ON NON-PROJECTIVE BLOCK COMPONENTS OF LEFSCHETZ CHARACTERS FOR SPORADIC GEOMETRIES

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Abstract. This work examines the possible projectivity of 2-modular block parts of non-projective Lefschetz characters over 2-local geometries of several sporadic groups. Previously known results on M_{12}, J_2, and HS are mentioned for completeness. The main new results are on the sporadic groups Suz, Co_3, Ru, $O'N$, and He. For each group, the Lefschetz character is calculated, and its 2-modular block parts are examined for projectivity. In each case it is confirmed that a non-principal block part contains a non-projective summand. The case of $O'N$ is additionally found to have a non-projective summand in its principal block part. Nineteen of the sporadic groups (including many previously known cases) are categorized into three classes based on projectivity properties of their Lefschetz characters.

1. Introduction

Of the 26 sporadic groups, 11 have 2-local geometries with mod 2 projective Lefschetz modules, and most of these have been calculated previously [RSY90]. Of the remaining 15 with non-projective Lefschetz modules, the Lefschetz characters in terms of irreducibles already have been calculated for three of these groups. On the other hand, there are 7 of the remaining 12 sporadics whose 2-modular irreducibles are not yet known. Thus there remain 5 sporadic groups with non-projective Lefschetz modules whose 2-modular irreducibles are known [Bre99], but whose Lefschetz characters in characteristic 2 have not yet been discussed in the literature.

In this work, we calculate the Lefschetz characters and their 2-modular block decompositions for these five sporadic groups, namely Suz, $O'N$, He, Co_3, and Ru. We find that the Lefschetz character for $O'N$ has a *non*-projective constituent in its principal block part, while the principal block parts for the other four groups each has the same character as a projective module.

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2. Notation and Assumed Results

Definitions. The (reduced) *Lefschetz module* of a complex Δ with action by a group G is the virtual $\mathbb{Z}G$-module given by the alternating sum of the chain groups (cf. [RSY90, p. 281]). Its degree term is the (reduced) *Euler characteristic*: $\widetilde{\chi}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i \dim C_i(\Delta)$. We may reduce this module mod 2, and then analyze it using the 2-modular representation theory of G.

Definition 2.1 (Lefschetz character). *The* Lefschetz character *of the natural 2-local geometry*[1] Δ *of* G *is the sequence of Euler characteristic values of the subgeometries fixed by representatives* g *of each conjugacy class of* G: $Tr(g, \widetilde{L}) = \widetilde{\chi}(\Delta^g)$. *We denote the Lefschetz character of* G *by* $\widetilde{\Lambda}_G$, *or just* $\widetilde{\Lambda}$ *when the group is unambiguous.*

Let B *be a particular 2-modular block for* G. *Then we define* $\widetilde{\Lambda}_B^G$ *to be the component of* $\widetilde{\Lambda}_G$ *coming from Brauer characters in* B.

Definition 2.2 (Projective). *We define* $\Phi(\varphi_i)$ *as the* character *of the projective cover* $P(\varphi_i)$ *of the 2-modular irreducible Brauer character* φ_i. *We say a module has* projective character *if its character can be written as a sum of* $\Phi(\varphi_i)$.

Definition 2.3 (Virtual projective). *A* virtual projective module *is a* $\mathbb{Z}$-*linear combination of projective indecomposables* $P(\varphi)$. *We say a virtual module has* v-projective character *if the module has the same character as that of a virtual projective module, i.e. if its character equals some* $\mathbb{Z}$-*linear combination of* $\Phi(\varphi_i)$. *When describing the character itself with this property, we say the character is* v-projective. *We emphasize that if a module has v-projective character, that does not necessarily imply that the module is a virtual projective module.*

Tests for projectivity. We use two standard properties of projective modules below. For each of these, the contrapositive presents a quick check for failure of a Lefschetz module to be projective. The second condition is stronger.

Lemma 2.4 (The p-test). *If* $\widetilde{L}(\Delta)$ *is projective, then* $|G|_p$ *divides* $\widetilde{\chi}(\Delta)$.

Lemma 2.5 (The vanishing test). [Fei82, p.148] *For the module* $\widetilde{L}(\Delta)$, *its Lefschetz character* $\widetilde{\Lambda}$ *is v-projective if and only if* $\widetilde{\Lambda}$ *vanishes at the nontrivial 2-elements.*

If a Lefschetz character passes the vanishing test, this indicates that the character is v-projective, i.e. can be expressed as a combination of $\Phi(\varphi_i)$.

3. Some remarks on the results of our study

[1] "2-local geometry" is defined in Benson-Smith [BS07, §7.2].

Partition of the sporadics based on their Lefschetz characters. We define Class I groups as having Lefschetz modules that are projective. We will refer to each group in this class as a "Lefschetz Module Projective."

A Class II group is one whose Lefschetz module is non-projective, yet whose Lefschetz module restricted to the principal block part has v-projective character (perhaps zero). This is the case with three groups previously studied in the literature, and we will show this to be the case with four more sporadics. We will refer to a sporadic group in this class as a "Principal Block Part V-Projective."

Finally, a Class III group contains a non-projective summand in the principle block part of its Lefschetz module—and this part has no cohomology[2] since Benson-Smith established that the Lefschetz module is acyclic. Though there are standard constructions in the literature [CR94] of such modules, this work concretely exhibits (for the first time) an example of such a module naturally occurring in the context of $\widetilde{L}(\Delta)$: in the O'Nan group. We will refer to a group in this class as a "Principal Block Part Non-Projective."

Other observations. In section 8.2, Benson and Smith [BS07] remark that reduced Lefschetz module appears always to involve an indecomposable in a non-principal block. After examining larger groups, we can reformulate their comment as follows.

Remark 3.1 (Non-principal block observation)**.** Each sporadic group affording a non-projective Lefschetz module appears to have a non-projective part in a non-principal block.

This observation holds in each case we study, and we conjecture that it will hold for the remaining seven sporadics.

In addition, Smith [Smi05] noticed a further pattern: the defect of the largest non-principal block part seems to equal the 2-power difference between $|\widetilde{\chi}(\Delta)|_2$ and $|G|_2$. Of the sporadic groups we study, only $O'N$ fails to follow this further pattern.[3]

Remark 3.2. In these 15 sporadic cases where $\widetilde{L}(\Delta)$ is not projective, the vanishing test 2.5 guarantees that there will be at least one 2-element b for which Δ^b is not contractible. We mention an experimental observation: this nonzero value for the Euler characteristic often matches the Euler characteristic value for a well-known non-contractible geometry. This hints that

[2] In particular, the variety of this principal block part must lie in the *representation theoretic nucleus*, as introduced by Benson-Carlson-Robinson [BCR90, p. 68] (see also Benson [Ben02, prop. 1.1].

[3] For an approach to this pattern via vertices of modules, see Sawabe [Saw06].

perhaps our geometric structure Δ^b is homotopy equivalent to that well-known geometry. We will suggest such possible geometries for seven groups in our study.[4]

4. Review of known cases

We present the Lefschetz characters of M_{12}, J_2, and HS, and then proceed to display their 2-modular block decompositions.

The Mathieu group M_{12}. Benson-Wilkerson [BW95, p. 44] computed $\widetilde{\Lambda}_{M_{12}} = \chi_{14} + 2\chi_{15}$. These characters lie in the unique non-principal block. We express this as a vector with columns indexed by M_{12} conjugacy classes:
$\widetilde{\Lambda}_{M_{12}} = [\, \overbrace{496}^{2^4\cdot 31}, \overset{\mathbf{2A}}{-\mathbf{4}}, \overset{\mathbf{2B}}{\mathbf{0}}$, -8, -5, 0, 0, 1, -1, 0, 0, 0, 1, 1, 1].

The vanishing test indicates that Δ^{2A} is not contractible. From the viewpoint of Remark 3.2, the value -4 in $\widetilde{\Lambda}_{M_{12}}$ matches the degree of the Lefschetz module of the building Σ of a subgroup $L_2(4)$ in the centralizer of a 2A element. This suggests that Δ^{2A} may be homotopy equivalent to this Σ. Finally, one concise way of expressing the Lefschetz character via possible v-projective and non-projective parts is: $\widetilde{\Lambda}_{M_{12}} = \Phi(\varphi_6) + \chi_{15}$.

The Janko group J_2. Benson and Smith [BS07, after Theorem 8.7.1] found $\widetilde{\Lambda}_{J_2} = \chi_{16} + \chi_{17} + \chi_{19}$, noting that these characters lie in the unique non-principal block. We calculate $\widetilde{\Lambda}_{J_2} = [\, \overbrace{736}^{2^5\cdot 23}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{-\mathbf{4}}$, 16, -5, 0, 1, 1, -4, -4, 0, -1, 1, 0, 1, 1, 0, 0, 0, 1, 1]. Again from the viewpoint of Remark 3.2, the value -4 at class 2B matches the degree of $\widetilde{L}(\Sigma)$ for building Σ of an $L_2(4)$ in C(2B), the centralizer of conjugacy class 2B. We can express the Lefschetz character as: $\widetilde{\Lambda}_{J_2} = \Phi(\varphi_8) + \chi_{16}$.

The Higman-Sims group HS. Klaus Lux calculated the Lefschetz character for HS in a private communication to Stephen D. Smith:
$\widetilde{\Lambda}_{HS} = [\overbrace{42624}^{2^7\cdot 333}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{-\mathbf{16}}$, -36, 0, 0, 0, 24, -1, 4, 2, 0, 1, 0, 0, 0, 0, -1, -1, -1, 0, -1, 0, 0]. In the viewpoint of Remark 3.2, the value of -16 is the degree of $\widetilde{L}(\Sigma)$ for Σ the building of $Sp_4(2)$ in C(2B).

Block decomposition. In this case we have some contribution from both blocks. For the principal block, we find that $\widetilde{\Lambda}^{HS}_{pr} = 2\Phi(\varphi_8)$, so this is v-projective. The "closest" we could come to expressing the block 2 part as a v-projective character is $\widetilde{\Lambda}^{HS}_{b2} = \Phi(\varphi_6) + \Phi(\varphi_7) + \Phi(\varphi_9) + \chi_{24}$.

We separate the Lefschetz character values according to the 2-modular blocks: $\widetilde{\Lambda}^{HS}_{pr} = [\, \overbrace{26624}^{2^{11}\cdot 13}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{\mathbf{0}}$, -16, 0, 0, 0, 24, 24, 4, 0, 0, -4, 0, 0, 0, 0,

[4] After a preprint version of this paper was circulated, Silvia Onofrei and John Maginnis [MO07] established many such homotopy equivalences.

0, 4, 4, 0, -6, 0, 0] and $\widetilde{\Lambda}^{HS}_{b2}$= [$\overbrace{16000}^{2^7\cdot 125}$, $\overset{\mathbf{2A}}{\mathbf{0}}$, $\overset{\mathbf{2B}}{-\mathbf{16}}$, -20, 0, 0, 0, 0, -25, 0, 2, 0, 5, 0, 0, 0, 0, -1, -5, -5, 0, 5, 0, 0]. We see by the vanishing test that the non-projective part of $\widetilde{\Lambda}_{HS}$ is contained completely in the non-principal block, which is consistent with Remark 3.1.

5. The Suzuki group

We study the maximal subgroups of Suz which contain a Sylow 2-group [CCN+85, p. 131]:

$$\begin{aligned} &H_1 \cong 2_-^{1+6}\cdot U_4(2),\\ &H_2 \cong 2^{2+8} : (A_5 \times S_3), \text{ and}\\ &H_4 \cong 2^{4+6} : 3A_6. \end{aligned}$$

Geometry. The diagram for the natural geometry of Suz is given in Ronan-Smith [RS80, p. 288] as:

$$\overset{1}{\bullet}\!=\!=\!\overset{2}{\bullet}\!=\!=\!\overset{4}{\bullet}$$

We proceed as in Benson-Smith [BS07, Def. 5.3.2]. The set $I := \{1, 2, 4\}$ indexes the "simplex of subgroups" $\mathcal{H}_I$ of Suz. This simplex is determined by the H_i, with $H_J := \bigcap_{i\in J} H_i$ for all nonempty subsets $J \subseteq I$. For simplicity, we will use the notation H_{ij} (without braces) to represent the subgroup $H_{\{i,j\}}$, etc.

The Lefschetz character in this geometry is an alternating sum of the permutation representations on vertices (with stabilizers H_1, H_2, and H_4), edges (with stabilizers H_{12}, H_{14}, H_{24}), and 2-simplices with stabilizer H_{124}.

Inducing from subgroups. We calculate Lefschetz characters by inducing trivial representations through a chain of subgroups: from intersections of H_1, H_2, and H_4 up to the maximal subgroups themselves, and then up to the whole group.[5]

We then combine these in an alternating sum. Since H_1, H_2, and H_4 are maximal subgroups of Suz, the permutation characters on their cosets are readily available in GAP [Gro99].[6] The permutation characters for the *intersections* of the maximal subgroups, however, are not contained in GAP. Instead, selected residues (i.e. links of simplices) described by the diagram tell us which characters of the intersections are needed in the calculation. We will investigate $\mathrm{Res}(H_1)$, in order to induce the trivial characters of H_{12}, H_{14}, and H_{124} up to H_1. Then we use $\mathrm{Res}(H_4)$ to induce from H_{24} up to H_4. GAP can then induce the resulting characters from H_1 and H_4 up to Suz, as well as the trivial characters for the maximal subgroups H_1, H_2, H_4 up to Suz.

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[5]This process is described and several examples are given in section 4 of [RSY90].

[6]I am grateful to Professor Robert A. Wilson of Queen Mary University of London for showing me the step-by-step details of this process.

Using the diagram, we find that inducing the trivial character of H_{12} up to H_1 will be the inflation of the trivial character of $2^4\colon A_5$ induced up to $U_4(2)$. The Atlas gives this character to be $1a + 6a + 20a$. So we have (by the transitivity of induction):

$$[1a]_{H_{12}}\uparrow^{Suz} = ([1a]_{H_{12}}\uparrow^{H_1})_{H_1}\uparrow^{Suz} = [1a + 6a + 20a]_{H_1}\uparrow^{Suz} .$$

The diagram also tells us that $[1a]_{H_{14}}\uparrow^{H_1}$ is the inflation of the trivial character of $2^{\cdot}(A_4 \times A_4).2$ induced up to $U_4(2)$, which is $1a + 20a + 24a$, so:

$$[1a]_{H_{14}}\uparrow^{Suz} = ([1a]_{H_{14}}\uparrow^{H_1})_{H_1}\uparrow^{Suz} = [1a + 20a + 24a]_{H_1}\uparrow^{Suz} .$$

Similarly $[1a]_{H_{124}}\uparrow^{H_{12}}$ is the inflation of $[1a]_{A_4}\uparrow^{A_5} = 1a + 4a$, giving us:

$$[1a]_{H_{124}}\uparrow^{Suz} = (([1a]_{H_{124}}\uparrow^{H_{12}})_{H_{12}}\uparrow^{H_1})_{H_1}\uparrow^{Suz} = ([1a + 4a]_{H_{12}}\uparrow^{H_1})_{H_1}\uparrow^{Suz} .$$

There is a mild complication here, in that GAP does not contain the character tables of maximal subgroups such as $2^4\colon A_5$ of $U_4(2)$, so it is not set up to do the induction $[4a]_{H_{12}}\uparrow^{H_1}$. But this can be calculated using the viewpoint of Smith [Smi90]. We display the result in the style of the Atlas:

$$[1a]_{H_{124}}\uparrow^{Suz} = ([1a]_{H_{124}}\uparrow^{H_1})_{H_1}\uparrow^{Suz} = [1a + 6a + 20aa + 24a + 64a]_{H_1}\uparrow^{Suz} .$$

$\mathrm{Res}(H_1)$ has taken care of all but one of the intersections of maximal subgroups whose trivial characters we need to induce up to Suz. The last one needed is H_{24}, which we examine in $\mathrm{Res}(H_2)$. We have:

$$[1a]_{H_{24}}\uparrow^{Suz} = ([1a]_{H_{24}}\uparrow^{H_2})_{H_2}\uparrow^{Suz} = [1a + 4a]_{H_2}\uparrow^{Suz} .$$

We are now ready to calculate the Lefschetz character of Suz as an alternating sum, with the sign at a simplex σ given by $(-1)^{\dim\sigma}$. To summarize our above findings:

$$\begin{array}{rcl}
[1a]_{H_{124}}\uparrow^{Suz} &=& [1a + 6a + 20aa + 24a + 64a]_{H_1}\uparrow^{Suz} \\
-[1a]_{H_{12}}\uparrow^{Suz} &=& -[1a + 6a + 20a]_{H_1}\uparrow^{Suz} \\
-[1a]_{H_{14}}\uparrow^{Suz} &=& -[1a + 20a + 24a]_{H_1}\uparrow^{Suz} \\
[1a]_{H_1}\uparrow^{Suz} &=& [1a]_{H_1}\uparrow^{Suz} \\ \hline
-[1a]_{H_{24}}\uparrow^{Suz} &=& -[1a + 4a]_{H_2}\uparrow^{Suz} \\
[1a]_{H_2}\uparrow^{Suz} &=& [1a]_{H_2}\uparrow^{Suz} \\ \hline
[1a]_{H_4}\uparrow^{Suz} &=& [1a]_{H_4}\uparrow^{Suz}
\end{array}$$

We can do some quick cancellation to simplify:

$$\widetilde{\Lambda}_{Suz} = [64a]_{H_1}\uparrow^{Suz} - [4a]_{H_2}\uparrow^{Suz} + [1a]_{H_4}\uparrow^{Suz} - 1.$$

The result is: $\widetilde{\Lambda}_{Suz} = $ [4189184, 0, -64, 3968, -352, -73, 0, 0, 0, 8, -16, 9, 0, 0, 0, 0, -1, -1, 0, 0, 0, 2, 2, 0, 1, -1, 0, 0, 0, -1, 0, -1, -1, -1, 2, 2, 3, 0, 0, 0, -1, -1, 0]. *This is the Lefschetz character in vector form, i.e., the columns are indexed by the conjugacy classes.*

Block decomposition. As a sum of irreducible complex characters, we get $\widetilde{\Lambda}_{Suz}$=[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 2, 2, 1, 2, 2, 2, 2, 2, 3]. *This is the Lefschetz character in scalar product form, i.e., the columns are indexed by the irreducible complex characters of Suz.*

We use these coefficients of irreducible characters along with the decomposition matrix to decompose $\widetilde{\Lambda}_{Suz}$ into v-projective and non-projective parts. We find that $\widetilde{\Lambda}_{pr}^{Suz} = \Phi(\varphi_{13}) + \Phi(\varphi_{14})$, a v-projective character.

For block 2 (of defect 3), the "closest" we can come to decomposing $\widetilde{\Lambda}_{b2}^{Suz}$ into v-projectives is: $\widetilde{\Lambda}_{b2}^{Suz} = \Phi(\varphi_{15})+\Phi(\varphi_{16})+\chi_{43}$. Hence the block 2 part of the Lefschetz character of Suz is non-projective, verifying our observation in Remark 3.1. Suz is categorized into Class II. Furthermore, the non-principal block part has defect 3, which matches the 2-power difference between $|G|_2 = 2^{13}$ and $|\widetilde{\chi}(\Delta)|_2 = |\widetilde{\Lambda}_{Suz}|_2 = 2^{10}$, continuing the pattern noticed by Smith [Smi05].

We summarize our findings with the following theorem.

Theorem 5.1. $\widetilde{\Lambda}_{pr}^{Suz} = \Phi(\varphi_{13}) + \Phi(\varphi_{14})$. *Thus Suz is a Class II "Principal Block Part V-Projective" group.*

The Lefschetz character values separated into blocks. The vectors of Lefschetz character values (indexed by the conjugacy classes) given just by characters in each respective block are:
$\widetilde{\Lambda}_{pr}^{Suz}$ = [2834432, $\overset{\mathbf{2A}}{\mathbf{0}}$, $\overset{\mathbf{2B}}{\mathbf{0}}$, 512, -352, -64, 0, 0, 0, 0, 32, 32, 0, 0, 0, 0, 0, -8, 0, 0, 0, -4, -4, 0, 0, -4, 0, 0, 0, 0, 0, 3, 3, 0, -4, -4, 2, 0, 0, 0, 1, 1, 0] and $\widetilde{\Lambda}_{b2}^{Suz}$= [1354752, **0**, **−64**, 3456, 0, -9, 0, 0, 0, 8, -48, -23, 0, 0, 0, 0, -1, 7, 0, 0, 0, 6, 6, 0, 1, 3, 0, 0, 0, -1, 0, -4, -4, -1, 6, 6, 1, 0, 0, 0, -2, -2, 0]. Here we have used bold face for the values at elements of order 2 in order to make clear the results of the vanishing test 2.5. From the viewpoint of Remark 3.2, this value of -64 is also the degree of $\widetilde{L}(\Sigma)$ for the building Σ of $L_3(4)$ in C(2B), suggesting a possibly homotopy equivalence.

6. The O'Nan group

For our remaining four groups starting with $O'N$, we present a somewhat more abbreviated study with fewer details than with Suz. The maximal subgroups of interest are [CCN+85, p. 132]:

$H_1 \cong 4_2 \cdot L_3(4){:}2_1$, and
$H_3 \cong 4^3 \cdot L_3(2)$.

With only two maximal subgroups, the process of obtaining the Lefschetz character is much simpler. There is no need for a diagram, and we have only one intersection to find: H_{13}. We can induce the trivial character of H_{13} via:

$$[1a]_{H_{13}}\!\uparrow^{O'N} = ([1a]_{H_{13}}\!\uparrow^{H_3})_{H_3}\!\uparrow^{O'N} = [1a + 6a]_{H_3}\!\uparrow^{O'N} .$$

One easy cancellation leads to the final calculation:

$$\widetilde{\Lambda}_{O'N} = [1a]_{H_1}\uparrow^{O'N} - [6a]_{H_3}\uparrow^{O'N} - 1.$$

Using GAP, we obtain the Lefschetz character (negating to obtain positive degree) $\widetilde{\Lambda}_{O'N} = [\overbrace{254294272}^{2^8 \cdot 993337}, \overset{\mathbf{2A}}{\mathbf{8960}}$, -44, 0, 0, -8, -4, -48, -13, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1].

The principal block. We might expect that the principal block part has v-projective character as it did in Suz. But we find:

$\widetilde{\Lambda}_{pr}^{O'N} = [\overbrace{147299328}^{2^{10} \cdot 143847}, \overset{\mathbf{2A}}{\mathbf{7168}}$, -2736, 0, 0, 708, -32, 2388, 344, 0, 0, 28, 176, 0, -28, -156, -156, 0, 0, 0, 0, 80, 80, 80, 0, 0, 0, 0, -520, -520].

Notice that $|O'N|_2 = 2^9$ does indeed divide $|\widetilde{\Lambda}_{pr}^{O'N}|$, so the p-test 2.4 passes.

However, the vanishing test 2.5 fails, and so $\widetilde{\Lambda}_{pr}^{O'N}$ is actually *not* projective. Indeed this block part is in some sense very far from being v-projective, as the "closest" expression (in terms of minimizing the coefficients of non-projective components) we can find is:

$$\begin{aligned}\widetilde{\Lambda}_{pr}^{O'N} &= 20\Phi(\varphi_3) + 12\Phi(\varphi_7) + 16\Phi(\varphi_8) \\ &\quad + 2\chi_8 + 2\chi_9 + 18\chi_{11} + 4\chi_{12} + 4\chi_{13} + 4\chi_{14} + 2\chi_{15} \\ &\quad + 8\chi_{18} + 6\chi_{19} + 12\chi_{20} + 18\chi_{21} + 18\chi_{22} + 14\chi_{25}.\end{aligned}$$

Block 2—defect 3. We compute $\widetilde{\Lambda}_{b2}^{O'N} = [\overbrace{1386240}^{2^8 \cdot 5415}, \overset{\mathbf{2A}}{\mathbf{1792}}$, 1140, 0, 0, 60, 28, 324, -12, 0, 0, -28, -24, 0, 28, 60, 60, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -18, -18]. This block part is also non-projective, and we can write $\widetilde{\Lambda}_{b2}^{O'N} = 8\Phi(\varphi_2) + 4\Phi(\varphi_4) + 8\Phi(\varphi_5) + 14\chi_7$.

The defect 0 blocks. The Lefschetz character of $O'N$ has several constituents from blocks of defect 0, which are of course projective: $\widetilde{\Lambda}_{b3}^{O'N} = 97\varphi_9$, $\widetilde{\Lambda}_{b4}^{O'N} = 97\varphi_{10}$, $\widetilde{\Lambda}_{b5}^{O'N} = 115\varphi_{11}$, $\widetilde{\Lambda}_{b6}^{O'N} = 115\varphi_{12}$, $\widetilde{\Lambda}_{b7}^{O'N} = 115\varphi_{13}$. These do not seem very illuminating for our study.

Theorem 6.1. *By the vanishing test 2.5, $\widetilde{\Lambda}_{pr}^{O'N}$ is non-projective. Thus $O'N$ is categorized into Class III—it is a "Principal Block Part Non-Projective" sporadic group.*

Observations of unusual behavior. The principal block part of the Lefschetz character of $O'N$ contains a non-projective summand, making $O'N$ the charter member of Class III. This discovery for O'Nan provides the first "natural" example in the literature on sporadic geometries of such a principal block part with no cohomology.

The principal block part of $O'N$ is the only block part we find in our study to pass the p-test 2.4, yet fail the vanishing test 2.5. This may be related to $O'N$ being the only group in our study with only one conjugacy class of order 2. Furthermore, the value of $\widetilde{\Lambda}_{O'N}$ at that involution is 8960. This is an unfamiliar Euler characteristic value, and indeed $O'N$ is the only group for which we do not make a homotopy equivalence conjecture as in Remark 3.2.

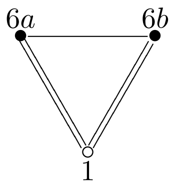
Finally, note that the largest non-principal block of $O'N$ has defect 3 but the 2-power difference between $|\widetilde{\Lambda}|_2$ and $|G|_2$ is only 1. This is the only example we study that violates the pattern observed by Smith [Smi05].

7. The Held group

Our next group is the Held group He. The relevant maximal 2-local subgroups are [CCN+85, p. 131]:

$H_1 \cong 2_+^{1+6}.L_3(2)$,
$H_{6a} \cong 2^6 : 3^{\cdot}S_6$, and
$H_{6b} \cong 2^6 : 3^{\cdot}S_6$.

The diagram is given in Ronan-Smith [RS80, p. 288] as:



We obtain the subgroups

$$H_{1,6a} = H_{1,6b} = S_3 2^2 2_+^{1+6},\ H_{1,6a,6b} = \frac{\frac{2}{2^2}}{2_+^{1+6}},\ \text{and}\ H_{6a,6b} = \frac{3(S_4 \times 2)}{2^6}.$$

Using previously described techniques, we find:[7]

$$\widetilde{\Lambda}_{He} = [8a]_{H_1}\!\uparrow^{He} - [5d + 9b]_{H_{6a}}\!\uparrow^{He} + [1a]_{H_{6b}}\!\uparrow^{He} - 1.$$

With columns indexed by the conjugacy classes of He, we calculate:

$\widetilde{\Lambda}_{He} = [\ \overbrace{1120384}^{2^7\cdot 8753}, \overset{\mathbf{2A}}{-\mathbf{64}}, \overset{\mathbf{2B}}{\mathbf{0}}$, -197, -8, -8, 0, 0, 9, -1, 0, -1, -1, -1, 6, 6, 0, 1, 1, 0, -1, -1, 0, 0, 3, -1, -1, -1, -1, -1, -1, -1, -1].

Block decompositions. $\widetilde{\Lambda}^{He}_{pr} = [\overbrace{760832}^{2^{10}\cdot 743}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{\mathbf{0}}$, 152, -40, 0, 0, 0, -68, 0, 0, 16, 16, 58, -12, -12, 0, 0, 0, 0, 0, 0, 0, 0, -8, 14, 14, -2, -2, -5, -5, 0, 0]. Using the 2-modular table, we can express this as $\widetilde{\Lambda}^{He}_{pr} = \Phi(\varphi_{10}) + \Phi(\varphi_{11}) + \Phi(\varphi_{12})$.

For block 2 (of defect 3), we have $\widetilde{\Lambda}^{He}_{b2} = [\overbrace{58496}^{2^7\cdot 457}, \overset{\mathbf{2A}}{-\mathbf{64}}, \overset{\mathbf{2B}}{\mathbf{0}}$, -13, 32, -8, 0, 0, 21, -1, 0, -17, -17, 39, 18, 18, 0, 1, 1, 0, -1, -1, 0, 0, -3, -1, -1, -6, -6, 4, 4, -1, -1]. This fails tests 2.4 and 2.5, so we use the decomposition matrix to see how "close" this block part is to being v-projective: $\widetilde{\Lambda}^{He}_{b2} = 3\Phi(\varphi_{14}) + \chi_{15}$. From

[7] I am grateful to Silvia Onofrei and John Maginnis for correcting an error at this point in my original calculation.

the viewpoint of Remark 3.2, this value of -64 is also the degree of $\widetilde{L}(\Sigma)$ for the building Σ of $L_3(4)$ in C(2B). The parts in blocks 3 and 4 must be projective, since they have defect 0. We have $\widetilde{\Lambda}_{b3}^{He} = 7\varphi_{15}$, and $\widetilde{\Lambda}_{b4}^{He} = 7\varphi_{16}$.

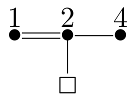
Theorem 7.1. $\widetilde{\Lambda}_{pr}^{He} = \Phi(\varphi_{10}) + \Phi(\varphi_{11}) + \Phi(\varphi_{12})$. *Hence He is a Class II "Principal Block Part V-Projective" group.*

8. The Conway group Co_3

The maximal subgroups we examine are [CCN+85, p. 134]:

$H_1 \cong 2^{\cdot}Sp_6(2)$,
$H_2 \cong 2^2.\,[2^7.\,3^2].\,S_3$, and
$H_4 \cong 2^{4\cdot}A_8$.

The diagram for the natural geometry [BS07, §8.13] is:



This gives us the needed intersections $H_{12} = 2^2.[2^6]:(S_3 \times S_3)$, $H_{14} = 2.2^6{:}L_3(2)$, $H_{124} = 2.2^6{:}S_4$, and $H_{24} = 2^4.2^4{:}(S_3 \times S_3)$. These lead us to:

$$\widetilde{\Lambda}_{Co_3} = [216a + 280b]_{H_1}\uparrow^{Co_3} - [14a + 20a]_{H_4}\uparrow^{Co_3} + [1a]_{H_2}\uparrow^{Co_3} - 1.$$

Now we use GAP to obtain $\widetilde{\Lambda}_{Co_3} = [\ \overbrace{50378624}^{2^7\cdot 393583}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{\mathbf{-496}}$, -2080, -784, 125, 0, 0, 24, 19, 0, 0, 0, 8, 5, 2, 0, 0, 0, 8, -1, 0, -1, -1, -1, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, -1, -1, -1, -1, 0, 0, 0]. From the viewpoint of Remark 3.2, we note that -496 is the degree of $\widetilde{L}(\Sigma)$ for the 2-local geometry Σ of an M_{12} in C(2B).

The Principal block. We compute $\widetilde{\Lambda}_{pr}^{Co_3} = [\ \overbrace{34263040}^{2^{12}\cdot 8365}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{\mathbf{0}}$, -11840, 544, -512, 0, 0, 440, -40, 0, 0, 0, 0, 0, -56, 0, 0, 0, 28, 106, 0, 0, 20, 20, 0, 0, 0, 0, -40, 44, 0, 0, 0, -8, 0, 0, -37, -37, 0, 0, 0], which is v-projective: $\widetilde{\Lambda}_{pr}^{Co_3} = \Phi(\varphi_9) + \Phi(\varphi_{10}) + 6\Phi(\varphi_{12}) + 8\Phi(\varphi_{14})$.

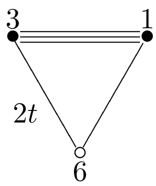
Block 2—defect 3. $\widetilde{\Lambda}_{b2}^{Co_3} = [\ \overbrace{13006720}^{2^7\cdot 101615}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{\mathbf{-496}}$, 11296, -2384, 445, 0, 0, -80, 155, 0, 0, 0, 8, 5, 34, 0, 0, 0, 4, -83, 0, -1, -21, -21, 0, 0, 0, 0, 16, -19, 0, 0, 0, -17, -1, -1, 36, 36, 0, 0, 0]. This character fails either test. We can express this as: $\widetilde{\Lambda}_{b2}^{Co_3} = 4\Phi(\varphi_{11}) + 6\Phi(\varphi_{13}) + 11\Phi(\varphi_{16}) + \chi_{32} + 2\chi_{38}$.

Block 3—defect 1. $\widetilde{\Lambda}_{b3}^{Co_3} = [\ \overbrace{3108864}^{2^{12}\cdot 759}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{\mathbf{0}}$, -1536, 1056, 192, 0, 0, -336, -96, 0, 0, 0, 0, 0, 24, 0, 0, 0, -24, -24, 0, 0, 0, 0, 0, 0, 0, 0, 24, -24, 0, 0, 0, 24, 0, 0, 0, 0, 0, 0, 0]. We can express this as $\widetilde{\Lambda}_{b3}^{Co_3} = 12\Phi(\varphi_{15})$.

Theorem 8.1. $\widetilde{\Lambda}_{pr}^{Co_3} = \Phi(\varphi_9) + \Phi(\varphi_{10}) + 6\Phi(\varphi_{12}) + 8\Phi(\varphi_{14})$*, making* Co_3 *a Class II "Principal Block Part V-Projective" group.*

9. The Rudvalis group

The relevant maximal 2-local subgroups are [CCN+85, p. 126]:
$H_1 \cong 2^{\cdot}2^{4+6}{:}S_5$,
$H_3 \cong 2^{3+8}{:}L_3(2)$, and
$H_6 \cong (2^6{:}U_3(3)){:}2 \cong 2^6G_2(2)$. The diagram is:



in the notation of [RS80, p. 288]. The diagram gives us the needed intersections, of the following structures:

$$H_{16} = \frac{\frac{S_3}{2^{2+1+2}}}{2^6},\ H_{36} = \frac{\frac{S_3}{2^{1+4}}}{2^6},\ H_{136} = \frac{\frac{2}{2^{1+4}}}{2^6},\text{ and } H_{13} = \frac{\frac{S_3}{2^2}}{2^{3+8}}.$$

These subgroups allow us to calculate:

$$\widetilde{\Lambda}_{Ru} = [32ab]_{H_6}\!\uparrow^{Ru} - [6a]_{H_3}\!\uparrow^{Ru} + [1a]_{H_1}\!\uparrow^{Ru} - 1.$$

Indexed by conjugacy classes, we obtain: $\widetilde{\Lambda}_{Ru}$= [$\overbrace{10113024}^{2^{12}\cdot 2469}$, $\overset{\mathbf{2A}}{\mathbf{0}}$, $\overset{\mathbf{2B}}{\mathbf{64}}$, -96, 0, 0, 0, 0, 24, -1, 0, 5, 0, 0, 0, 0, -1, 0, 0, -1, 1, 1, 1, -1, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, -1]. This character does not vanish at the 2B element, telling us that the Lefschetz module is non-projective. From the perspective of Remark 3.2, we see that 64 matches the degree of $\widetilde{L}(\Sigma)$, where Σ is the building of $Sz(8)$ in C(2B).

Block decomposition. $\widetilde{\Lambda}_{pr}^{Ru}$ = [$\overbrace{6881280}^{2^{16}\cdot 105}$, $\overset{\mathbf{2A}}{\mathbf{0}}$, $\overset{\mathbf{2B}}{\mathbf{0}}$, -48, 0, 0, 0, 0, 280, 80, 0, 28, 0, 0, 0, 0, 0, 0, 0, -36, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -14, -14], which is v-projective: $\widetilde{\Lambda}_{pr}^{Ru} = \Phi(\varphi_5) + \Phi(\varphi_8)$. Thus Ru belongs in Class II.

For block 2 (of defect 2), we find $\widetilde{\Lambda}_{b2}^{Ru}$ = [$\overbrace{3231744}^{2^{12}\cdot 789}$, $\overset{\mathbf{2A}}{\mathbf{0}}$, $\overset{\mathbf{2B}}{\mathbf{64}}$, -48, 0, 0, 0, 0, -256, -81, 0, -23, 0, 0, 0, 0, -1, 0, 0, 35, 1, 1, 1, -3, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, 13, 13]. This does not vanish at element 2B, so $\widetilde{\Lambda}_{b2}^{Ru}$ is non-projective. We minimize the number of non-projective characters and express the Lefschetz character of this block part as $\widetilde{\Lambda}_{b2}^{Ru} = \Phi(\varphi_6) + \Phi(\varphi_7) + 6\Phi(\varphi_9) + \chi_{36}$.

Theorem 9.1. $\widetilde{\Lambda}_{pr}^{Ru} = \Phi(\varphi_5) + \Phi(\varphi_8)$*. So Ru is a Class II "Principal Block Part V-Projective" group.*

10. Summary

We present a partition of the sporadic groups according to our findings, as described in Section 3 (boldface indicates groups originally classified by this work).

Class	**Sporadic Groups**
I: Lefschetz Module Projective	M_{11}, J_1, M_{22}, M_{23}, J_3, M_{24}, M^cL, Co_2, Ly, J_4, Th
II: Principal Block Part V-Projective	M_{12}, J_2, HS, **Suz**, **He**, $\mathbf{Co_3}$, **Ru**
III: Principal Block Part Non-Projective	**O'N**

The groups Co_1, Fi_{22}, Fi_{23}, Fi'_{24}, HN, B, and M have yet to be classified.[8] The 2-modular decomposition matrices [Bre99] of these groups are not yet known. We do know by the p-test that their Lefschetz modules are non-projective, so they will be in Class II or Class III.

11. Future directions

For the block parts of Lefschetz modules that we found to have v-projective character, we suggest that the method of Steinberg module inversion by Webb [Web87] as extended by Grodal [Gro02] could possibly be used to show that the block part is actually a projective module in each case. The case of $O'N$ should be investigated further; we found a number of unusual features (see Section 6) that made this group stand out. There are still seven sporadic groups whose Lefschetz characters could be computed (but they could not be fully decomposed mod 2 since the 2-modular irreducibles of these groups are not yet known). For even further directions, some different geometries for each group could be studied, as indicated by Benson-Smith [BS07]. The block decomposition corresponding to primes $p \neq 2$ could also be explored.

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