_2|\lambda|}{d+M+1}$, so that (6.4) becomes

$$||U_i|| \leq C|\lambda|^{-\frac{M+1}{2(d+M+1)}} \tag{6.5}$$

As explained in section 5, this is the exponent in Theorem 1.1.

We now turn our attention to V and V'. Since they are done the same way except with the axes reversed, we will focus our attention on V. Since the $x > 0$ and $x < 0$ are done the same way, we will restrict consideration to the half of V for which $x > 0$. Let k be as in Lemma 2.4 (iii). If $k = 0$, then (2.14) is identical to (2.13) and one may use the same argument that was used to analyze the U_i above and get the optimal estimate (6.5) again for V.

Assume now that $k > 0$. The following argument is quite similar to the correpsonding argument in [R]. As above let $j' = \frac{\log_2|\lambda|}{d+M+1}$. For $j > j'$, one can use Lemma 3.1 and as in (6.1) we have

$$||V_j|| < C2^{-\frac{M+1}{2}j} \tag{6.6}$$

Adding this over $j > j'$ one obtains

$$||\sum_{j>j'} V_j|| < C|\lambda|^{-\frac{M+1}{2(d+M+1)}}$$

Now consider $j \leq j'$. The kernel of $T_jT_j^*$ is given by $K(x_1, x_2)$, where

$$K(x_1, x_2) = \int e^{i\lambda S(x_2,y)-i\lambda S(x_1,y)}\phi_j''(x_1,y)\phi_j''(x_2,y)\,dy \tag{6.7}$$

We use Van der Corput's lemma to bound (6.7). Observe that the $k+1$th derivative of the phase has absolute value

$$|\lambda||\frac{\partial^{k+1}S}{\partial y^{k+1}}(x_2,y) - \frac{\partial^{k+1}S}{\partial y^{k+1}}(x_1,y)| = |\lambda||\int_{x_1}^{x_2} \frac{\partial^{k+2}S}{\partial x\partial y^{k+1}}(x,y)dx|$$

$$> C|\lambda||x_1 - x_2|2^{-jd+jkM} \tag{6.8}$$

Equation (6.8) follows from (2.14). Hence by Van der Corput

$$|K(x_1, x_2)| < C|\lambda|^{-\frac{1}{k+1}}|x_1 - x_2|^{-\frac{1}{k+1}}2^{j\frac{d-kM}{k+1}} \tag{6.9}$$

The cutoffs in (6.7) do not interfere with this application of Van der Corput; the condition $j \leq j_0$ insures that they are wide enough. We now apply Schur's test to $K(x_1, x_2)$ using (6.9). We have

$$\int |K(x_1,x_2)|dx_2 < C|\lambda|^{-\frac{1}{k+1}}2^{j\frac{d-kM}{k+1}}\int_{|x_2-x_1|<C2^{-j}} |x_1 - x_2|^{-\frac{1}{k+1}}\,dx_2$$

$$< C|\lambda|^{-\frac{1}{k+1}}2^{j\frac{d-kM-k}{k+1}}$$

By symmetry, the x_1 integral satisfies the same bounds, so Schur's test implies that

$$||V_jV_j^*|| < C|\lambda|^{-\frac{1}{k+1}}2^{j\frac{d-kM-k}{k+1}}$$

Therefore

$$||V_j|| < C|\lambda|^{-\frac{1}{2(k+1)}}2^{j\frac{d-kM-k}{2(k+1)}} \tag{6.10}$$

Recall from section 2 that k was the order of the zero of $p(y)$ at $y = 0$, which is exactly the same as saying that the lowest point on the segment of the Newton polygon of $\frac{\partial^2 S}{\partial x\partial y}$ of slope $-\frac{1}{M}$ has coordinates (k', k) for a k' satisfying $k' + Mk = d$. Since the line $y = x$ intersects the interior of this segment, we have $k < k'$, and thus $k(M+1) < d$. We rewrite this in the form

$$d - kM - k > 0 \tag{6.11}$$

Dividing (6.11) by $2(k+1)$ shows that the coefficient in (6.10) is positive. In particular, adding (6.10) over $j \leq j'$ gives

$$\sum_{j\leq j'} ||V_j|| < C|\lambda|^{-\frac{1}{2(k+1)}} 2^{j' \frac{d-kM-k}{2(k+1)}} \tag{6.12}$$

Substituting $2^{j'} = |\lambda|^{\frac{1}{d+M+1}}$ back into (6.12), one obtains

$$\sum_{j\leq j'} ||V_j|| < C|\lambda|^{-\frac{1}{2(k+1)} + \frac{d-kM-k}{2(d+M+1)(k+1)}} \tag{6.13}$$

The exponent in (6.13) simplifies to $-\frac{M+1}{2(d+M+1)}$, so we conclude

$$\sum_{j\leq j'} ||V_j|| < C|\lambda|^{-\frac{M+1}{2(d+M+1)}} \tag{6.14}$$

Combining (6.7) and (6.14) gives

$$||V|| < C|\lambda|^{-\frac{M+1}{2(d+M+1)}} \tag{6.15}$$

This is the desired estimate for V and we are done. So we have now found the sharp estimates for each T_i, U_i and V. The T_i' and U_i' are dealt with in the same way as the T_i and U_i respectively, and V' is analyzed like V but with the roles of the axes reversed. Hence we are done proving Theorem 1.1 in the case where the line $y = x$ intersects the interior of a bounded edge of the reduced Newton polygon of S.

The cases when the line $y = x$ intersects the reduced Newton polygon of S at a vertex or on one of the unbounded edges do not require an elaborate decomposition into rectangles. Since they are straightforward and very much like the analysis of V above, we will be brief here. In the case where the line $y = x$ intersects at a vertex $v = (a, a)$, denote the slopes of the adjacent edges by $-\frac{1}{m}$ and $-\frac{1}{m'}$. Let M be any number between m and m'. We divide the support of $\phi(x, y)$ into 4 regions via the curves $|y| = |x|^M$. We consider the region where $x > 0$ and $|y| < |x|^M$; the other 3 are done similarly. We write this portion of T, call it T_0, as $T_0 = \sum T_j$, where for $x > 0$

$$T_j f(x) = \int_{|y|<|x|^M} e^{i\lambda S(x,y)} \phi(x, y) f(y)\, dy \tag{6.16}$$

(One does not need to define T_j using cutoff functions for the following to work). There is an appropriate j'' such that if $j > j''$, one gets optimal estimates for $||T_j||$ by applying Lemma 3.1 exactly as in the discussion of V above for $j > j'$. If $j \leq j''$, one obtains estimates $||T_i T_j^*||, ||T_i^* T_j|| < C|\lambda|^{-\frac{1}{a}} 2^{-\epsilon|i-j|}$ by applying Van der Corput's lemma to the kernel of $T_i T_j^*$ or $T_i^* T_j$ and then invoking Schur's test; the argument is essentially the same as the one used to analyze $V_j V_j^*$ above for $j < j'$. By almost-orthogonality one then gets

$||T_0|| < C|\lambda|^{-\frac{1}{2a}}$. The other 3 regions are done similarly and we have $||T|| < C|\lambda|^{-\frac{1}{2a}}$, the desired estimate.

In the case where the line $y = x$ intersects one of the unbounded edges in its interior, the argument is very direct. Take the case where it intersects a horizontal edge $y = a$ in its interior. Then we write $T = \sum_j T_j$, where this time

$$T_j f(x) = \int_{2^{-j} \leq |x| < 2^{-j+1}} e^{i\lambda S(x,y)} \phi(x,y) f(y) dy \tag{6.17}$$

By applying Van der Corput to the kernel of $T_j T_j^*$ as in the analysis of V, for some $\epsilon' > 0$ one gets that

$$||T_j|| < C 2^{-j\epsilon'} |\lambda|^{-\frac{1}{2a}}$$

Adding this over j gives $||T|| < C|\lambda|^{-\frac{1}{2a}}$, the desired estimate. The case where the line $y = x$ intersects the vertical edge is done in the analogous fashion with the dyadic decomposition being in the y direction.

Since all possible cases have now been covered, we have proven all the estimates of Theorem 1.1. The estimates obtained can be seen to be sharp as follows. Let $A_{ij} \times B_{ij}$ be the support of the kernel of the operator denoted by U_{ij} in section 6. The phase function $\lambda S(x,y)$ will not vary by more than $\frac{\pi}{4}$ on $A_{ij} \times B_{ij}$ if $j \geq j' + N$ for some integer N, where j' is as in (6.4). As a result, if we let $f(y)$ be the characteristic function of $B_{i\ j+N}$ and $g(x)$ be the characteristic function of $A_{i\ j+N}$, we have $|\int Tf(x)g(x)\,dx| > C|\lambda|^{\frac{d+1}{d+M+1}} ||f||_2 ||g||_2$, proving the sharpness of the estimates.

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