Stability of Sublevel Set Estimates and Sharp L^2 Regularity of Radon Transforms in the Plane

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1: Introduction

In this paper, we consider operators of the following form, acting on functions on $\mathbf{R}^2$:

$$Tf(x) = \int_{\mathbf{R}} f(\gamma(x,t))\phi(x,t)\,dt \tag{1.1}$$

Here $\phi(x,t)$ is a smooth function supported on a small neighborhood of the origin in $\mathbf{R}^2 \times \mathbf{R}$ with $\phi(0,0) \neq 0$, and $\gamma(x,t)$ is a smooth function defined on the support of $\phi(x,t)$ satisfying

$$\gamma(x,0) = x,\ \frac{\partial\gamma}{\partial t}(x,t) \neq 0 \tag{1.2}$$

Thus $Tf(x)$ is the average of f over a curve "centered at x". The condition $\frac{\partial\gamma}{\partial t}(x,t) \neq 0$ ensures that the averaging is smooth; T doesn't degenerate into a fractional or singular Radon transform. Our goal will be to prove sharp L^2 estimates for T. In the semitranslation-invariant case, this has been done for real-analytic $\gamma(x,t)$ by Phong and Stein [PS], and for general $\gamma(x,t)$ (not just semi-translation invariant) this was done up to ϵ derivatives by Seeger [Se]. In this paper we will relate L^2 regularity of T to uniform sublevel set estimates for a certain determinant function that arises. The estimates will be sharp for a significant class of T many of which are not semitranslation-invariant; for such operators the results of this paper are not covered by [Se] or [PS].

There has also been quite a bit of work on sharp L^2 estimates for Radon transforms along curves in higher dimensions, such as that of Greenleaf and Seeger [GrSe1]-[GrSe5], and Comech and Cuccagna [CoCu1]-[CoCu2]. For L^p to L^q estimates there have also been a number of papers written; the author is most familiar with those of Christ [Ch], Oberlin [O1]-[O3], and [G2]. We refer the reader to [GrSe5] for more details on these and various other papers on related topics.

Since we seek L^2 estimates, it makes sense to look at the adjoint of T. T^* is of the form

$$T^*f(x) = \int_{\mathbf{R}} f(\gamma^*(x,t))\phi^*(x,t)\,dt \tag{1.3}$$

Here $\phi^*(x,t)$ is a cutoff function, and $\gamma^*(x,t)$ is defined by

$$\gamma(\gamma^*(x,t)) = \gamma^*(\gamma(x,t)) = x \tag{1.4}$$

Another important function for the purposes of this paper is $b^x(t,u)$, defined by

$$y = b^x(t,u) \iff \gamma(x,t) = \gamma(y,t+u) \tag{1.5}$$

Equivalently,

$$b^x(t,u) = \gamma^*(\gamma(x,t), t+u) \tag{1.6}$$

Denote the Jacobian determinant of $b^x(t,u)$ in t and u by $J^x(t,u)$. In [CNSW] it is shown that a necessary and sufficient condition for the operator T to have any L^2 smoothing is their "curvature condition", which can be stated in terms of $J^x(t,u)$ as follows.

Definition: T is said to satisfy the curvature condition at x if there exists a multiindex (r,s) such that

$$\frac{\partial^{r+s} J^x}{\partial t^r \partial u^s}(0,0) \neq 0 \tag{1.7}$$

We will not prove the equivalence of (1.7) to the conditions of [CNSW] here; it suffices to say that (1.7) is most similar to the Jacobian condition of [CNSW]. Notice that (1.7) is equivalent to there existing M, $\epsilon > 0$ such that for any sufficiently small ball B containing x we have the following for each $a > 0$:

$$|\{(t,u) \in B : J^x(t,u) < a\}| < Ma^\epsilon \tag{1.8}$$

One can also reverse the roles of γ and γ^*, defining $(b^*)^x(t,u) = \gamma(\gamma^*(x,t), t+u)$ and $(J^*)^x(t,u)$ to be the determinant of $(b^*)^x(t,u)$ in t and u. The curvature condition at x can be reexpressed as the existence of a multiindex $(r.s)$ such that

$$\frac{\partial^{r+s} (J^*)^x}{\partial t^r \partial u^s}(0,0) \neq 0 \tag{1.7$'$}$$

Similarly, (1.8) is equivalent to the existence of M', $\epsilon' > 0$ such that for a small enough ball B' containing x we have

$$|\{(t,u) \in B' : (J^*)^x(t,u) < a\}| < M'a^{\epsilon'} \tag{1.8$'$}$$

Our main theorem is a way of expressing L^2 regularity of T in terms of the largest ϵ or ϵ' for which (1.8) or (1.8$'$) holds uniformly (i.e with a fixed M or M') for x in a neighborhood of the origin. These estimates will be sharp for a significant class of T. The theorem is as follows.

Theorem 1.1: Suppose T satisfies the curvature condition at the origin. Suppose there exist neighborhoods B_1 and B_2 of the origin in $\mathbf{R}^2$ and constants M, $\epsilon > 0$ such that (1.8) holds with $B = B_2$ for all x in B_1. Then if the support of $\phi(x,t)$ is sufficiently small, T is bounded from H^s to $H^{s+\frac{\epsilon}{2(1+\epsilon)}}$. If the line $y = x$ intersects the Newton polygon of $J^0(t,u)$ in the interior of its vertical edge $x = p$ and this ϵ can be taken to be $\frac{1}{p}$, then the estimate is sharp; T is not bounded from H^s to $H^{s+\delta}$ for $\delta > \frac{\epsilon}{2(1+\epsilon)}$. The analogous statements hold if (1.8) is replaced by (1.8$'$).

Observe that if the line $y = x$ intersects the Newton polygon of $J^0(t,u)$ in the interior of its vertical edge $x = p$, then it is well-known (see [PSSt] or [G1] for example) that the ϵ in (1.8) at the origin can be taken to be $\frac{1}{p}$. Hence Theorem 1.1 says that if the level set estimate at the origin holds uniformly in x in a neighborhood of the origin, then we automatically have sharp estimates. By Theorem 2.3 in Karpushkin [K], this always holds in the real-analytic case, so as a consequence we have:

Theorem 1.2: If $\gamma(x,t)$ is real-analytic, and the line $y = x$ intersects the Newton polygon of $J^0(t,u)$ in the interior of its vertical edge $x = p$, then we have the sharp estimate that T is bounded from H^s to $H^{s+\frac{1}{2(p+1)}}$ for each s. The corresponding statement holds when $J^0(t,u)$ is replaced by $(J^*)^0(t,u)$.

To help understand what the condition on $J^0(t,u)$ means, reparameterizing (1.1) if necessary we can assume that $\gamma(x,t)$ is of the form

$$\gamma(x,t) = (x_1 + t, \gamma_2(x,t)) \tag{1.9}$$

As a result, the definition (1.5) becomes

$$y = b^x(t,u) \iff (x_1 + t, \gamma_2(x,t)) = (y_1 + t + u, \gamma_2(y, t+u))$$

In particular the first component of $b^x(t,u)$ is $y_1 = x_1 - u$ and we have

$$J^x(t,u) = \frac{\partial b_2^x}{\partial t}(t,u) \tag{1.10}$$

(Here b_2^x denotes the second component of b^x). Similarly, we have

$$(J^*)^x(t,u) = -\frac{\partial (b^*)_2^x}{\partial t}(t,u) \tag{1.10$'$}$$

We now give a explicit description of when the hypotheses of Theorem 1.2 hold; this will also show how to construct concrete examples. First, observe that there are coordinates such that the curves $t \to \gamma^*(x,t)$ are horizontal for x on the y-axis. In other words for such an $x = (0, x_2)$ we have the following (again assuming we've reparameterized our curves so that (1.9) holds:)

$$\gamma^*((0,x_2), -x_1) = (x_1, x_2) \tag{1.11}$$

Equivalently,

$$\gamma((x_1,x_2), -x_1) = (0, x_2) \tag{1.12}$$

Thus in the real-analytic situation, γ must be of the following form, for some real-analytic function $\delta(x_1, x_2, t)$:

$$\gamma((x_1,x_2),t) = (x_1 + t, x_2 + (x_1 + t)\delta(x_1,x_2,t)) \tag{1.13}$$

Conversely, every γ of the form (1.13) gives a Radon transform in a coordinate system where the curves for γ^* are horizontal for x on the y-axis. It is not hard to determine which

$\delta(x_1, x_2, t)$ give an operator which satisfies the conditions of Theorem 1.2 for $(J^*)^0(t,u)$. For we have

$$(b^*)^0(t,u) = \gamma(\gamma^*((0,0),t), t+u) = \gamma((-t,0), t+u) = (u, u\delta(-t,0,t+u)) \qquad (1.14)$$

In view of (1.10′), the statements of Theorem 1.2 corresponding to $(J^*)^0(t,u)$ hold exactly when the Newton polygon of $\frac{\partial}{\partial t}(u\delta(-t,0,t+u))$ intersects the line $y = x$ in the interior of its vertical edge. The statements of Theorem 1.2 corresponding to $J^0(t,u)$ have an analogous realization with the roles of γ^* and γ reversing.

Interesting examples of when the conditions of Theorem 1.2 hold occur in the setting of line complexes:

Real-Analytic Line Complexes.

We assume $\gamma(x,t)$ is of the following form, for a real-analytic function $a(x_1, x_2)$:

$$\gamma(x,t) = (x_1 + t, x_2 + a(x_1,x_2)\,t) \qquad (1.15)$$

In this case, denoting $\gamma^*((0,0),t)$ by (y_1, y_2) we have

$$(b^*)^0(t,u) = (y_1 + t + u,\ y_2 + a(y_1,y_2)(t+u)) \qquad (1.16)$$

In particular,

$$(b^*)^0_2(t,u) = y_2 + a(y_1,y_2)(t+u) \qquad (1.17)$$

Note that $\gamma(\gamma^*((0,0),t),t) = (0,0)$. Taking second components of this we have

$$y_2 + a(y_1,y_2)t = 0 \qquad (1.18)$$

Hence

$$(b^*)^0_2(t,u) = a(y_1,y_2)u = -\frac{y_2 u}{t} = -\frac{\gamma^*((0,0),t)\,u}{t} \qquad (1.19)$$

The second to last equality follows from (1.18). We conclude by (1.10′) that

$$(J^*)^0(t,u) = \frac{\partial}{\partial t}\frac{\gamma^*((0,0),t)\,u}{t} \qquad (1.20)$$

Hence the conditions of Theorem 1.2 hold if $t \to \gamma^*((0,0),t)$ has a zero of order four or greater at the origin. This can also be expressed in terms of the function $a(0,t)$. For (1.18) can be rewritten as

$$\gamma_2^*((0,0),t) = -a(-t, \gamma_2^*((0,0),t))\,t \qquad (1.21)$$

Thus expanding a in the second component we have

$$\gamma_2^*((0,0),t) = -a(-t,0)\,t + O(\gamma_2^*((0,0),t)\,t) \qquad (1.22)$$

Hence we have

$$|\gamma_2^*((0,0),t)| \sim |a(-t,0)\,t| \tag{1.23}$$

So the order of the zero of $a(t,0)$ at $t=0$ is the same as that of $\frac{\gamma_2^*((0,0),t)}{t}$. As a result, the hypotheses of Theorem 1.2 hold as long as $a(t,0)$ has a zero of order three or greater at the origin. In the terminology of singularity theory, this is equivalent to the operator T having a one-sided type k singularity for some $k \geq 2$. (see [GrSe5] for background material.) For $k \leq 3$, sharp estimates for such operators have been proven in [GrSe2] [GrSe3] [GrSe4]. Thus combining those results with this paper, we get sharp estimates for all real-analytic line complexes in the plane.

Relation to Fefferman-Phong Metrics.

In [G3], the author defined a metric associated to a Radon transform with certain properties that resemble those of the metric Fefferman and Phong use [FP] in their study of subelliptic PDE's. In fact, in many situations the L^2 Sobolev smoothing of a Radon or Radon-like transform can be expressed in terms of the inradii of balls of this metric in a way analogous to that of [FP]. (See [G4] for examples and more discussion.)

Furthermore, it can be shown in the real-analytic case in two dimensions, that the measure of the set $\{\gamma^*(\gamma(x,t),t+u) : |t|,\ |u| < r\}$ is at least c times that of the whole ball of radius r for some constant c, so that the set $\{\gamma^*(\gamma(x,t),t+u) : |t|,\ |u| < r\}$ is in some sense comparable to the ball itself. By the right-hand assumption of (1.2), the vector $\frac{\partial}{\partial u}\gamma^*(\gamma(x,t),t+u)$ is always of magnitude ~ 1, and the vector $\frac{\partial}{\partial t}\gamma^*(\gamma(x,t),t+u)$ is of magnitude at most ~ 1. Thus the determinant $J^x(t,u)$ is comparable to the inradius of the "ball"

$$\{c_1\frac{\partial}{\partial t}\gamma^*(\gamma(x,t),t+u) + c_2\frac{\partial}{\partial u}\gamma^*(\gamma(x,t),t+u) : |c_1|,|c_2| < 1\} \tag{1.24}$$

The analogous statement holds for $(J^*)^x(t,u)$. Thus Theorems 1.1 and 1.2 can be viewed as statements about the distribution functions of inradii of infinitesimal Fefferman-Phong-like balls, giving a theorem analogous to that of [FP], [G4], and others.

2. The Main Estimates

Let $\rho(t)$ be a function on $\mathbf{R}$ whose Fourier transform is a nonnegative function supported on $[-2,2]$ that is equal to one on $[-1,1]$. Let $\zeta(x) = \rho(x_1)\rho(x_2)$. Since the operator $f \rightarrow \zeta * Tf$ is bounded from H^s to H^{s+c} for any c, to prove Theorem 1.1 it suffices to show $\zeta * Tf - Tf = (\zeta - \delta) * Tf$ is bounded from H^s to $H^{s+\frac{\epsilon}{2(\epsilon+1)}}$, where ϵ is as in Theorem 1.1. Let $\rho_m(t) = 2^m\rho(2^m t)$ and $\rho_m^0(t) = 2^m\rho(2^m t) - 2^{m+1}\rho(2^{m+1}t)$. Then we have

$$\zeta(x) - \delta(x) = \sum_{m=0}^{\infty} \rho_m(x_1)\rho_m^0(x_2) + \sum_{m=0}^{\infty} \rho_m^0(x_1)\rho_{m+1}(x_2) \tag{2.1}$$

If we write $\psi_m(x) = \rho_m(x_1)\rho_m^0(x_2)$ and $\tilde{\psi}_m = \rho_m^0(x_1)\rho_{m+1}(x_2)$, then we have

$$\zeta * Tf - Tf = \sum_{m=0}^{\infty} \psi_m * Tf + \sum_{m=0}^{\infty} \tilde{\psi}_m * Tf \tag{2.2}$$

The two sums in (2.2) are analyzed in the same way, so we will restrict ourselves to proving boundedness of the first sum. Let $\phi_m(x) = (1 - \triangle)^{\frac{\epsilon}{2(\epsilon+1)}} \psi_m(x)$. Write

$$U_m f = \phi_m * Tf$$

$$U = \sum_m U_m \tag{2.3}$$

Our goal is therefore to show that U is bounded on L^2. To do this, we use Cotlar-Stein almost-orthogonality on the sum (2.3). First we show boundedness of the individual U_m's.

Lemma 2.1: Under the hypotheses of Theorem 1.1, we have $||U_m||_{L^2 \to L^2} < C$.

Proof: Let ϕ_m^c denote the operator given by convolving by ϕ_m; in other words $\phi_m^c g = \phi_m * g$. Thus $U_m U_m^* = \phi_m^c TT^* (\phi_m^c)^*$. We will apply Schur's test to $U_m U_m^*$. In other words, denoting the kernel of $U_m U_m^*$ by $K(x, y)$, we will show that $\int |K(x, y)|\, dy, \int |K(x, y)|\, dx < C$. Since $U_m U_m^*$ is self adjoint, it suffices to do the y integral. Let $L(x, y)$ be the kernel of TT^*. Then $K(x, y)$ is given by

$$K(x, y) = \int \phi_m(x - a) L(a, b) \bar{\phi}_m(b - y)\, da\, db \tag{2.4}$$

Observe that the Fourier transform of ϕ_m is given by $(1 + |\xi|^2)^{\frac{\epsilon}{2(1+\epsilon)}} \sigma(2^{-m}\xi)$ for a smooth function σ supported away from the line $\xi_2 = 0$. As a result we may find a smooth Φ_m such that

$$\frac{d\Phi_m}{dx_2} = \phi_m \tag{2.5}$$

Furthermore, we have

$$\int |\phi_m(x)|\, dx < C 2^{\frac{m\epsilon}{2(\epsilon+1)}}, \quad \int |\Phi_m(x)|\, dx < C 2^{m(-1 + \frac{\epsilon}{2(\epsilon+1)})} \tag{2.6}$$

We now examine the function $L(x, y)$, the kernel of TT^*. By (1.1) and (1.3), we have

$$TT^* f(x) = \int f(\gamma^*(\gamma(x, t), u)) \phi^*(\gamma(x, t), u) \phi(x, t)\, dt\, du \tag{2.7}$$

Changing variables $(t, u)_{old} = (t, t + u)_{new}$, letting $\eta(x, t, u)$ denote $\phi^*(\gamma(x, t), t + u)\phi(x, t)$ we have

$$\begin{aligned} TT^* f(x) &= \int f(\gamma^*(\gamma(x, t), t + u)) \eta(x, t, u)\, dt\, du \\ &= \int f(b^x(t, u)) \eta(x, t, u)\, dt\, du \end{aligned} \tag{2.8}$$

The function $\eta(x, t, u)$ is just a smooth cutoff function. We conclude that the kernel $L(x, y)$ of TT^* is given by

$$\int \delta(y - b^x(t, u)) \eta(x, t, u)\, dt\, du \tag{2.9}$$

We now decompose $L(x,y) = L_1(x,y) + L_2(x,y)$ in the following way. Letting $\alpha(t)$ be a nonnegative bump function on $\mathbf{R}$ that is equal to 1 in a neighborhood of the origin, we define

$$L_1(x,y) = \int \delta(y - b^x(t,u))\eta(x,t,u)\alpha(2^{\frac{2m}{\epsilon+1}}[J^x(t,u)]^2)\,dt\,du \tag{2.10a}$$

$$L_2(x,y) = \int \delta(y - b^x(t,u))\eta(x,t,u)(1-\alpha)(2^{\frac{2m}{\epsilon+1}}[J^x(t,u)]^2)\,dt\,du \tag{2.10b}$$

We correspondingly write $K(x,y) = K_1(x,y) + K_2(x,y)$, where

$$K_i(x,y) = \int \phi_m(x-a)L_i(a,b)\bar{\phi}_m(b-y)\,da\,db \tag{2.11}$$

By (2.6), we have

$$\int |K_1(x,y)|\,dy < C2^{m\frac{\epsilon}{1+\epsilon}} \sup_a \int |L_1(a,b)|\,db \tag{2.12a}$$

For $K_2(x,y)$, we integrate (2.11) by parts in b_2, and obtain

$$|K_2(x,y)| < C\int |\phi_m(x-a)\nabla_b L_2(a,b)\bar{\Phi}_m(b-y)|\,da\,db$$

Therefore, using (2.6) again, we have

$$\int |K_2(x,y)|\,dy < C2^{-m\frac{1}{1+\epsilon}} \sup_a \int |\nabla_b L_2(a,b)|\,db \tag{2.12b}$$

Hence in order to prove Lemma 2.1 by using Schur's test on $K(x,y)$, we must prove the following inequalities (we switch variable names from (a,b) to (x,y)):

$$\int |L_1(x,y)|\,dy < C2^{-m\frac{\epsilon}{1+\epsilon}} \tag{2.13a}$$

$$\int |\nabla_y L_2(x,y)|\,dy < C2^{m\frac{1}{1+\epsilon}} \tag{2.13b}$$

(2.13a) follows immediately from (2.10a) and the definition (1.8) of ϵ. So we turn our attention to (2.13b). In what follows we will be taking various derivatives of delta functions; if one wishes to be completely rigorous, one can take a sequence of smooth δ_N converging to a delta function, apply the following steps with δ_N in place of the delta function, and take limits. First, we observe that $\partial_{y_j} L_2(x,y)$ is given by

$$\int [\frac{\partial}{\partial y_j}\delta(y - b^x(t,u))]\eta(x,t,u)(1-\alpha)(2^{\frac{2m}{\epsilon+1}}[J^x(t,u)]^2)\,dt\,du \tag{2.14}$$

We replace the y_j derivative by a derivative in t and u so that we may integrate by parts. Namely, we seek functions $g_{1j}(t,u)$ and $g_{2j}(t,u)$ such that

$$\frac{\partial}{\partial y_j}\delta(y - b^x(t,u)) = g_{1j}(t,u)\frac{\partial}{\partial t}\delta(y - b^x(t,u)) + g_{2j}(t,u)\frac{\partial}{\partial u}\delta(y - b^x(t,u)) \tag{2.15}$$

Letting $\mathbf{e_j}$ denote the standard jth coordinate vector, $\mathbf{g_j}(t,u)$ denote the column vector $(g_{1j}(t,u), g_{2j}(t,u))$, and $Db^x(t,u)$ denote the derivative matrix of $b^x(t,u)$, we have

$$Db^x(t,u)\,\mathbf{g_j}(t,u) = \mathbf{e_j} \tag{2.16}$$

Thus each $g_{ij}(t,u)$ is of the form $\frac{h_{ij}(t,u)}{J^x(u,v)}$ for some smooth functions $h_{ij}(t,u)$. Hence (2.14) can be written as the sum of two terms, which we write as follows, after an integration by parts:

$$\int \delta(y - b^x(t,u))\frac{\partial}{\partial t}\{\frac{h_{1j}(t,u)}{J^x(t,u)}\eta(x,t,u)(1-\alpha)(2^{\frac{2m}{\epsilon+1}}[J^x(t,u)]^2)\}\,dt\,du \tag{2.17a}$$

$$\int \delta(y - b^x(t,u))\frac{\partial}{\partial u}\{\frac{h_{2j}(t,u)}{J^x(t,u)}\eta(x,t,u)(1-\alpha)(2^{\frac{2m}{\epsilon+1}}[J^x(t,u)]^2)\}\,dt\,du \tag{2.17b}$$

The two terms are dealt with the same way, so we only consider (2.17a), which we denote by $I(x,y)$. We write $h_{1j}(t,u)\eta(x,t,u)$ as $\zeta(x,t,u)$, take absolute values in (2.17a) and integrate, obtaining

$$\int |I(x,y)|\,dy \leq \int |\frac{\partial}{\partial t}\{\frac{1}{J^x(t,u)}\zeta(x,t,u)(1-\alpha)(2^{\frac{2m}{\epsilon+1}}[J^x(t,u)]^2)\}|\,dt\,du \tag{2.18}$$

As a result, in order to prove (2.13b), it suffices to verify that

$$\int |\frac{\partial}{\partial t}\{\frac{1}{J^x(t,u)}\zeta(x,t,u)(1-\alpha)(2^{\frac{2m}{\epsilon+1}}[J^x(t,u)]^2)\}|\,dt\,du < C2^{m\frac{1}{1+\epsilon}} \tag{2.19}$$

By the curvature condition (1.7), there is a positive integer a, a positive number δ, and a direction v_1 such that the following holds for x in a neighborhood of the origin:

$$|\frac{\partial^a J^x}{\partial v_1^a}(t,u)| > \delta \tag{2.20a}$$

We can also let v_2 be a direction close to v_1, but not a multiple of v_1, such that

$$|\frac{\partial^a J^x}{\partial v_2^a}(t,u)| > \delta \tag{2.20b}$$

Since $\frac{\partial}{\partial t} = c_1\frac{\partial}{\partial v_1} + c_2\frac{\partial}{\partial v_2}$ for some constants c_1 and c_2, in order to prove (2.19) it suffices to show that for each i we have

$$\int |\frac{\partial}{\partial v_i}\{\frac{1}{J^x(t,u)}\zeta(x,t,u)(1-\alpha)(2^{\frac{2m}{\epsilon+1}}[J^x(t,u)]^2)\}|\,dt\,du < C2^{m\frac{1}{1+\epsilon}} \tag{2.21}$$

If the derivative in (2.21) lands on the cutoff $\zeta(x,t,u)$, one obtains a term of absolute value at most $C2^{\frac{m}{\epsilon+1}}$ by virtue of the fact that $\frac{1}{|J^x(t,u)|} < C'2^{\frac{m}{\epsilon+1}}$ on the support of the integrand, so we focus on the terms obtained when the derivative lands on the other two factors. If it lands on the $\frac{1}{J^x(t,u)}$ factor, we get a term of absolute value at most

$$C\int_{|J^x(t,u)|>c2^{-\frac{m}{\epsilon+1}}} \frac{|\partial_{v_i} J^x(t,u)|}{|J^x(t,u)|^2}\, dt\, du \tag{2.22}$$

We integrate this first in the v_i direction. By the condition (2.20), the domain of integration consists of finitely many intervals, on each of which $\partial_{v_i} J^x(t,u)$ is monotone (in the v_i direction). On each of these intervals, $\frac{|\partial_{v_i} J^x(t,u)|}{|J^x(t,u)|^2}$ integrates to the absolute value of the difference between the values of $\frac{1}{J^x(t,u)}$ at the two endpoints of the interval, which can be no more than $C2^{\frac{m}{\epsilon+1}}$. Hence (2.22) is bounded by $C2^{\frac{m}{\epsilon+1}}$, which is what is required for (2.21).

We turn our attention to the term where the derivative in (2.21) lands on the $(1-\alpha)$ factor. We get a term of absolute value at most

$$2^{\frac{2m}{\epsilon+1}} \int |\zeta(x,t,u)||\partial_{v_i} J^x(t,u)||\alpha'(2^{\frac{2m}{\epsilon+1}}[J^x(t,u)]^2)|\, dt\, du \tag{2.23}$$

Since α' is compactly supported, for some constants c and C this is bounded by

$$C2^{\frac{2m}{\epsilon+1}} \int_{J^x(t,u)<c2^{-\frac{m}{\epsilon+1}}} |\partial_{v_i} J^x(t,u)|\, dt\, du \tag{2.24}$$

We integrate (2.24), first in the v_i direction. Because of the condition (2.20), like before there are boundedly many intervals of integration in the v_i integral in (2.24), and over each such interval the integrand integrates to the absolute value of the difference of the values of $J^x(t,u)$ at the endpoints of the interval, which is at most $C2^{-\frac{m}{\epsilon+1}}$. Hence the whole integral (2.24) is bounded by $C2^{\frac{m}{\epsilon+1}}$.

Since (2.24) was the final term we needed to consider, we have verified (2.19) and thus (2.13b). This completes the proof of Lemma 2.1.

We now prove the almost-orthogonality estimates:

Lemma 2.2:

$$||U_m U_n^*||_{L^2\to L^2},\ ||U_m^* U_n||_{L^2\to L^2} < C2^{-|m-n|} \tag{2.25}$$

Proof: For $|m-n|\geq 3$, $U_m^* U_n$ is just the zero operator by the definition of ϕ_n used here, so it suffices to prove the estimates for $U_m U_n^*$. Replacing $U_m U_n^*$ by its adjoint if necessary, we may assume that $m \geq n$. Analgous to (2.4), the kernel of $U_m U_n^*$, which we denote by $M_{m,n}(x,y)$, is given by

$$M_{m,n}(x,y) = \int \phi_m(x-a)L(a,b)\bar{\phi}_n(b-y)\, da\, db \tag{2.26}$$

We integrate (2.21) by parts in a_2, obtaining

$$M_{m,n}(x,y) = \int \Phi_m(x-a)\frac{\partial L}{\partial a_2}(a,b)\bar{\phi}_n(b-y)\,da\,db \tag{2.27}$$

Thus we need to look at a derivative of the function $L(a,b)$, the kernel of TT^*. Differentiating (1.1), we see that $\frac{\partial(Tf)}{\partial x_2}$ can be written is $T_1\frac{\partial f}{\partial x_1} + T_2\frac{\partial f}{\partial x_2}$, where T_1 and T_2 are Radon transform operators of the form (1.1) except with different cutoff functions from $\phi(x,t)$. One has a similar statement for T^*, so we can write $\frac{\partial(TT^*f)}{\partial x_2}$ as the sum of four operators as follows:

$$\frac{\partial(TT^*f)}{\partial x_2} = \sum_{i=1}^{2}\sum_{j=1}^{2} V_{ij}W_{ij}^*\frac{\partial f}{\partial x_j} \tag{2.28}$$

Here V_{ij}, W_{ij} are of the form (1.1) except with a different cutoff functions. As a result, the kernel $M_{m,n}(x,y)$ can be written as the sum of four kernels of the form

$$\int \Phi_m(x-a)L_{ij}(a,b)\frac{\partial\bar{\phi}_n}{\partial y_j}(b-y)\,da\,db \tag{2.29}$$

Here L_{ij} is the kernel of $V_{ij}W_{ij}^*$. The operator with kernel (2.29) can thus be written as $X_{ij}Y_{ij}^*$, where

$$X_{ij}f = \Phi_m * V_{ij}f,\;\; Y_{ij}f = \frac{\partial\bar{\phi}_n}{\partial x_j} * W_{ij}f \tag{2.30}$$

Lemma 2.1 gives L^2 to L^2 bounds for X_{ij} and Y_{ij}. For the differences between these operators and T are the differing cutoffs in the definition of V_{ij} or W_{ij}, which don't matter, and the fact that ϕ_m is replaced by Φ_m for the X_{ij}, and ϕ_n is replaced by $\frac{\partial\phi_n}{\partial x_j}$ for the Y_{ij}. In the case of the X_{ij}, changing ϕ_m to Φ_m will introduce a factor of 2^{-m}, while changing ϕ_n to $\frac{\partial\phi_n}{\partial x_j}$ will introduce a factor of 2^n, but otherwise the arguments of Lemma 2.1 still hold. Hence we have

$$||X_{ij}||_{L^2\to L^2} < C2^{-m},\;\; ||Y_{ij}||_{L^2\to L^2} < C2^{n} \tag{2.31}$$

As a result,

$$||X_{ij}Y_{ij}^*||_{L^2\to L^2} < C2^{n-m} \tag{2.32}$$

Since $U_mU_n^*$ is the sum of the four terms $X_{ij}Y_{ij}^*$, Lemma 2.2 follows.

3. Sharpness of the Estimates

In this section we assume the Newton polygon condition of Theorem 1.1 is satisfied, so that the leftmost vertex of the Newton polygon is of the form (p,q) with $p > q$. Assume that T is bounded from H^s to $H^{s+\delta}$. In order to show the sharpness aspect of Theorem 1.1, we must show that $\delta \leq \frac{1}{2(p+1)}$. As in section 1, we assume that we are in coordinates such that

$$\gamma(x,t) = (x+t, \gamma_2(x,t)) \tag{3.1}$$

$$\gamma((0,0),t)=(t,0) \tag{3.2}$$

We do a finite Taylor expansion of $J^0(t,u)$, obtaining

$$J^0(t,u)=\sum_{m,n<M} f_{mn}t^m u^n + O(|t|^M+|u|^M) \tag{3.3}$$

Denote the sum in (3.3) by $S_M(t,u)$. It is not hard to show (see [G] or [PS] for example) that there exists a constant c_0 and a $d>0$ such that when $|t|<|u|^d$ we have

$$S_M(t,u)=c_0t^pu^q+o(t^pu^q) \tag{3.4}$$

As a result, on the set $|t|<|u|^d$ we have

$$J^0(t,u)=c_0t^pu^q+o(t^pu^q)+O(|t|^M+|u|^M) \tag{3.5}$$

Note that the size of the $o(t^pu^q)$ term will depend on M. Observe that since T is bounded from L^2 to L^2_δ, so is T^*. We fix a large integer N and let $t_0, u_0>0$ be such that $t_0=u_0^N$. Let $g(x)$ be the characteristic function of the set E defined by

$$E=\{(x_1,x_2):|x_1|<u_0,\, |x_2-\gamma_2^*((0,0),-x_1)|<4c_0t_0^{p+1}u_0^q\} \tag{3.6}$$

Note that $\gamma^*((0,0),-x_1)$ is of the form $(x_1,\gamma_2^*((0,0),-x_1))$, so that the set E has vertical slices of diameter $8c_0t_0^{p+1}u_0^q$ centered at points $\gamma_2^*((0,0),-x_1)$ with $|x_1|<u_0$. We will examine the consequences of the assumption that $||T^*g||_{L^2_\delta}<C||g||_{L^2}$, when we first let t_0 go to zero with N fixed, and then let N go to infinity. We will see that this implies that $\delta\le\frac{1}{2(p+1)}$, proving the sharpness part of Theorem 1.1. We have the following lemma:

Lemma 3.1: There are constants $c_1,c_2>0$ independent of N such that for u_0 sufficiently small, depending on N, we have

$$|T^*g(t,u')|>c_1u_0 \text{ if } |t|<t_0 \text{ and } |u'|<c_0t_0^{p+1}u_0^q \tag{3.7}$$

$$T^*g(t,u')=0 \text{ if } |t|<t_0 \text{ and } |u'|>c_2t_0^{p+1}u_0^q \tag{3.8}$$

Proof: We start with proving (3.7). Suppose (t,u') is such that $|t|<t_0$ and $|u'|<c_0t_0^{p+1}u_0^q$. In view of (1.3), in order to prove (3.7), it suffices to show that for $|u|<\frac{u_0}{2}$, we have $\gamma^*((t,u'),u)\in E$. Since $t<<u$, it suffices to show that for $|u|<\frac{u_0}{4}$, we have $\gamma^*((t,u'),t+u)\in E$.

We start with the case where $u'=0$. In our coordinates (3.2) holds and we have

$$\gamma^*((t,0),t+u)=\gamma^*(\gamma((0,0),t),t+u)=b^0(t,u)=(b_1^0(t,u),b_2^0(t,u)) \tag{3.9}$$

Note that $b_1^0(t,u)=-u$ and hence $|b_1^0(t,u)|<u_0$. As a result, we will verify (3.7) if we can show that

$$|b_2^0(t,u)-\gamma_2^*((0,0),u)|<2c_0t_0^{p+1}u_0^q \tag{3.10}$$

Note that $\gamma_2^*((0,0),u) = b_2^0(0,u)$, so by the mean value theorem, for some $|t'| < t$ we have

$$b_2^0(t,u) - \gamma_2^*((0,0),u) = t\frac{\partial b_2^0}{\partial t}(t',u) = tJ^0(t',u) \tag{3.11}$$

The last inequality follows from (1.10). Since $|t'| \leq |t| < t_0$ and $|u| < \frac{u_0}{4}$, by (3.5) we have

$$|tJ^0(t',u)| < 2c_0 t_0^{p+1} u_0^q \tag{3.12}$$

Here, in (3.5) we take M to be sufficiently large so that the error term is small; note that M depends on the N for which $t_0 = u_0^N$. Combining (3.11) and (3.12) gives (3.10). Thus we have shown (3.7) in the case where $u' = 0$.

We now assume that u' is arbitrary with $|u'| < c_0 t_0^{p+1} u_0^q$. Observe that $|\gamma_1^*((t,u'),t+u)| = |u| < u_0$ again, so in order to verify (3.7) it suffices to show that

$$|\gamma_2^*((t,u'),t+u) - \gamma_2^*((0,0),u)| < 4c_0 t_0^{p+1} u_0^q \tag{3.13}$$

Observe that because $\gamma_2^*((t,u'),0) = u'$ for all (t,u') we have

$$\frac{\partial \gamma_2^*}{\partial u'}((t,u'),t+u) = 1 + O(|t+u|) \tag{3.14}$$

Using (3.14) and (3.10), and recalling that $\gamma^*((t,0),t+u) = b_2^0(t,u)$, we have

$$\begin{aligned} |\gamma_2^*((t,u'),t+u) - \gamma_2^*((0,0),u)| &\leq |\gamma_2^*((t,u'),t+u) - \gamma_2^*((t,0),t+u)| \\ &+|\gamma_2^*((t,0),t+u) - \gamma_2^*((0,0),u)| \\ &< 2|u'| + 2c_0 t_0^{p+1} u_0^q \\ &< 4c_0 t_0^{p+1} u_0^q \end{aligned}$$

Thus (3.13) holds and (3.7) follows.

We proceed to proving (3.8). In view of (1.3), in order to prove (3.8) it suffices to show that there is a constant c_2 such that $g(\gamma^*((t,u'),u))) = 0$ for all u and all (t,u') with $|t| < t_0$ and $|u'| > c_2 t_0^{p+1} u_0^{q+1}$. If $|u| > 2u_0$, the first component of $\gamma^*((t,u'),u)$, given by $t-u$, is of absolute value greater than u_0 and thus $g(\gamma^*((t,u'),u))) = 0$. Hence (3.8) will be proven if we can show that if for $|u| < 2u_0$ we have

$$|\gamma_2^*((t,u'),u) - \gamma_2^*((0,0),u-t)| > 4c_0 t_0^{p+1} q_0^q \tag{3.15}$$

By (3.14) and the fact that $|u'| > c_2 t_0^{p+1} u_0^q$, we have

$$\begin{aligned} |\gamma_2^*((t,u'),u) - \gamma_2^*((0,0),u-t)| &> |\gamma_2^*((t,u'),u) - \gamma_2^*((t,0),u)| - |\gamma_2^*((t,0),u) - \gamma_2^*((0,0),u-t)| \\ &> \frac{c_2}{2} t_0^{p+1} u_0^q - |\gamma_2^*((t,0),u) - \gamma_2^*((0,0),u-t)| \end{aligned} \tag{3.16}$$

Next, observe that $\gamma_2^*((t,0),u) = b_2^0(t,u-t)$ and $\gamma^*((0,0),u-t) = b_2^0(0,u-t)$, so by the mean-value theorem, for some t' with $|t'| < |t|$ we have

$$\gamma_2^*((t,0),u) - \gamma_2^*((0,0),u-t) = t\frac{\partial b_2^0}{\partial t}(t',u-t) = tJ^0(t',u-t) \tag{3.17}$$

By (3.5) and the fact that $|t| < t_0, |u-t| < 3u_0$, we have for a constant C that

$$|J^0(t',u-t)| < Ct_0^p u_0^q \tag{3.18}$$

As before, we are assuming M is sufficiently large such that the error term in (3.5) is small for $t_0 = u_0^N$; hence M depends on N. Combining (3.18), (3.17), and (3.16), we conclude that

$$|\gamma_2^*((t,u'),u) - \gamma_2^*((0,0),u-t)| > \frac{c_2}{2}t_0^{p+1}u_0^q - Ct_0^{p+1}u_0^q \tag{3.19}$$

Thus so long as c_2 is chosen such that $\frac{c_2}{2} - C > 4c_0$, (3.15) will hold and therefore (3.8) will as well. This completes the proof of Lemma 3.1.

We now let U be the convolution operator whose multiplier is $|\xi|^\delta$. By the assumption that T^* is bounded from L^2 to L^2_δ, we have that UT^* is bounded on L^2. In particular, if g is the characteristic function of E as above, then there is a uniform constant C such that for any t_0 and u_0 we have

$$||UT^*g||_{L^2} < C||g||_{L^2} \tag{3.20}$$

Observe that if x is such that $T^*g(x) = 0$, then for some number c_3, $UT^*g(x)$ is given by

$$UT^*g(x) = c_3\int |x-y|^{-2-\delta}T^*g(y)\,dy \tag{3.21}$$

In particular, by Lemma 3.1 this holds for x in $[-\frac{t_0}{2}, \frac{t_0}{2}] \times [c_2t_0^{p+1}u_0^q, (c_2+1)t_0^{p+1}u_0^q]$. Since $\phi(0,0) \neq 0$, without loss of generality we can assume the integrand in (3.21) is nonnegative or nonpositive. As a result, (3.21) implies that

$$|UT^*g(x)| > C\int_{\{y:|y_1-x_1|,\ |y_2|<c_0t_0^{p+1}u_0^q\}} |x-y|^{-2-\delta}T^*g(y)\,dy \tag{3.22}$$

By (3.7) of Lemma 3.1, for such x we thus have

$$|UT^*g(x)| > Cu_0\int_{\{y:|y_1-x_1|,\ |y_2|<c_0t_0^{p+1}u_0^q\}} |x-y|^{-2-\delta} \tag{3.23}$$

Since we are assuming that x is such that $x_2 \in [c_2t_0^{p+1}u_0^q, (c_2+1)t_0^{p+1}u_0^q]$, on the domain of the integral (3.23) we have $|x-y| \sim |x_2 - y_2| \sim t_0^{p+1}u_0^q$, so we have

$$|UT^*g(x)| > Cu_0 \times t_0^{2(p+1)}u_0^{2q} \times t_0^{-(p+1)(2+\delta)}u_0^{-q(2+\delta)}$$

$$Ct_0^{-(p+1)\delta}u_0^{1-q\delta} \tag{3.24}$$

Integrating this over x in $[-\frac{t_0}{2},\frac{t_0}{2}]\times[c_2t_0^{p+1}u_0^q,(c_2+1)t_0^{p+1}u_0^q]$, we have

$$||UT^*g||_{L^2} > Ct_0^{-(p+1)\delta}u_0^{1-q\delta}\times t_0^{\frac{1}{2}}\times t_0^{\frac{p+1}{2}}u_0^{\frac{q}{2}} \tag{3.25}$$

On the other hand, g is the characteristic function of E, given by (3.6). As a result, we have

$$||g||_{L^2} < Cu_0^{\frac{1}{2}}\times t_0^{\frac{p+1}{2}}u_0^{\frac{q}{2}} \tag{3.26}$$

Since we are assuming UT^* is bounded on L^2, we have

$$t_0^{-(p+1)\delta}u_0^{1-q\delta}\times t_0^{\frac{1}{2}}\times t_0^{\frac{p+1}{2}}u_0^{\frac{q}{2}} < Cu_0^{\frac{1}{2}}\times t_0^{\frac{p+1}{2}}u_0^{\frac{q}{2}} \tag{3.27}$$

Equivalently,

$$t_0^{\frac{1}{2}-(p+1)\delta}u_0^{\frac{1}{2}-q\delta} < C \tag{3.28}$$

Recalling now that we are assuming $u_0=t_0^{\frac{1}{N}}$ for some fixed large N, (3.28) implies that

$$t_0^{\frac{1}{2}-(p+1)\delta+\frac{1}{2N}-\frac{q}{N}\delta} < C \tag{3.29}$$

Letting t_0 go to zero, we must have

$$\frac{1}{2}-(p+1)\delta+\frac{1}{2N}-\frac{q}{N}\delta \geq 0 \tag{3.30}$$

Equivalently,

$$\delta \leq \frac{\frac{1}{2}+\frac{1}{2N}}{p+1+\frac{q}{N}} \tag{3.31}$$

Taking the limit as N goes to infinity, we have

$$\delta \leq \frac{1}{2(p+1)} \tag{3.32}$$

Since δ was an arbitrary number such that T is bounded from H^s to $H^{s+\delta}$, and (p,q) was defined to be the leftmost vertex of the Newton polygon of $J^0(t,u)$, (3.32) proves the sharpness part of Theorem 1.1, and we are done.

4. References

[C] M. Christ, *Convolution, curvature, and combinatorics: a case study*, Internat. Math. Res. Notices, **19** (1998), 1033-1048.

[CNSW] M. Christ, A. Nagel, E. Stein, S. Wainger, *Singular and maximal Radon transforms: analysis and geometry*, Annals of Mathematics (2) **150** (1999), no. 2, 489-577.

[CoCu1] A. Comech, S. Cuccagna, *Integral operators with two-sided cusp singularities*, Internat. Math. Res. Notices **23** (2000), 1225-1242.

[CoCu2] A. Comech, S. Cuccagna, *On L^p continuity of singular Fourier integral operators*, Trans. Amer. Math. Soc. **355** (2003), no. 6, 2453-2476.

[FP] C. Fefferman, D.H. Phong, *Subelliptic eigenvalue problems*, Conference on harmonic analysis in honor of Antoni Zygmund, 590-606, Wadsworth Math. Series, Wadsworth, Belmont, California, 1983.

[G1] M. Greenblatt, *Newton polygons and local integrability of negative powers of smooth functions in the plane*, preprint, 2002.

[G2] M. Greenblatt, *Metrics and smoothing of translation-invariant Radon transforms along curves*, Journal of Functional Analysis, to appear.

[G3] M. Greenblatt, *Boundedness of singular Radon transforms under a finite-type condition*, Amer. Journal of Math **123** (2001), 1009-1053.

[G4] M. Greenblatt *An analogue to a theorem of Fefferman and Phong for averaging operators along curves with fractional integral kernel*, submitted.

[GrSe1] A. Greenleaf, A. Seeger, *Fourier integral operators with fold singularities*, J. Reine ang. Math **455** (1994), 35-56.

[GrSe2] A. Greenleaf, A. Seeger, *Fourier integral operators with cusp singularities*, Amer. J. Math **120** (1998) 1077-1119.

[GrSe3] A. Greenleaf, A. Seeger, *On Oscillatory integral operators with folding canonical relations*, Studia Math. **132** (1999), 125-139.

[GrSe4] A. Greenleaf, A. Seeger, *Oscillatory integral operators with lower order degeneracies*, Duke Math. J. **112** (2002) 397-420.

[GrSe5] A. Greenleaf, A. Seeger, *Oscillatory and Fourier integral operators with degenerate canonical relation*, Publicacions Matematiques special issue: Proceedings of the El Escorial Conference 2000 (2002), 93-141.

[K] V. N. Karpushkin, *Uniform estimates for volumes*, Tr. Math. Inst. Steklova **221** (1998), 225-231.

[O1] D. Oberlin, *Convolution estimates for some measures on curves*, Proc. Amer. Math. Soc. **99** (1987), 56-60.

[O2] D. Oberlin, *A convolution estimate for a measure on a curve in $\mathbf{R}^4$*, Proc. Amer. Math. Soc. **125** (1997), 1355-1361.

[O3] D. Oberlin, *A convolution estimate for a measure on a curve in* $\mathbf{R}^4$*, II*, Proc. Amer. Math. Soc. **127** (1999), 217-221.

[PS] D. H. Phong, E. M. Stein, *The Newton polyhedron and oscillatory integral operators*, Acta Mathematica **179** (1997), 107-152.

[PSSt] D. H. Phong, E. M. Stein, J. Sturm, *On the growth and stability of real-analytic functions*, American J. Math **121** (1999), 519-554.

[Se] A. Seeger, *Radon transforms and finite type conditions*, Journal of the American Mathematical Society **11** (1998) no.4, 869-897.

[V] A. N. Varchenko, *Newton polyhedra and estimates of oscillatory integrals*, Functional Anal. Appl. **18** (1976), no. 3, 175–196.