On Lefschetz Characters of 2-local Geometries for Some Sporadic Groups

R. Philip Grizzard
University of Illinois at Chicago
Ph.D. Thesis Defense

February 13, 2007

Outline

1. Introduction and Background
   - Historical Framework
   - Definitions
   - Example Calculation
   - Methods
2. Description of the process for the Suzuki group
   - Geometry
   - Inducing Subgroups
   - Block Theory
3. Summaries for other sporadic groups
   - $O'N$
   - He
   - Co_3
   - Ru
4. Future Directions

The Classification of Finite Simple Groups

The classification of finite simple groups was announced in 1983 and its proof was formally finalized in 2004. The theorem states that any finite simple group must belong to one of the following classes.

- The infinite family of cyclic groups Z_p of prime order
- The infinite family of alternating groups A_n for $n \geq 5$
- The infinite family of groups of Lie type, including classical matrix groups, exceptional types, and twisted types
- 26 "sporadic" groups

The class of *sporadic* groups is essentially the collection of finite simple groups that did not fit into any infinite family. This has sparked investigation to find common properties among these sporadic groups.

The Classification of Finite Simple Groups

The classification of finite simple groups was announced in 1983 and its proof was formally finalized in 2004. The theorem states that any finite simple group must belong to one of the following classes.

- The infinite family of cyclic groups Z_p of prime order
- The infinite family of alternating groups A_n for $n \geq 5$
- The infinite family of groups of Lie type, including classical matrix groups, exceptional types, and twisted types
- 26 "sporadic" groups

The class of *sporadic* groups is essentially the collection of finite simple groups that did not fit into any infinite family. This has sparked investigation to find common properties among these sporadic groups.

Background for this problem

- Jacques Tits developed building geometries to better understand groups of Lie type.
- For arbitrary finite groups, Brown and Quillen introduced the study of the partially ordered set of p-subgroups for an arbitrary prime p. In the case of G of Lie type in characteristic p, this poset is homotopy equivalent to the Tits building geometry and gives a Lefschetz module which is projective over $\mathbb{F}_p$.
- Webb considered more general subgroup posets than Brown and Quillen had, including some posets Ronan-Smith had studied for sporadic groups.
- Throughout our study, we will work in characteristic $p = 2$, the prime which generally provides the richest geometric structures.

Background for this problem

- Jacques Tits developed building geometries to better understand groups of Lie type.
- For arbitrary finite groups, Brown and Quillen introduced the study of the partially ordered set of p-subgroups for an arbitrary prime p. In the case of G of Lie type in characteristic p, this poset is homotopy equivalent to the Tits building geometry and gives a Lefschetz module which is projective over $\mathbb{F}_p$.
- Webb considered more general subgroup posets than Brown and Quillen had, including some posets Ronan-Smith had studied for sporadic groups.
- Throughout our study, we will work in characteristic $p = 2$, the prime which generally provides the richest geometric structures.

- Ryba-Smith-Yoshiara showed via Webb's method that 11 sporadic groups give projective Lefschetz modules for p=2.
- More recently, Benson-Smith studied the geometry of the remaining 15 sporadic groups, and observed some interesting patterns in the 2-modular block theory.
- Of these 15 sporadics with non-projective Lefschetz modules, the Lefschetz characters already have been calculated for three of these groups: M_{12}, J_2, and HS.
- Of the remaining 12, there are 7 groups for which we currently do not have enough information to perform a full analysis.

- Ryba-Smith-Yoshiara showed via Webb's method that 11 sporadic groups give projective Lefschetz modules for $p=2$.
- More recently, Benson-Smith studied the geometry of the remaining 15 sporadic groups, and observed some interesting patterns in the 2-modular block theory.
- Of these 15 sporadics with non-projective Lefschetz modules, the Lefschetz characters already have been calculated for three of these groups: M_{12}, J_2, and HS.
- Of the remaining 12, there are 7 groups for which we currently do not have enough information to perform a full analysis.
- This work determines the Lefschetz character in terms of 2-modular irreducibles for the 5 cases which are within range of present computation: Suz, $O'N$, He, Co_3, and Ru.

- Ryba-Smith-Yoshiara showed via Webb's method that 11 sporadic groups give projective Lefschetz modules for $p=2$.
- More recently, Benson-Smith studied the geometry of the remaining 15 sporadic groups, and observed some interesting patterns in the 2-modular block theory.
- Of these 15 sporadics with non-projective Lefschetz modules, the Lefschetz characters already have been calculated for three of these groups: M_{12}, J_2, and HS.
- Of the remaining 12, there are 7 groups for which we currently do not have enough information to perform a full analysis.
- This work determines the Lefschetz character in terms of 2-modular irreducibles for the 5 cases which are within range of present computation: Suz, $O'N$, He, Co_3, and Ru.

- Ryba-Smith-Yoshiara showed via Webb's method that 11 sporadic groups give projective Lefschetz modules for p=2.
- More recently, Benson-Smith studied the geometry of the remaining 15 sporadic groups, and observed some interesting patterns in the 2-modular block theory.
- Of these 15 sporadics with non-projective Lefschetz modules, the Lefschetz characters already have been calculated for three of these groups: M_{12}, J_2, and HS.
- Of the remaining 12, there are 7 groups for which we currently do not have enough information to perform a full analysis.
- This work determines the Lefschetz character in terms of 2-modular irreducibles for the 5 cases which are within range of present computation: Suz, $O'N$, He, Co_3, and Ru.

Definitions

- For any group, we define a *2-local subgroup* to be the normalizer of a 2-group.
- A *2-local geometry* is defined as a geometry of the 2-local subgroups. We choose the geometry Benson-Smith used: for each group, the natural geometry is a simplicial complex determined by certain maximal 2-local subgroups.
- A *principal indecomposable* module is an indecomposable direct summand of the group algebra. Each principal indecomposable is the projective cover $P(\varphi)$ of a 2-modular irreducible φ.
- There are many equivalent definitions of a projective module. We define a *projective module* (in characteristic 2) as a module that can be written as a sum of projective indecomposables $P(\varphi)$.

- A *virtual module* is a formal $\mathbb{Z}$-linear combination of modules. Observe this can be written as a formal difference of two ordinary modules (i.e., modules with positive coefficients).
- Thus a virtual projective module is a $\mathbb{Z}$-linear combination (i.e., $\pm$ combination) of projective indecomposables $P(\varphi)$.
- We define $\Phi(\varphi)$ as the *character* of the module $P(\varphi)$.

- A *virtual module* is a formal $\mathbb{Z}$-linear combination of modules. Observe this can be written as a formal difference of two ordinary modules (i.e., modules with positive coefficients).
- Thus a virtual projective module is a $\mathbb{Z}$-linear combination (i.e., $\pm$ combination) of projective indecomposables $P(\varphi)$.
- We define $\Phi(\varphi)$ as the *character* of the module $P(\varphi)$.
- For a virtual module, we say it has **v-projective character** if the module has the same character as that of a virtual projective module, i.e. if its character equals some $\pm$ combination of $\Phi(\varphi_i)$. We say that a character with this property is *v-projective*.

- A *virtual module* is a formal $\mathbb{Z}$-linear combination of modules. Observe this can be written as a formal difference of two ordinary modules (i.e., modules with positive coefficients).
- Thus a virtual projective module is a $\mathbb{Z}$-linear combination (i.e., $\pm$ combination) of projective indecomposables $P(\varphi)$.
- We define $\Phi(\varphi)$ as the *character* of the module $P(\varphi)$.
- For a virtual module, we say it has **v-projective character** if the module has the same character as that of a virtual projective module, i.e. if its character equals some $\pm$ combination of $\Phi(\varphi_i)$. We say that a character with this property is *v-projective*.
- Since each group we study has order divisible by the characteristic $p = 2$, the group algebra breaks up into a direct sum of *block* components. Each principal indecomposable module $P(\varphi)$ appears in just one block.

- A *virtual module* is a formal $\mathbb{Z}$-linear combination of modules. Observe this can be written as a formal difference of two ordinary modules (i.e., modules with positive coefficients).
- Thus a virtual projective module is a $\mathbb{Z}$-linear combination (i.e., $\pm$ combination) of projective indecomposables $P(\varphi)$.
- We define $\Phi(\varphi)$ as the *character* of the module $P(\varphi)$.
- For a virtual module, we say it has **v-projective character** if the module has the same character as that of a virtual projective module, i.e. if its character equals some $\pm$ combination of $\Phi(\varphi_i)$. We say that a character with this property is *v-projective*.
- Since each group we study has order divisible by the characteristic $p = 2$, the group algebra breaks up into a direct sum of *block* components. Each principal indecomposable module $P(\varphi)$ appears in just one block.

- A *virtual module* is a formal $\mathbb{Z}$-linear combination of modules. Observe this can be written as a formal difference of two ordinary modules (i.e., modules with positive coefficients).
- Thus a virtual projective module is a $\mathbb{Z}$-linear combination (i.e., $\pm$ combination) of projective indecomposables $P(\varphi)$.
- We define $\Phi(\varphi)$ as the *character* of the module $P(\varphi)$.
- For a virtual module, we say it has **v-projective character** if the module has the same character as that of a virtual projective module, i.e. if its character equals some $\pm$ combination of $\Phi(\varphi_i)$. We say that a character with this property is *v-projective*.
- Since each group we study has order divisible by the characteristic $p = 2$, the group algebra breaks up into a direct sum of *block* components. Each principal indecomposable module $P(\varphi)$ appears in just one block.

- Let G be a group, and let χ_i be the set of irreducible complex (ordinary) characters in a 2-block of G. The *defect* of the block is the difference between the exponent of 2 in $|G|_2$ and the exponent of 2 in $\min(|\chi_i|_2)$.
- The (reduced) *Lefschetz module* of a simplicial complex Δ is the virtual module given by the alternating sum of the chain groups:

$$\widetilde{L}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i C_i(\Delta).$$

- Its degree term is the (reduced) *Euler characteristic:*

$$\widetilde{\chi}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i \dim C_i(\Delta).$$

- Let G be a group, and let χ_i be the set of irreducible complex (ordinary) characters in a 2-block of G. The *defect* of the block is the difference between the exponent of 2 in $|G|_2$ and the exponent of 2 in $\min(|\chi_i|_2)$.
- The (reduced) *Lefschetz module* of a simplicial complex Δ is the virtual module given by the alternating sum of the chain groups:

$$\widetilde{L}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i C_i(\Delta).$$

- Its degree term is the (reduced) *Euler characteristic:*

$$\widetilde{\chi}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i \dim C_i(\Delta).$$

Notation (Fixed Point Notation)

Let $g \in G$ and let Δ be a simplicial complex acted upon by G. Then we use the symbol Δ^g to denote the set of simplices in Δ fixed by g.

- The (reduced) *Lefschetz character* is the sequence of Euler characteristic values on the geometry fixed by representatives of each conjugacy class of G: $Tr(g, \widetilde{L}) = \widetilde{\chi}(\Delta^g)$.
- For convenience, we will denote the Lefschetz character of G by $\widetilde{\Lambda}_G$.
- Note that the Euler characteristic $\widetilde{\chi}_G$ is simply the degree of $\widetilde{\Lambda}_G$, i.e. the value at 1.

Notation (Fixed Point Notation)

Let $g \in G$ and let Δ be a simplicial complex acted upon by G. Then we use the symbol Δ^g to denote the set of simplices in Δ fixed by g.

- The (reduced) *Lefschetz character* is the sequence of Euler characteristic values on the geometry fixed by representatives of each conjugacy class of G: $Tr(g, \widetilde{L}) = \widetilde{\chi}(\Delta^g)$.
- For convenience, we will denote the Lefschetz character of G by $\widetilde{\Lambda}_G$.
- Note that the Euler characteristic $\widetilde{\chi}_G$ is simply the degree of $\widetilde{\Lambda}_G$, i.e. the value at 1.

Example: $L_3(2)$

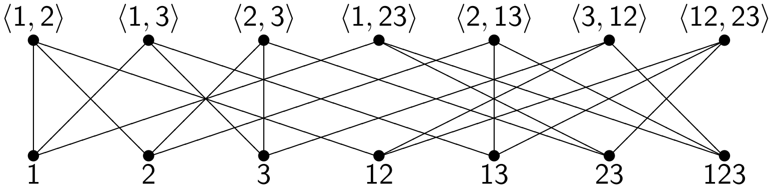
- As an introductory example, we calculate the Lefschetz character for the simple Lie type group $G = L_3(2)$ of order 168.
- In its natural geometry, "points" are conjugates of point stabilizers, "lines" are conjugates of line stabilizers, and "point-in-line pairs" are conjugates of point-line stabilizers.
- Thus the natural geometry of $L_3(2)$ is $\mathbb{P}_2(2)$ of seven points (1-spaces), seven lines (2-spaces), and 21 point-in-line pairs (which we call "flags").
- For simplicity, we rename the vectors:

  $$(1,0,0) \leftrightarrow \text{“1”}$$
  $$(0,1,0) \leftrightarrow \text{“2”}$$

  the sum $(1,1,0) \leftrightarrow$ "12", and so on.

Example: $L_3(2)$

Then the natural geometry for $L_3(2)$ can be displayed with points and lines each represented by points (C_0) and flags represented by lines (C_1):



The Lefschetz character will have a value for each conjugacy class of $L_3(2)$, of orders 1, 2, 3, 4, and 7. We choose a representative g of each conjugacy class and calculate the Euler characteristic of the subset of the geometry that is fixed by g.

Example: $L_3(2)$

- We start with the identity element 1, which fixes the entire geometry of 7 points, 7 lines, and 21 flags. So we have

$$\widetilde{\chi} = -1 + |C_0| - |C_1| \quad (1)$$
$$= -1 + (7 + 7) - 21 \quad (2)$$
$$= -8 \quad (3)$$

- The order 2 class can be represented by the element $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. We multiply on the right, so for example

$$1 \cdot g = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = 1.$$

Thus g fixes the point 1. Similarly, g also fixes points 2 and 12. Since it fixes the points 1, 2, and 12, it fixes the line $\langle 1, 2 \rangle$.

Example: $L_3(2)$

- We start with the identity element 1, which fixes the entire geometry of 7 points, 7 lines, and 21 flags. So we have

$$\begin{aligned} \widetilde{\chi} &= -1 + |C_0| - |C_1| & (1) \\ &= -1 + (7+7) - 21 & (2) \\ &= -8 & (3) \end{aligned}$$

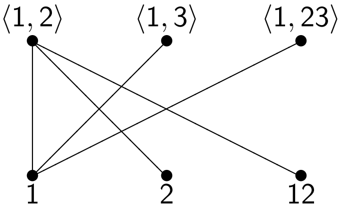
- The order 2 class can be represented by the element $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. We multiply on the right, so for example

$$1 \cdot g = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = 1.$$

  Thus g fixes the point 1. Similarly, g also fixes points 2 and 12. Since it fixes the points 1, 2, and 12, it fixes the line $\langle 1, 2 \rangle$.

Example: $L_3(2)$, $|g| = 2$

- g swaps points 3 and 13, and thus fixes the line $\langle 1, 3 \rangle$. Since g fixes point 1, and swaps point 23 with 123, it also fixes the line $\langle 1, 23 \rangle$.
- g normalizes no other lines. The geometry fixed by g is:



- Thus g fixes 3 points and 3 lines, so in the complex of the poset, $|{C_0}^g| = 6$. Since g also fixes 5 flags, we have $|{C_1}^g| = 5$.
- Thus the Lefschetz character for this conjugacy class is $Tr(g, \widetilde{L}) = \widetilde{\chi}(\Delta^g) = -1 + 6 - 5 = 0$.

Example: $L_3(2)$

- Similar calculations with a representative from each conjugacy class give us the Lefschetz character of $L_3(2)$:

Order of g	1	2	3	4	7
$\mathrm{Tr}\left(g, \widetilde{L}\right)$	-8	0	1	0	-1

- Since the Lefschetz module is actually a virtual module, we can negate the Lefschetz character so that its dimension is positive:

Order of g	1	2	3	4	7
$\mathrm{Tr}\left(g, \widetilde{L}\right)$	8	0	-1	0	1

- Experts will recognize this as the *Steinberg* character of $L_3(2)$, which is projective. In fact, we can use the Brauer atlas to see that, restricted to 2-modular characters, $\widetilde{\Lambda}_{L_3(2)} = \Phi(\varphi_4) = \varphi_4$.

Example: $L_3(2)$

- Similar calculations with a representative from each conjugacy class give us the Lefschetz character of $L_3(2)$:

Order of g	1	2	3	4	7
$\mathrm{Tr}\left(g, \widetilde{L}\right)$	-8	0	1	0	-1

- Since the Lefschetz module is actually a virtual module, we can negate the Lefschetz character so that its dimension is positive:

Order of g	1	2	3	4	7
$\mathrm{Tr}\left(g, \widetilde{L}\right)$	8	0	-1	0	1

- Experts will recognize this as the *Steinberg* character of $L_3(2)$, which is projective. In fact, we can use the Brauer atlas to see that, restricted to 2-modular characters, $\widetilde{\Lambda}_{L_3(2)} = \Phi(\varphi_4) = \varphi_4$.

Methods

- The manual calculation of $\widetilde{\Lambda}_{L_3(2)}$ would be prohibitively cumbersome to make for the much larger sporadic groups.
- Recall that the Lefschetz character is an alternating sum of chain groups. A group orbit on chains is the induced module $1_H\uparrow^G$, where H is the stabilizer of a chain in the orbit.
- So we can compute the Lefschetz module as an alternating sum of induced modules. In fact, we work with the character as an alternating sum of induced characters.

- For example, the previous manual computation for $L_3(2)$ could be written in the form:

$$-1 + 1_{H_1}\uparrow^G + 1_{H_2}\uparrow^G - 1_{H_{12}}\uparrow^G$$

where H_1, H_2, H_{12} are stabilizers of points, lines, and point-in-line pairs, respectively.

- Thus we can translate the geometric information into representation theory and use the program GAP to do much of the Lefschetz character computation for us.
- For each group G, once we compute $\widetilde{\Lambda}_G$, we will decompose it as a combination of irreducible 2-modular characters of G.

Tests for Projectivity

The Lefschetz character can give us conclusive information about the actual Lefschetz module via the following two tests.

Lemma (The p-test)

If $\widetilde{L}(\Delta)$ is projective, then $|G|_p$ divides $\widetilde{\chi}(\Delta)$.

The converse to the p-test is not true in general.

Lemma (The vanishing test)

For the module $\widetilde{L}(\Delta)$, its Lefschetz character $\widetilde{\Lambda}$ is v-projective if and only if $\widetilde{\Lambda}$ vanishes at the nontrivial 2-elements. (Brauer)

Tests for Projectivity

The Lefschetz character can give us conclusive information about the actual Lefschetz module via the following two tests.

Lemma (The p-test)

If $\widetilde{L}(\Delta)$ is projective, then $|G|_p$ divides $\widetilde{\chi}(\Delta)$.

The converse to the p-test is not true in general.

Lemma (The vanishing test)

For the module $\widetilde{L}(\Delta)$, its Lefschetz character $\widetilde{\Lambda}$ is v-projective if and only if $\widetilde{\Lambda}$ vanishes at the nontrivial 2-elements. (Brauer)

How We Use the Projectivity Tests

- The vanishing test is stronger than the *p*-test, as it can be used both to prove that a module is non-projective ("failing" the test), and to prove that the module has v-projective character ("passing" the test).
- We will use the coefficients of irreducible 2-modular characters and the decomposition matrices website to break each module into block parts.
- Using the projectivity tests, we will find which block parts of $\widetilde{\Lambda}$ give v-projective characters and which block parts give characters of non-projective modules.

Status of the Research

- Ryba-Smith-Yoshiara found 11 of the sporadic groups to be projective, and used the p-test to show that the other 15 sporadics have non-projective Lefschetz modules.
- Benson-Smith studied the remaining 15, and noticed a pattern based on a few early examples. We can formulate a refined version of their observation.

Conjecture (Refined Benson-Smith Conjecture)

All 15 sporadic groups affording non-projective Lefschetz modules have a non-projective part in a non-principal block of nonzero defect.

Status of the Research

- Ryba-Smith-Yoshiara found 11 of the sporadic groups to be projective, and used the p-test to show that the other 15 sporadics have non-projective Lefschetz modules.
- Benson-Smith studied the remaining 15, and noticed a pattern based on a few early examples. We can formulate a refined version of their observation.

Conjecture (Refined Benson-Smith Conjecture)

All 15 sporadic groups affording non-projective Lefschetz modules have a non-projective part in a non-principal block of nonzero defect.

Status of the Research

- Ryba-Smith-Yoshiara found 11 of the sporadic groups to be projective, and used the *p*-test to show that the other 15 sporadics have non-projective Lefschetz modules.
- Benson-Smith studied the remaining 15, and noticed a pattern based on a few early examples. We can formulate a refined version of their observation.

Conjecture (Refined Benson-Smith Conjecture)

All 15 sporadic groups affording non-projective Lefschetz modules have a non-projective part in a non-principal block of nonzero defect.

This Work

In this work, we will study the 5 sporadic groups Suz, $O'N$, He, Co_3, and Ru.

- For each group, we use the Dynkin-like diagram of its geometry to guide the sequence of inductions for the Lefschetz character.
- Using the 2-modular decomposition matrix, we break the Lefschetz character into block parts and examine the projectivity of each block part. We find whether the block part is non-projective or has v-projective character.
- We confirm the refined Benson-Smith conjecture for each of the five cases in our study.

This Work

In this work, we will study the 5 sporadic groups Suz, $O'N$, He, Co_3, and Ru.

- For each group, we use the Dynkin-like diagram of its geometry to guide the sequence of inductions for the Lefschetz character.
- Using the 2-modular decomposition matrix, we break the Lefschetz character into block parts and examine the projectivity of each block part. We find whether the block part is non-projective or has v-projective character.
- We confirm the refined Benson-Smith conjecture for each of the five cases in our study.

The Suzuki Group

The first sporadic group we examine is the group *Suz* of order $\mathbf{2^{13}} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, discovered by Suzuki. We show more details of this case, as later cases follow similarly.

- Since we are investigating the 2-local geometry, we study the maximal subgroups which contain a fixed Sylow 2-group, numbering them by the ranks of the centers of their largest normal 2-subgroups:

$$H_1 \cong 2_-^{1+6} \cdot U_4(2), \; H_2 \cong 2^{2+8}{:}(A_5 \times S_3), \; H_4 \cong 2^{4+6}{:}3A_6.$$

Notation

For simplicity, we use the notation $H_{12} = H_1 \cap H_2$, $H_{14} = H_1 \cap H_4$, $H_{24} = H_2 \cap H_4$, *and* $H_{124} = H_1 \cap H_2 \cap H_4$.

The Suzuki Group

The first sporadic group we examine is the group *Suz* of order $\mathbf{2^{13}} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, discovered by Suzuki. We show more details of this case, as later cases follow similarly.

- Since we are investigating the 2-local geometry, we study the maximal subgroups which contain a fixed Sylow 2-group, numbering them by the ranks of the centers of their largest normal 2-subgroups:

$$H_1 \cong 2_{-}^{1+6} \cdot U_4(2),\ H_2 \cong 2^{2+8}{:}(A_5 \times S_3),\ H_4 \cong 2^{4+6}{:}3A_6.$$

Notation

For simplicity, we use the notation $H_{12} = H_1 \cap H_2$, $H_{14} = H_1 \cap H_4$, $H_{24} = H_2 \cap H_4$, *and* $H_{124} = H_1 \cap H_2 \cap H_4$.

The Suzuki Diagram

- The common intersection H_{124} has order $2^{13} \cdot 3$. It contains a Sylow 2-subgroup with index 3.
- For the natural geometry, we use the intersection complex described by Benson-Smith. Incidence is determined by containing a common Sylow 2-group.
- The **vertices** are G-conjugates of H_1, H_2, and H_4. We call these vertices of *type* 1, 2, and 4, respectively.
- An **edge** is a pair of vertices whose groups intersect in at least a Sylow group. For example, the pair $\{H_1, H_2\}$ corresponds to $H_1 \cap H_2 = H_{12} \supseteq H_{124} \supseteq \mathrm{Syl}_2$. Similarly, we get edges from pairs of conjugates of $\{H_1, H_4\}$ and $\{H_2, H_4\}$.
- **Faces** are triples (one of each type) whose intersection contains a Sylow 2-group.

The Suzuki Diagram

- The common intersection H_{124} has order $2^{13} \cdot 3$. It contains a Sylow 2-subgroup with index 3.
- For the natural geometry, we use the intersection complex described by Benson-Smith. Incidence is determined by containing a common Sylow 2-group.
- The **vertices** are G-conjugates of H_1, H_2, and H_4. We call these vertices of *type* 1, 2, and 4, respectively.
- An **edge** is a pair of vertices whose groups intersect in at least a Sylow group. For example, the pair $\{H_1, H_2\}$ corresponds to $H_1 \cap H_2 = H_{12} \supseteq H_{124} \supseteq \mathrm{Syl}_2$. Similarly, we get edges from pairs of conjugates of $\{H_1, H_4\}$ and $\{H_2, H_4\}$.
- **Faces** are triples (one of each type) whose intersection contains a Sylow 2-group.

- Information on local structure of the geometry is encoded in the diagram:

- Removing a node from the diagram gives us the link (residue) of that vertex. For example, if we remove node 1, the diagram describes all vertices of type 2 and 4 that are incident to vertices of type 1.
- The double bonds each represent generalized quadrangles, but they are different types. The left bond is type $Sp_4(2)$, while the right bond is type $U_4(2)$.

Background on Inducing Subgroups

- We will compute the Lefschetz character by inducing the trivial representation of each stabilizing subgroup up to Suz.
- Since H_1, H_2, and H_4 are maximal subgroups of Suz, we are easily be able to induce these maximal subgroups up to Suz using GAP.
- GAP does not have the permutation characters for the intersection subgroups H_{12}, H_{14}, H_{24}, and H_{124}, however.
- Hence we will use selected residues described by the diagram to find which characters of the intersections are needed in the calculation of induced representations.

Background on Inducing Subgroups

- We will compute the Lefschetz character by inducing the trivial representation of each stabilizing subgroup up to *Suz*.
- Since H_1, H_2, and H_4 are maximal subgroups of *Suz*, we are easily be able to induce these maximal subgroups up to *Suz* using GAP.
- GAP does not have the permutation characters for the intersection subgroups H_{12}, H_{14}, H_{24}, and H_{124}, however.
- Hence we will use selected residues described by the diagram to find which characters of the intersections are needed in the calculation of induced representations.

- By the transitivity of induction, we know that, for example,

$$[1a]_{H_{12}}\uparrow^{Suz} = ([1a]_{H_{12}}\uparrow^{H_1})_{H_1}\uparrow^{Suz} .$$

- For the Lefschetz character calculation, the maximal subgroups H_1, H_2, and H_4 correspond to C_0-spaces (i.e., orbits of vertices).
- H_{12}, H_{14}, and H_{24} correspond to C_1-spaces (orbits of edges). Thus these will have a negative sign in the alternating sum.
- The triple intersection H_{124} corresponds to the C_2-space (the orbit of faces).

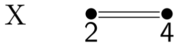
- By the transitivity of induction, we know that, for example,

$$[1a]_{H_{12}}\uparrow^{Suz} = ([1a]_{H_{12}}\uparrow^{H_1})_{H_1}\uparrow^{Suz} .$$

- For the Lefschetz character calculation, the maximal subgroups H_1, H_2, and H_4 correspond to C_0-spaces (i.e., orbits of vertices).
- H_{12}, H_{14}, and H_{24} correspond to C_1-spaces (orbits of edges). Thus these will have a negative sign in the alternating sum.
- The triple intersection H_{124} corresponds to the C_2-space (the orbit of faces).

Inducing H_{12} up to H_1

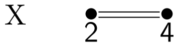
- We use the diagram of $\text{Res}(H_1)$ to see what subgroup of $U_4(2)$ to induce from:



- As given in Benson-Smith, the residue of H_1 is of type $U_4(2) \cong \Omega_6^-(2)$, the 6-dimensional orthogonal group of minus type. This group acts on a 6-space over $\mathbb{F}_2$.
- The 6-space is realized as $O_2(H_1)/Z(O_2(H_1))$. Then H_{12} stabilizes the image of $Z(O_2(H_2))$, which is an isotropic 1-space in the 6-space quotient.

Inducing H_{12} up to H_1

- We use the diagram of $\mathrm{Res}(H_1)$ to see what subgroup of $U_4(2)$ to induce from:



- As given in Benson-Smith, the residue of H_1 is of type $U_4(2) \cong \Omega_6^-(2)$, the 6-dimensional orthogonal group of minus type. This group acts on a 6-space over $\mathbb{F}_2$.
- The 6-space is realized as $O_2(H_1)/Z(O_2(H_1))$. Then H_{12} stabilizes the image of $Z(O_2(H_2))$, which is an isotropic 1-space in the 6-space quotient.

- Using the Atlas terminology, a 1-space stabilized by a Sylow 2-group in the $\Omega_6^-(2)$ geometry is an "isotropic point." In the $U_4(2)$ entry of the Atlas, we see that "isotropic point" corresponds to $2^4{:}A_5$.
- Therefore, inducing H_{12} up to H_1 will have the same permutation character as inducing $2^4{:}A_5$ up to $U_4(2)$.
- The Atlas gives this character to be $1a + 6a + 20a$. So we have

$$[1a]_{H_{12}}\uparrow^{Suz} = ([1a]_{H_{12}}\uparrow^{H_1})_{H_1}\uparrow^{Suz} = [1a + 6a + 20a]_{H_1}\uparrow^{Suz} .$$

- Using the Atlas terminology, a 1-space stabilized by a Sylow 2-group in the $\Omega_6^-(2)$ geometry is an "isotropic point." In the $U_4(2)$ entry of the Atlas, we see that "isotropic point" corresponds to $2^4{:}A_5$.
- Therefore, inducing H_{12} up to H_1 will have the same permutation character as inducing $2^4{:}A_5$ up to $U_4(2)$.
- The Atlas gives this character to be $1a + 6a + 20a$. So we have

$$[1a]_{H_{12}}\uparrow^{Suz} = ([1a]_{H_{12}}\uparrow^{H_1})_{H_1}\uparrow^{Suz} = [1a + 6a + 20a]_{H_1}\uparrow^{Suz} .$$

Summary of Inductions

Similarly, we find the following characters to induce up to Suz.

$$
\begin{aligned}
[1a]_{H_{124}}\uparrow^{Suz} &= \ \ [1a+6a+20aa+24a+64a]_{H_1}\uparrow^{Suz}\\
-[1a]_{H_{12}}\uparrow^{Suz} &= -[1a+6a+20a]_{H_1}\uparrow^{Suz}\\
-[1a]_{H_{14}}\uparrow^{Suz} &= -[1a+20a+24a]_{H_1}\uparrow^{Suz}\\
[1a]_{H_1}\uparrow^{Suz} &= \ \ [1a]_{H_1}\uparrow^{Suz}\\
\hline
-[1a]_{H_{24}}\uparrow^{Suz} &= -[1a+4a]_{H_2}\uparrow^{Suz}\\
[1a]_{H_2}\uparrow^{Suz} &= \ \ [1a]_{H_2}\uparrow^{Suz}\\
\hline
[1a]_{H_4}\uparrow^{Suz} &= \ \ [1a]_{H_4}\uparrow^{Suz}
\end{aligned}
$$

With some quick cancellation, we find

$$\widetilde{\Lambda}_{Suz} = [64a]_{H_1}\uparrow^{Suz} - [4a]_{H_2}\uparrow^{Suz} + [1a]_{H_4}\uparrow^{Suz} - 1.$$

The Lefschetz Character Calculation

We now have the Lefschetz character in a form that GAP can compute. With columns **indexed by the conjugacy classes** of *Suz*, we express the Lefschetz character as:

- $\widetilde{\Lambda}_{Suz} = [\overbrace{4189184}^{2^{10}\cdot 4091}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{\mathbf{-64}}$, 3968, -352, -73, 0, 0, 0, 8, -16, 9, 0, 0, 0, 0, -1, -1, 0, 0, 0, 2, 2, 0, 1, -1, 0, 0, 0, -1, 0, -1, -1, -1, 2, 2, 3, 0, 0, 0, -1, -1, 0].
- Note that the degree $\widetilde{\chi}_{Suz} = 2^{10} \cdot 4091$ is not divisible by the 2-part of the group order $|Suz|_2 = 2^{13}$, so the p-test fails.
- The vanishing test also fails because of the -64 value at class 2B.
- Either of these tells us that $\widetilde{L}_{Suz}(\Delta)$ is non-projective.

The Lefschetz Character Calculation

We now have the Lefschetz character in a form that GAP can compute. With columns **indexed by the conjugacy classes** of *Suz*, we express the Lefschetz character as:

- $\widetilde{\Lambda}_{Suz} = [\overbrace{4189184}^{2^{10}\cdot 4091}, \overset{\textbf{2A}}{\mathbf{0}}, \overset{\textbf{2B}}{\mathbf{-64}}$, 3968, -352, -73, 0, 0, 0, 8, -16, 9, 0, 0, 0, 0, -1, -1, 0, 0, 0, 2, 2, 0, 1, -1, 0, 0, 0, -1, 0, -1, -1, -1, 2, 2, 3, 0, 0, 0, -1, -1, 0].
- Note that the degree $\widetilde{\chi}_{Suz} = 2^{10} \cdot 4091$ is not divisible by the 2-part of the group order $|Suz|_2 = 2^{13}$, so the p-test fails.
- The vanishing test also fails because of the -64 value at class 2B.
- Either of these tells us that $\widetilde{L}_{Suz}(\Delta)$ is non-projective.

Block Theory

The decomposition matrices website tells us that there are just two 2-modular blocks for *Suz*: the principal block and a much smaller block of defect 3.

Notation

Let G be a group, and let B be a particular block part of the 2-modular decomposition matrix of G. Then we define $\widetilde{\Lambda}_B^G$ to be $\widetilde{\Lambda}_G$ restricted to characters in block B.

We use the coefficients of irreducible characters along with the decomposition matrices to break down $\widetilde{\Lambda}_{pr}^{Suz}$ and $\widetilde{\Lambda}_{b2}^{Suz}$ into v-projective and non-projective parts:

- $\widetilde{\Lambda}_{pr}^{Suz} = \Phi(\varphi_{13}) + \Phi(\varphi_{14})$.
- $\widetilde{\Lambda}_{b2}^{Suz} = \Phi(\varphi_{15}) + \Phi(\varphi_{16}) + \chi_{43}$.

Block Theory

The decomposition matrices website tells us that there are just two 2-modular blocks for *Suz*: the principal block and a much smaller block of defect 3.

Notation

Let G be a group, and let B be a particular block part of the 2-modular decomposition matrix of G. Then we define $\widetilde{\Lambda}_B^G$ to be $\widetilde{\Lambda}_G$ restricted to characters in block B.

We use the coefficients of irreducible characters along with the decomposition matrices to break down $\widetilde{\Lambda}_{pr}^{Suz}$ and $\widetilde{\Lambda}_{b2}^{Suz}$ into v-projective and non-projective parts:

- $\widetilde{\Lambda}_{pr}^{Suz} = \Phi(\varphi_{13}) + \Phi(\varphi_{14})$. So this is v-projective.
- $\widetilde{\Lambda}_{b2}^{Suz} = \Phi(\varphi_{15}) + \Phi(\varphi_{16}) + \chi_{43}$.

Block Theory

The decomposition matrices website tells us that there are just two 2-modular blocks for *Suz*: the principal block and a much smaller block of defect 3.

Notation

Let G be a group, and let B be a particular block part of the 2-modular decomposition matrix of G. Then we define $\widetilde{\Lambda}_B^G$ to be $\widetilde{\Lambda}_G$ restricted to characters in block B.

We use the coefficients of irreducible characters along with the decomposition matrices to break down $\widetilde{\Lambda}_{pr}^{Suz}$ and $\widetilde{\Lambda}_{b2}^{Suz}$ into v-projective and non-projective parts:

- $\widetilde{\Lambda}_{pr}^{Suz} = \Phi(\varphi_{13}) + \Phi(\varphi_{14})$. So this is v-projective.
- $\widetilde{\Lambda}_{b2}^{Suz} = \Phi(\varphi_{15}) + \Phi(\varphi_{16}) + \chi_{43}$. This is the closest we could come to writing $\widetilde{\Lambda}_{b2}^{Suz}$ as a combination of $\Phi(\varphi_i)$'s.

Block Theory

The decomposition matrices website tells us that there are just two 2-modular blocks for *Suz*: the principal block and a much smaller block of defect 3.

Notation

Let G be a group, and let B be a particular block part of the 2-modular decomposition matrix of G. Then we define $\widetilde{\Lambda}_B^G$ *to be* $\widetilde{\Lambda}_G$ *restricted to characters in block B.*

We use the coefficients of irreducible characters along with the decomposition matrices to break down $\widetilde{\Lambda}_{pr}^{Suz}$ and $\widetilde{\Lambda}_{b2}^{Suz}$ into v-projective and non-projective parts:

- $\widetilde{\Lambda}_{pr}^{Suz} = \Phi(\varphi_{13}) + \Phi(\varphi_{14})$. So this is v-projective.
- $\widetilde{\Lambda}_{b2}^{Suz} = \Phi(\varphi_{15}) + \Phi(\varphi_{16}) + \chi_{43}$. This is the closest we could come to writing $\widetilde{\Lambda}_{b2}^{Suz}$ as a combination of $\Phi(\varphi_i)$'s.

Block Theory

The decomposition matrices website tells us that there are just two 2-modular blocks for *Suz*: the principal block and a much smaller block of defect 3.

Notation

Let G be a group, and let B be a particular block part of the 2-modular decomposition matrix of G. Then we define $\widetilde{\Lambda}_B^G$ to be $\widetilde{\Lambda}_G$ restricted to characters in block B.

We use the coefficients of irreducible characters along with the decomposition matrices to break down $\widetilde{\Lambda}_{pr}^{Suz}$ and $\widetilde{\Lambda}_{b2}^{Suz}$ into v-projective and non-projective parts:

- $\widetilde{\Lambda}_{pr}^{Suz} = \Phi(\varphi_{13}) + \Phi(\varphi_{14})$. So this is v-projective.
- $\widetilde{\Lambda}_{b2}^{Suz} = \Phi(\varphi_{15}) + \Phi(\varphi_{16}) + \chi_{43}$. This is the closest we could

come to writing $\widetilde{\Lambda}_{b2}^{Suz}$ as a combination of $\Phi(\varphi_i)$'s.

Character Values Separated into Blocks

We can divide up the original Lefschetz character into its contributions from each block. Indexed by the conjugacy classes of *Suz*, we have:

- $\widetilde{\Lambda}_{pr}^{Suz}$=[$\overbrace{2834432}^{2^{14}\cdot 173}$, $\overset{2A}{\mathbf{0}}$, $\overset{2B}{\mathbf{0}}$, 512, -352, -64, 0, 0, 0, 0, 32, 32, 0, 0, 0, 0, 0, -8, 0, 0, 0, -4, -4, 0, 0, -4, 0, 0, 0, 0, 0, 3, 3, 0, -4, -4, 2, 0, 0, 0, 1, 1, 0].
- $\widetilde{\Lambda}_{b2}^{Suz}$=[$\overbrace{1354752}^{2^{10}\cdot 1323}$, $\overset{2A}{\mathbf{0}}$, $\overset{2B}{-\mathbf{64}}$, 3456, 0, -9, 0, 0, 0, 8, -48, -23, 0, 0, 0, 0, -1, 7, 0, 0, 0, 6, 6, 0, 1, 3, 0, 0, 0, -1, 0, -4, -4, -1, 6, 6, 1, 0, 0, 0, -2, -2, 0].

Finally, we mention that we have verified the refined Benson-Smith conjecture that the non-projective part appears in a non-principal block of positive defect.

Character Values Separated into Blocks

We can divide up the original Lefschetz character into its contributions from each block. Indexed by the conjugacy classes of *Suz*, we have:

- $\widetilde{\Lambda}_{pr}^{Suz}$=[$\overbrace{2834432}^{2^{14}\cdot 173}$, $\overset{\textbf{2A}}{\mathbf{0}}$, $\overset{\textbf{2B}}{\mathbf{0}}$, 512, -352, -64, 0, 0, 0, 0, 32, 32, 0, 0, 0, 0, 0, -8, 0, 0, 0, -4, -4, 0, 0, -4, 0, 0, 0, 0, 0, 3, 3, 0, -4, -4, 2, 0, 0, 0, 1, 1, 0].
- $\widetilde{\Lambda}_{b2}^{Suz}$=[$\overbrace{1354752}^{2^{10}\cdot 1323}$, $\overset{\textbf{2A}}{\mathbf{0}}$, $\overset{\textbf{2B}}{-\mathbf{64}}$, 3456, 0, -9, 0, 0, 0, 8, -48, -23, 0, 0, 0, 0, -1, 7, 0, 0, 0, 6, 6, 0, 1, 3, 0, 0, 0, -1, 0, -4, -4, -1, 6, 6, 1, 0, 0, 0, -2, -2, 0].

Finally, we mention that we have verified the refined Benson-Smith conjecture that the non-projective part appears in a non-principal block of positive defect.

Character Values Separated into Blocks

We can divide up the original Lefschetz character into its contributions from each block. Indexed by the conjugacy classes of *Suz*, we have:

- $\widetilde{\Lambda}_{pr}^{Suz} = [\, \overbrace{2834432}^{2^{14}\cdot 173}, \underset{}{\overset{\underline{\textbf{2A}}}{\mathbf{0}}}, \overset{\underline{\textbf{2B}}}{\mathbf{0}}$, 512, -352, -64, 0, 0, 0, 0, 32, 32, 0, 0, 0, 0, 0, -8, 0, 0, 0, -4, -4, 0, 0, -4, 0, 0, 0, 0, 0, 3, 3, 0, -4, -4, 2, 0, 0, 0, 1, 1, 0].
- $\widetilde{\Lambda}_{b2}^{Suz} = [\, \overbrace{1354752}^{2^{10}\cdot 1323}, \overset{\underline{\textbf{2A}}}{\mathbf{0}}, \overset{\underline{\textbf{2B}}}{-\mathbf{64}}$, 3456, 0, -9, 0, 0, 0, 8, -48, -23, 0, 0, 0, 0, -1, 7, 0, 0, 0, 6, 6, 0, 1, 3, 0, 0, 0, -1, 0, -4, -4, -1, 6, 6, 1, 0, 0, 0, -2, -2, 0].

Finally, we mention that we have verified the refined Benson-Smith conjecture that the non-projective part appears in a non-principal block of positive defect.

$O'N$

This group of order $|O'N| = \mathbf{2^9} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$ was discovered by O'Nan. It has subgroups:

$$H_1 \cong 4_2 \cdot L_3(4){:}2_1 \text{ and } H_3 \cong 4^3 \cdot L_3(2).$$

Using a similar process as with *Suz*, we find the Lefschetz character indexed by irreducible complex characters:

- $\widetilde{\Lambda}_{O'N} = [\overbrace{254294272}^{2^8 \cdot 993337}, \overset{\underline{\mathbf{2A}}}{\mathbf{8960}}$, -44, 0, 0, -8, -4, -48, -13, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1].

$O'N$ Block Theory

In addition to the principal block, there is a block of defect 3. Separating the character values corresponding to characters from the principal block, we find:

- $\widetilde{\Lambda}_{pr}^{O'N} = [\overbrace{147299328}^{2^{10}\cdot 143847}, \underset{}{\overset{\mathbf{2A}}{\mathbf{7168}}}$, -2736, 0, 0, 708, -32, 2388, 344, 0, 0, 28, 176, 0, -28, -156, -156, 0, 0, 0, 0, 80, 80, 80, 0, 0, 0, 0, -520, -520].
- This passes the p-test, but interestingly enough, fails the vanishing test!
- So unlike with Suzuki, there is a non-projective component in the principal block.
- $O'N$ is the first known example with this property discovered "in nature," i.e. in the sense of occurring in a natural geometry.

$O'N$ Block Theory

In addition to the principal block, there is a block of defect 3. Separating the character values corresponding to characters from the principal block, we find:

- $\widetilde{\Lambda}_{pr}^{O'N} = [\overbrace{147299328}^{2^{10}\cdot 143847}, \overset{\mathbf{2A}}{\mathbf{7168}}$, -2736, 0, 0, 708, -32, 2388, 344, 0, 0, 28, 176, 0, -28, -156, -156, 0, 0, 0, 0, 80, 80, 80, 0, 0, 0, 0, -520, -520].
- This passes the *p*-test, but interestingly enough, fails the vanishing test!
- So unlike with Suzuki, there is a non-projective component in the principal block.
- $O'N$ is the first known example with this property discovered "in nature," i.e. in the sense of occurring in a natural geometry.

$O'N$ Block Theory

For block 2, we find

- $\widetilde{\Lambda}_{b2}^{O'N} = [\overbrace{1386240}^{2^8 \cdot 5415}, \overset{\mathbf{2A}}{\mathbf{1792}}, 1140, 0, 0, 60, 28, 324, -12, 0, 0, -28, -24, 0, 28, 60, 60, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -18, -18\ ].$

Either test shows us that this part is non-projective, verifying the refined Benson-Smith conjecture.

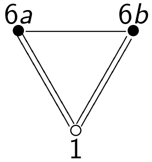
- There are five blocks of defect 0 which are not very suggestive for our study.

He

Held discovered this group of order $|He| = \mathbf{2^{10}} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$.

$$H_1 \cong 2_+^{1+6}.L_3(2),\ H_{6a} \cong 2^6{:}3{}^{\cdot}S_6,\ H_{6b} \cong 2^6{:}3{}^{\cdot}S_6$$

are its subgroups, and its natural geometry is described by the diagram:



- $\widetilde{\Lambda}_{He} = [\ \overbrace{1120384}^{2^7\cdot 8753}, \overset{\mathbf{2A}}{\mathbf{3296}}, \overset{\mathbf{2B}}{\mathbf{448}}$, 118, -29, 48, 32, 16, 9, 2, -5, -1, -1, -1, 6, 6, 0, 1, 0, -1, -1, -1, 0, 0, 3, -1, -1, -1, -1, -1, -1, -1, -1].

He

Restricted to the principal block, we have:

- $\widetilde{\Lambda}^{He}_{pr} = [\overbrace{802176}^{2^7 \cdot 6267}, \overset{\mathbf{2A}}{\mathbf{2656}}, \overset{\mathbf{2B}}{\mathbf{448}}, 348, -69, 32, 32, 16, -49, -8, -5, -3, -3, 46, -10, -10, 0, 11, 2, -1, -11, -11, 0, 0, -7, 14, 14, -2, -2, -6, -6, -3, -3\ ].$

Thus as with $O'N$, the principal block part of $\widetilde{\Lambda}_{He}$ is non-projective.

For block 2 (of defect 3), we have:

- $\widetilde{\Lambda}^{He}_{b2} = [\overbrace{60160}^{2^8 \cdot 235}, \overset{\mathbf{2A}}{\mathbf{640}}, \overset{\mathbf{2B}}{\mathbf{0}}, 58, 40, 16, 0, 0, 10, 10, 0, 2, 2, 37, 16, 16, 0, -10, -2, 0, 10, 10, 0, 0, -2, -3, -3, -5, -5, 5, 5, 2, 2\ ]$

This block part is also non-projective, verifying the refined Benson-Smith conjecture.

- There are two blocks of defect 0 which are not very suggestive for our study.

He

Restricted to the principal block, we have:

- $\widetilde{\Lambda}^{He}_{pr} = [\overbrace{802176}^{2^7 \cdot 6267}, \overset{\textbf{2A}}{\mathbf{2656}}, \overset{\textbf{2B}}{\mathbf{448}}, 348, -69, 32, 32, 16, -49, -8, -5, -3, -3, 46, -10, -10, 0, 11, 2, -1, -11, -11, 0, 0, -7, 14, 14, -2, -2, -6, -6, -3, -3\,].$

Thus as with $O'N$, the principal block part of $\widetilde{\Lambda}_{He}$ is non-projective. For block 2 (of defect 3), we have:

- $\widetilde{\Lambda}^{He}_{b2} = [\overbrace{60160}^{2^8 \cdot 235}, \overset{\textbf{2A}}{\mathbf{640}}, \overset{\textbf{2B}}{\mathbf{0}}, 58, 40, 16, 0, 0, 10, 10, 0, 2, 2, 37, 16, 16, 0, -10, -2, 0, 10, 10, 0, 0, -2, -3, -3, -5, -5, 5, 5, 2, 2\,]$

This block part is also non-projective, verifying the refined Benson-Smith conjecture.

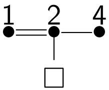
- There are two blocks of defect 0 which are not very suggestive for our study.

Co_3

This group of order $|Co_3| = \mathbf{2^{10}} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ was discovered by Conway. Its subgroups are:

$$H_1 \cong 2^{\cdot} Sp_6(2),\; H_2 \cong 2^{2}.\,[2^7.\,3^2].\,S_3,\; H_4 \cong 2^{4\cdot} A_8,$$

and its natural geometry is described by the diagram:



- $\widetilde{\Lambda}_{Co_3} = [\, \overbrace{50378624}^{2^7\cdot 393583}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{\overline{-\mathbf{496}}}$, -2080, -784, 125, 0, 0, 24, 19, 0, 0, 0, 8, 5, 2, 0, 0, 0, 8, -1, 0, -1, -1, -1, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, -1, -1, -1, -1, 0, 0, 0].

Co_3 Principal Block

This time there are three blocks, all of nonzero defect. For the principal block, we have:

- $\widetilde{\Lambda}_{pr}^{Co_3} = [\ \overbrace{34263040}^{2^{12}\cdot 8365}, \underset{}{\overset{\mathbf{2A}}{\mathbf{0}}}, \overset{\mathbf{2B}}{\mathbf{0}}$, -11840, 544, -512, 0, 0, 440, -40, 0, 0, 0, 0, 0, -56, 0, 0, 0, 28, 106, 0, 0, 20, 20, 0, 0, 0, 0, -40, 44, 0, 0, 0, -8, 0, 0, -37, -37, 0, 0, 0].

This passes both tests, and in fact we can use the decomposition matrix to see that it has a v-projective character:

- $\widetilde{\Lambda}_{pr}^{Co_3} = \Phi(\varphi_9) + \Phi(\varphi_{10}) + 6\Phi(\varphi_{12}) + 8\Phi(\varphi_{14})$.

Co_3 Principal Block

This time there are three blocks, all of nonzero defect. For the principal block, we have:

- $\widetilde{\Lambda}_{pr}^{Co_3} = [\ \overbrace{34263040}^{2^{12}\cdot 8365}, \underset{\mathbf{0}}{\underline{\mathbf{2A}}}, \underset{\mathbf{0}}{\underline{\mathbf{2B}}}$, -11840, 544, -512, 0, 0, 440, -40, 0, 0, 0, 0, 0, -56, 0, 0, 0, 28, 106, 0, 0, 20, 20, 0, 0, 0, 0, -40, 44, 0, 0, 0, -8, 0, 0, -37, -37, 0, 0, 0].

This passes both tests, and in fact we can use the decomposition matrix to see that it has a v-projective character:

- $\widetilde{\Lambda}_{pr}^{Co_3} = \Phi(\varphi_9) + \Phi(\varphi_{10}) + 6\Phi(\varphi_{12}) + 8\Phi(\varphi_{14})$.

Co_3 Blocks 2 and 3

For block 2 (of defect 3), we have:

- $\widetilde{\Lambda}_{b2}^{Co_3} = [\ \overbrace{13006720}^{2^7 \cdot 101615}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{-\mathbf{496}}$, 11296, -2384, 445, 0, 0, -80, 155, 0, 0, 0, 8, 5, 34, 0, 0, 0, 4, -83, 0, -1, -21, -21, 0, 0, 0, 0, 16, -19, 0, 0, 0, -17, -1, -1, 36, 36, 0, 0, 0].

This fails either test, so is non-projective, verifying the refined Benson-Smith conjecture.

This time we have a third block of defect 1:

- $\widetilde{\Lambda}_{b3}^{Co_3} = [\ \overbrace{3108864}^{2^{12} \cdot 759}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{\mathbf{0}}$, -1536, 1056, 192, 0, 0, -336, -96, 0, 0, 0, 0, 0, 24, 0, 0, 0, -24, -24, 0, 0, 0, 0, 0, 0, 0, 0, 24, -24, 0, 0, 0, 24, 0, 0, 0, 0, 0, 0, 0].

This block part does have v-projective character:

- $\widetilde{\Lambda}_{b3}^{Co_3} = 12\Phi(\varphi_{15})$.

Co_3 Blocks 2 and 3

For block 2 (of defect 3), we have:

- $\widetilde{\Lambda}_{b2}^{Co_3} = [\ \overbrace{13006720}^{2^7 \cdot 101615}, \overset{\textbf{2A}}{\mathbf{0}}, \overset{\textbf{2B}}{-\mathbf{496}}$, 11296, -2384, 445, 0, 0, -80, 155, 0, 0, 0, 8, 5, 34, 0, 0, 0, 4, -83, 0, -1, -21, -21, 0, 0, 0, 0, 16, -19, 0, 0, 0, -17, -1, -1, 36, 36, 0, 0, 0].

This fails either test, so is non-projective, verifying the refined Benson-Smith conjecture.

This time we have a third block of defect 1:

- $\widetilde{\Lambda}_{b3}^{Co_3} = [\ \overbrace{3108864}^{2^{12} \cdot 759}, \overset{\textbf{2A}}{\mathbf{0}}, \overset{\textbf{2B}}{\mathbf{0}}$, -1536, 1056, 192, 0, 0, -336, -96, 0, 0, 0, 0, 0, 24, 0, 0, 0, -24, -24, 0, 0, 0, 0, 0, 0, 0, 0, 24, -24, 0, 0, 0, 24, 0, 0, 0, 0, 0, 0, 0].

This block part does have v-projective character:

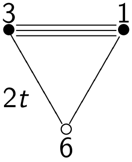
- $\widetilde{\Lambda}_{b3}^{Co_3} = 12\Phi(\varphi_{15})$.

Ru

This group of order $|Ru| = \mathbf{2^{14}} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$ was discovered by Rudvalis. It has subgroups:

$$H_1 \cong 2 \cdot 2^{4+6}{:}\, S_5,\ H_3 \cong 2^{3+8}{:}\, L_3(2),\ H_6 \cong 2^6\, G_2(2).$$

Its diagram is:



- $\widetilde{\Lambda}_{Ru} = [\ \overbrace{10113024}^{2^{12}\cdot 2469},\ \overset{\mathbf{2A}}{\mathbf{0}},\ \overset{\mathbf{2B}}{\mathbf{64}}$, -96, 0, 0, 0, 0, 24, -1, 0, 5, 0, 0, 0, 0, -1, 0, 0, -1, 1, 1, 1, -1, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, -1].

Ru Block Theory

There are only two blocks. For the principal block, we find:

- $\widetilde{\Lambda}_{pr}^{Ru} = [\ \overbrace{6881280}^{2^{16}\cdot 105}, \overset{\underline{\mathbf{2A}}}{\mathbf{0}}, \overset{\underline{\mathbf{2B}}}{\mathbf{0}}$, -48, 0, 0, 0, 0, 280, 80, 0, 28, 0, 0, 0, 0, 0, 0, 0, -36, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -14, -14].

This passes both tests, and indeed we find $\widetilde{\Lambda}_{pr}^{Ru} = \Phi(\varphi_5) + \Phi(\varphi_8)$. Thus this part has v-projective character.

For block 2 (of defect 2), we have:

- $\widetilde{\Lambda}_{b2}^{Ru} = [\ \overbrace{3231744}^{2^{12}\cdot 789}, \overset{\underline{\mathbf{2A}}}{\mathbf{0}}, \overset{\underline{\mathbf{2B}}}{\mathbf{64}}$, -48, 0, 0, 0, 0, -256, -81, 0, -23, 0, 0, 0, 0, -1, 0, 0, 35, 1, 1, 1, -3, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, 13, 13].

This fails either test, and hence is non-projective, as expected by the refined Benson-Smith conjecture.

Ru Block Theory

There are only two blocks. For the principal block, we find:

- $\widetilde{\Lambda}_{pr}^{Ru} = [\ \overbrace{6881280}^{2^{16}\cdot 105}, \overset{\textbf{2A}}{\mathbf{0}}, \overset{\textbf{2B}}{\mathbf{0}}$, -48, 0, 0, 0, 0, 280, 80, 0, 28, 0, 0, 0, 0, 0, 0, 0, -36, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -14, -14].

This passes both tests, and indeed we find $\widetilde{\Lambda}_{pr}^{Ru} = \Phi(\varphi_5) + \Phi(\varphi_8)$.
Thus this part has v-projective character.
For block 2 (of defect 2), we have:

- $\widetilde{\Lambda}_{b2}^{Ru} = [\ \overbrace{3231744}^{2^{12}\cdot 789}, \overset{\textbf{2A}}{\mathbf{0}}, \overset{\textbf{2B}}{\mathbf{64}}$, -48, 0, 0, 0, 0, -256, -81, 0, -23, 0, 0, 0, 0, -1, 0, 0, 35, 1, 1, 1, -3, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, 13, 13].

This fails either test, and hence is non-projective, as expected by the refined Benson-Smith conjecture.

Summary

Of the 26 sporadic groups, 19 of them can now be categorized according to projectivity of their Lefschetz modules for their 2-local geometries.

Class	**Sporadic Groups**
I: Lefschetz Module Projective	M_{11}, J_1, M_{22}, M_{23}, J_3, M_{24}, M^cL, Co_2, Ly, J_4, Th
II: Principal Block Part V-Projective	M_{12}, J_2, HS, **Suz**, $\mathbf{Co_3}$, **Ru**
III: Principal Block Part Non-Projective	**O'N**, **He**

- The 2-modular irreducibles for Co_1, Fi_{22}, Fi_{23}, Fi'_{24}, HN, B, and M are not known for $p = 2$. These groups have yet to be classified.
- We know by the p-test that their Lefschetz modules are non-projective, so they will be in Class II or Class III.

Summary

Of the 26 sporadic groups, 19 of them can now be categorized according to projectivity of their Lefschetz modules for their 2-local geometries.

Class	Sporadic Groups
I: Lefschetz Module Projective	M_{11}, J_1, M_{22}, M_{23}, J_3, M_{24}, M^cL, Co_2, Ly, J_4, Th
II: Principal Block Part V-Projective	M_{12}, J_2, HS, **Suz**, $\mathbf{Co_3}$, **Ru**
III: Principal Block Part Non-Projective	**O′N**, **He**

- The 2-modular irreducibles for Co_1, Fi_{22}, Fi_{23}, Fi'_{24}, HN, B, and M are not known for $p = 2$. These groups have yet to be classified.
- We know by the p-test that their Lefschetz modules are non-projective, so they will be in Class II or Class III.

Future Directions

There are several directions to pursue from this point.

- Examine patterns to see what separates the 8 sporadics with non-projective Lefschetz modules from the 11 with projective Lefschetz modules.
- Compare the Class III groups $O'N$ and He to the Class II groups by studying each non-projective block part.
- Compute the Lefschetz characters for the remaining seven sporadic groups. Divide the complex characters into blocks, and use the vanishing test to categorize them according to their block parts.
- Use a different geometry for each group, perhaps constructing natural geometries with $p \neq 2$.

□