

ON LEFSCHETZ CHARACTERS OF 2-LOCAL GEOMETRIES FOR  
SOME SPORADIC GROUPS

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To my wife Christine, for your unwavering support and cheerfulness despite some really rough times. I couldn't have made it through these seven years without you. We're going to Tucson, baby!

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## PREFACE

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## SUMMARY

This work examines the 2-modular block parts of non-projective Lefschetz characters over 2-local geometries of the sporadic groups. Results previously published by other authors on  $M_{12}$  and  $J_2$  are presented for completeness, as well as a previously calculated but unpublished result on  $HS$ . The main new results are on the sporadic groups  $Suz$ ,  $ON$ ,  $He$ ,  $Co_3$ , and  $Ru$ .

For each group, the Lefschetz character over its natural 2-local geometry is calculated. The 2-modular blocks of each group are used to examine which blocks in the Lefschetz character have the character of a projective module, and which do not. In each case it is confirmed that a non-principal block contains a non-projective summand.

The sporadic groups are then categorized into three classes based on the projectivity of the block parts of their Lefschetz modules. Class I is the set of 11 sporadics whose entire Lefschetz modules were known to be projective before this work. Class II is the set of sporadics that have non-projective Lefschetz modules, but whose Lefschetz modules restricted to principal block parts each have the same character as that of a projective module.  $M_{12}$ ,  $J_2$ , and  $HS$  were previously known to be in this class, and we find that  $Suz$ ,  $He$ ,  $Co_3$ , and  $Ru$  also belong to this class. Class III is the set of sporadics whose Lefschetz modules restricted to principal block parts each contain a non-projective summand. This work shows that  $ON$  is the first sporadic categorized into this class. Further research is proposed to investigate the vertex of each module, particularly in the unusual case of  $ON$ .

## CHAPTER 1

### INTRODUCTION

The Classification of Finite Simple Groups categorized the finite simple groups into four classes: the infinite family of cyclic groups of prime order, the infinite family of alternating groups, the infinite family of groups of Lie type, and 26 “sporadic” groups. The class of *sporadic* groups is essentially the collection of finite simple groups that did not fit into any infinite family. This has sparked investigation to find connections among these sporadic groups. In this thesis, we examine one type of similar behavior exhibited by many sporadic groups.

One useful way to study simple groups is by examining their geometries. A natural invariant of the geometry called the *Lefschetz module* is obtained by taking the (formal) alternating sum of the homology groups of the geometry. The trace of the group action on the Lefschetz module (actually a *virtual* module) is known as the *Lefschetz character*. This thesis focuses on the Lefschetz characters of particular geometries of sporadic groups.

Brown [9] and Quillen [24] introduced the study of the partially ordered set of  $p$ -subgroups of an arbitrary finite group  $G$  for an arbitrary prime  $p$ . In the case of  $G$  of Lie type in characteristic  $p$ , this poset is homotopy equivalent to the Tits building geometry and gives a Lefschetz module which is projective over  $\mathbb{F}_p$ . Webb [34] initiated similar study of geometries for sporadic groups, such as those in Ronan-Smith [26]. Ryba-Smith-Yoshiara [27] showed via Webb’s method that 11 sporadic groups give projective Lefschetz modules for  $p=2$  (the prime which generally provides the richest geometric structures). More recently, Benson-Smith [7] studied the geometry of the remaining 15 sporadic groups, and observed that the Lefschetz character always seems to have a non-projective constituent from a non-principal 2-block of small (relative to full defect) but positive defect [7, footnote after Theorem 8.2.1]. Some general theoretical approaches to this phenomenon have been suggested [Grodal, Smith, unpublished]. This thesis provides still more detailed practical determination of the Lefschetz character in terms of 2-modular irreducibles for the cases which are within range of present computation.

Here is a summary of the status of the literature. Landrock [21] had comprehensively described the nonprincipal 2-blocks of the sporadics. Of the 26 sporadic groups, 11 have projective Lefschetz modules (in characteristic 2), and most of these have been calculated previously [27]. Of the remaining 15 with non-projective Lefschetz modules, the Lefschetz characters in terms of irreducibles already have been calculated for three of these groups ( $M_{12}$ ,  $J_2$ , and  $HS$ ). References to each of these are given in the chapter “Known Cases,” along with a brief examination of the 2-modular block decomposition for each. On the other hand, there are 7 of the remaining 12 groups whose 2-modular irreducibles are not yet known. Thus there are 5 sporadic groups ( $Suz$ ,  $O’N$ ,  $He$ ,  $Co_3$ , and  $Ru$ ) with non-projective Lefschetz modules whose 2-modular irreducibles are known [8], but whose Lefschetz characters in characteristic 2 had not yet been discussed in the literature.

In this work, we calculate the Lefschetz characters and 2-modular block decompositions of the five sporadic groups  $Suz$ ,  $O’N$ ,  $He$ ,  $Co_3$ , and  $Ru$ . This completes these calculations for

all sporadic groups whose whose 2-modular irreducibles are known. In addition, we verify a refined version of the Benson-Smith conjecture for all currently possible cases. In particular, we calculate the characters and concretely examine their block decompositions. We find that the group  $O'N$  has a *non*-projective part in its principal block, while the other four have *v-projective* principal block parts. The former gives a new example “in nature” of a module whose corresponding variety is in the representation-theoretic *nucleus*, as defined by Benson-Carlson-Robinson [5, p. 68]. We are then able to categorize 19 of the 26 sporadics according to the projectivity of block parts of their Lefschetz characters.

The results facilitate further investigation into these particular groups and also provide a blueprint for making similar calculations for the seven sporadic groups that have yet to be categorized.

## CHAPTER 2

### BACKGROUND

Our results will require basic understanding of specific aspects of several areas including geometry, topology, homological algebra, representation theory, and finite group theory. Since most readers may be somewhat unfamiliar with at least one of these topics, here we provide the necessary background knowledge, including references for more detailed explanations.

#### 2.1 2-Local Geometry

Concise sketches of the background and history of geometries for simple groups can be found at [31, §2.1] and [7, §7.1]. Throughout this thesis, we will work in characteristic 2, the prime that provides the richest geometric structures. For any group, we define a “2-local subgroup” to be the normalizer of a 2-group. The “2-local geometry” is defined in the sense of Benson-Smith [7, §7.2]: for each group, the natural geometry is a simplicial complex determined by maximal 2-local subgroups each containing a common Sylow 2-group. The vertices in the geometry are identified with the maximal 2-local subgroups themselves, and simplices of higher dimension are determined by (and often can be identified with) suitable intersections of maximal subgroups.

#### 2.2 The Lefschetz Character

**Definition 2.2.1 (Virtual module)** *A virtual module is a formal  $\mathbb{Z}$ -linear combination (i.e., a formal  $\pm$  combination) of modules.*

The (reduced) *Lefschetz module* of a complex  $\Delta$  is the virtual module given by the alternating sum of the chain groups<sup>1</sup> [27, p. 281]:

$$\tilde{L}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i C_i(\Delta).$$

---

<sup>1</sup>We present this alternating sum in a slightly different manner than in [27, p. 281], to preserve consistency in the sign of each chain group regardless of the parity of  $\dim \Delta$ . Note that the reduced Lefschetz module is denoted with a tilde. Also, see Munkres [23, p. 27] for details about chain groups. There are two additional items to mention here. First, the more usual definition of the Lefschetz module uses the homology groups  $H_i(\Delta)$  in place of the chain groups  $C_i(\Delta)$ . The Hopf trace formula [23, p. 122] guarantees that these different definitions give at least the same character. For our purposes, the definition via  $C_i(\Delta)$  is more useful. Second, the classical (not reduced) Lefschetz module—denoted with no tilde—does not include the  $i=-1$  term (formally representing the “empty” simplex of dimension -1) in its definition:

$$L(\Delta) := \sum_{i=0}^{\dim \Delta} (-1)^i H_i(\Delta).$$

Its degree term is the (reduced) *Euler characteristic*:

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i \dim C_i(\Delta).$$

The chain groups  $C_i(\Delta)$  can be defined over  $\mathbb{Z}$ , and so give ordinary characters if we extend to a field such as  $\mathbb{Q}$ . But we can also reduce the groups over  $\mathbb{Z}_2$ , allowing us to define the module in characteristic 2.<sup>1</sup>

**Notation 2.2.2 (Fixed point notation)** *Let  $g \in G$  and let  $\Delta$  be a simplicial complex acted upon by  $G$ . Then we use the symbol  $\Delta^g$  to denote the set of simplices in  $\Delta$  fixed<sup>2</sup> by  $g$ .*

**Definition 2.2.3 (Lefschetz character)** *The Lefschetz character<sup>3</sup> is the sequence of Euler characteristic values on the geometry fixed by representatives of each conjugacy class of  $G$ :  $\text{Tr}(g, \tilde{L}) = \tilde{\chi}(\Delta^g)$ . Thus that the Lefschetz character is a class function whose value at 1 is the Euler characteristic of  $\Delta$ . For convenience, we will denote the Lefschetz character of  $G$  by  $\tilde{\Lambda}_G$ , or just  $\tilde{\Lambda}$  when the group is clear from the context.*

Though this work does deal exclusively with finite groups, some groups feel more finite than others<sup>4</sup>. So it may be useful to show a Lefschetz character calculation on the geometry of a simple group that is more well-known and significantly smaller than a sporadic group. We show the calculation for the Lie type group  $L_3(2)$ .

### 2.3 Example: $G = L_3(2)$

The Dynkin diagram for  $L_3(2)$  is  $\circ \text{---} \circ$ , since  $L_3(2)$  is the adjoint Chevalley group  $A_2(2)$  [12, p. x]. Its natural finite geometry<sup>5</sup> is the projective plane  $\mathbb{P}^2(2)$ , from the natural  $\mathbb{F}_2G$ -module  $V = (\mathbb{F}_2)^3$ . The poset  $\mathcal{P}(V)$  of nonzero, proper subspaces affords a collection of simplices called the *order complex*  $K(\mathcal{P}(V))$ .<sup>6</sup>

<sup>1</sup>The Brauer characters in the GAP tables correspond (by lifting of idempotents) to modular characters that would be visible if we reduced all groups over  $\mathbb{Z}_2$ . Hence in practice, we can do all calculations with ordinary characters.

<sup>2</sup>We consider geometric objects as sets. Thus *fixing* an object means normalizing it as a set, not necessarily centralizing every element of the set.

<sup>3</sup>Throughout this thesis, we will assume the reduced form for the Lefschetz module, Euler characteristic, and Lefschetz character without indication other than the tilde in  $\tilde{L}(\Delta)$ ,  $\tilde{\chi}$ , and  $\tilde{\Lambda}$ , respectively.

<sup>4</sup>For less expert readers looking for a reference on simple groups in general (as opposed to just the sporadics), [18] is a useful resource.

<sup>5</sup>This geometry is often viewed as the *Fano plane* (see for example [31, ch. 2]). Here we will view the geometry in a less tidy form that more clearly elucidates the inclusion relations of the chain groups.

<sup>6</sup> $K(\mathcal{P}(V))$  means inclusion-chains in the poset  $\mathcal{P}(V)$ . For background on posets, inclusion chains, and order complexes, see [31, ch.1].

**Notation 2.3.1 (Vector notation)** *For simplicity, we rename the vectors as follows. Instead of the usual  $(1,0,0)$  notation, the first standard basis vector is simply denoted 1. Thus  $(0,1,0)$  is 2, the sum  $(1,1,0)$  is denoted 12, and so on.*

Using this notation, we show the natural geometry of  $L_3(2)$  in Figure 2.1.

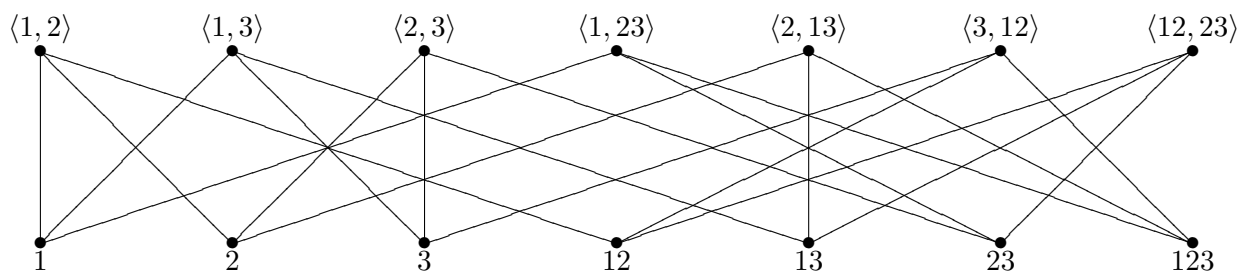


Figure 2.1. The natural geometry of  $L_3(2)$ .

The diagram shows the inclusion relations among the chain groups. The lines (each generated by two points) are on the top row, with the points along the bottom row. The inclusion of a point in a line is called a *flag* and is denoted by an edge in the diagram. Hence in the order complex of the poset, both lines and points are vertices  $C_0$  and flags are edges  $C_1$ . The calculation of the Lefschetz character on this geometry is as follows.

We start with the identity element, which is the calculation of the reduced Euler characteristic, i.e.  $Tr(1, \tilde{L}) = \tilde{\chi}(\Delta^1) = \tilde{\chi}(\Delta)$ . The identity fixes everything, so for  $\#P$  representing the number of points fixed,  $\#L$  for number of lines fixed, and  $\#F$  for number of flags fixed, we have

$$\tilde{\chi} = -1 + (\#P + \#L) - \#F \quad (2.1)$$

$$= -1 + (7 + 7) - 21 \quad (2.2)$$

$$= -8 \quad (2.3)$$

Next we consider the order 2 conjugacy class, which can be represented by the element  $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ . We multiply on the right, so for example

$$1 \cdot g = (1 \ 0 \ 0) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = (1 \ 0 \ 0) = 1.$$

Thus  $g$  fixes the point 1. Similarly,  $g$  also fixes points 2 and 12. It swaps points 3 and 13, and thus fixes line  $\langle 1, 3 \rangle$ . (Even though  $g$  does not fix the point 3 or the point 13, it does fix the line  $\langle 1, 3 \rangle = \{1, 3, 13\}$ , cf. Notation 2.2.2.) Since it fixes point 1, and swaps point 23 with 123, it also fixes line  $\langle 1, 23 \rangle$ . It normalizes no other lines. Thus it fixes 3 points and 3 lines, so in the complex of the poset,  $|C_0^g| = 6$ . By Figure 2.2, we can see that  $g$  also fixes 5 flags. Thus for this conjugacy class,  $Tr(g, L) = \tilde{\chi}(\Delta^g) = -1 + 6 - 5 = 0$ .

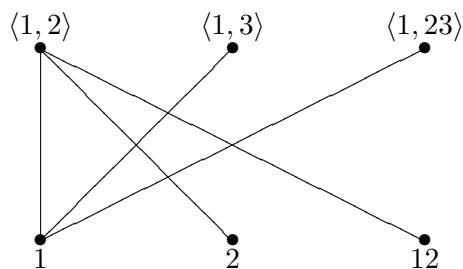


Figure 2.2. Flags fixed by an element of order 2.

Similar calculations for  $g \in G$  of order 3, 4, and 7 yield the values of the Lefschetz character of  $G$ . As we mentioned in Section 2.2, the Lefschetz module is actually a virtual module. Thus for convenience,<sup>1</sup> we negate the Lefschetz character so that its dimension is positive. The values are shown in Table 2.1.

---

<sup>1</sup>Slightly different definitions of the Lefschetz character found in the literature (such as the aforementioned [27, p. 281]) avoid this inconvenience.

TABLE 2.1

THE LEFSCHETZ CHARACTER OF  $L_3(2)$ .

Order of $g$	1	2	3	4	7
$\text{Tr}(g, \tilde{L})$	8	0	-1	0	1

Experienced group theorists will recognize this Lefschetz character as the *Steinberg character* of  $L_3(2)$ . Using the Atlas [12, p. 3], we see that this character equals the irreducible complex character  $\chi_6$ . This restricts in characteristic 2 to the irreducible 2-modular character  $\varphi_4$ , which is projective and hence is equal to its own projective cover. The Lefschetz module for  $L_3(2)$  is in fact projective,<sup>1</sup> as is the case for 11 of the 26 sporadic groups. By contrast, our work will focus on groups with non-projective Lefschetz modules. The following notation will be useful.

**Notation 2.3.2 (The Lefschetz character restricted to a block part)** *Let  $G$  be a sporadic group. Recall that we denote the Lefschetz character of the natural 2-local geometry of  $G$  by  $\tilde{\Lambda}_G$ . Let  $B$  be a particular block part of the 2-modular decomposition matrix of  $G$ . Then we define  $\tilde{\Lambda}_B^G$  to be  $\tilde{\Lambda}_G$  restricted to characters in block  $B$ .*

### Induced Representations and GAP

The above Lefschetz character computation for  $L_3(2)$  was illustrated geometrically with manual calculations. Such a process would be prohibitively cumbersome to make for the much larger sporadic groups. Fortunately, we can translate the geometric information into representation theory and use the program GAP [20] to do much of this work for us. We will use the standard fact that the Lefschetz character can be calculated as an alternating sum of induced modules of subgroups ([34, prop. 5.3], [19, lemma 5.2], [25, theorem 3]). Using the notation that follows, a group orbit on chains is the induced module  $1_H \uparrow^G$ , where  $H$  is the stabilizer of a chain in the orbit.

**Notation 2.3.3 (Induced character)** *To induce character  $\chi$  from subgroup  $H$  up to group  $G$ , we will use either the notation  $\text{Ind}_H^G(\chi)$  or the more abbreviated form  $[\chi]_H \uparrow^G$ .*

Throughout this thesis, when we informally describe “inducing a subgroup,” we mean inducing a character (usually the trivial character unless otherwise stated) from the specified

---

<sup>1</sup>Since we are restricting our focus to  $p = 2$ , “projective” will always mean “projective in characteristic 2.” We discuss projectivity in Section 2.5.

subgroup up to the group. GAP allows us to calculate Lefschetz characters by inducing representations through a chain of subgroups: from intersections of appropriate maximal 2-local subgroups up to the maximal subgroups themselves, and then up to the whole group. The role of the geometry is to determine the appropriate subgroups, which are the stabilizing subgroups of the various subgeometries under the group action. This process is described and several examples are given in section 4 of [27].

## 2.4 Cohomology

A geometry is *acyclic* if all of its reduced cohomology groups are 0, giving the geometry the same set of cohomology groups as a point. Similarly, a module is acyclic if and only if its reduced module cohomology groups are 0.

For any group  $G$ , the reduced Lefschetz module is acyclic if and only if  $G$  has a decomposition of cohomology in terms of its subgroups [Appendix A.2]. Obtaining decompositions is a theme in much of the literature (see for example [9], [24], [34], and most of [7]), and this requires the group to have an acyclic Lefschetz module.

In the terminology of Dwyer [16]—as adapted to a simplicial complex  $\Delta$  in Definition 6.6.6 of Benson-Smith—showing the normalizer decomposition of  $\Delta$  is *ample and normalizer-sharp* will give a decomposition of cohomology of  $G$  in terms of its subgroups. By our discussion above, this in turn produces a reduced Lefschetz module  $\tilde{L}(\Delta)$  which is acyclic. In chapter 8 of Benson-Smith, this ample/sharp property is checked for a geometry  $\Delta$  for each sporadic group. Thus each sporadic group has an acyclic reduced Lefschetz module, which gives zero module cohomology.

Hence we will not investigate group cohomology itself, but instead the acyclic module  $\tilde{L}(\Delta)$  and its corresponding character  $\tilde{\Lambda}$ . This is a different direction from most of the literature, where the cohomology decomposition afforded by the acyclicity of the Lefschetz module is of primary interest. Here we examine properties of the Lefschetz module for its own sake, i.e. as a module. In the literature, primarily Ryba-Smith-Yoshiara [27] and Benson-Smith [7, footnote after Theorem 8.2.1] suggest exploring this route.

## 2.5 2-modular Block Theory

We focus our study on specific distinctions provided by the Lefschetz characters on the natural geometry of each sporadic group. These distinctions are based on the concept of projectivity. Though there are many equivalent definitions<sup>1</sup> of a projective module, we will use the following definition.

**Definition 2.5.1 (Projective module)** *We define a projective module as a module that can be written as a sum of projective indecomposables<sup>2</sup>  $P(\varphi_i)$ .*

---

<sup>1</sup>See for example [15, pp. 369-371]

<sup>2</sup>Recall that we work exclusively in characteristic 2. A *principal indecomposable* module is an indecomposable direct summand of the group algebra. Each principal indecomposable is the projective cover  $P(\varphi)$  of a 2-modular irreducible  $\varphi$ . Projective covers are discussed in Appendix A.3.

**Definition 2.5.2 (Projective character)** We define  $\Phi(\varphi)$  as the character of the module  $P(\varphi)$ . We say a module has projective character if its character can be written as a sum of  $\Phi(\varphi_i)$ . Note that a projective module will always have projective character, but if a module has projective character, that does not necessarily imply that the module itself is a projective module.

### Virtual Projectivity

Since the Lefschetz module is actually a virtual module, we make precise what is meant by the projectivity of a virtual module.

**Definition 2.5.3 (Virtual projective module)** A virtual projective module is a  $\mathbb{Z}$ -linear combination of projective indecomposables  $P(\varphi)$ .

In this work, we focus more on the characters than the modules themselves. Thus we make the following definition which we will use throughout our study.

**Definition 2.5.4 (V-projective character)** We say a virtual module has v-projective character if the module has the same character as that of a virtual projective module, i.e. if its character equals some  $\mathbb{Z}$ -linear combination of  $\Phi(\varphi_i)$ . When describing the character itself with this property, we say the character is v-projective. Note also that if a module has v-projective character, that does not necessarily imply that the module is a virtual projective module.

Since each group we study has order divisible by the characteristic  $p = 2$ , the group algebra breaks up into a direct sum of *block* components. Each principal indecomposable module  $P(\varphi)$  appears in just one block. For each group  $G$ , once we have computed its Lefschetz character  $\tilde{\Lambda}$ , we will decompose  $\tilde{\Lambda}$  as a combination of irreducible characters of  $G$ . We will use these coefficients and the decomposition matrices website [8] to break each Lefschetz character into block parts,<sup>1</sup> and find which block parts of  $\tilde{\Lambda}$  give v-projective characters and which block parts do not.

### What the Character Tells Us About the Module

Our study is focused on Lefschetz characters, which do not give us complete information about the Lefschetz module. We could in principle compute explicit generators and relations for the module, but in practice this computation is out of range. However, there is much about the module that can be understood from knowing only its character. In terms of projectivity, information from the character can conclusively prove that a Lefschetz module is *not* a virtual projective, and hence is not a projective module. One way to show this is via the following well-known lemma [27, p. 282].

**Lemma 2.5.5 (The  $p$ -test)** If  $\tilde{L}(\Delta)$  is projective, then  $|G|_p$  divides  $\tilde{\chi}(\Delta)$ .

---

<sup>1</sup>For some brief background on block theory, see Appendix A.3.

Recall that the Euler characteristic  $\tilde{\chi}(\Delta)$  is the degree of the Lefschetz character  $\tilde{\Lambda}$ . Thus if  $|G|_p$  does not divide the degree of  $\tilde{\Lambda}$ , then the  $p$ -test tells us immediately that the Lefschetz module is non-projective.<sup>1</sup> The converse, however, does not hold in general. If  $|G|_p$  divides  $\tilde{\chi}(\Delta)$ , then it is an indication that the Lefschetz module might be projective, but not a conclusive proof.<sup>2</sup>

There is an additional test that the character can provide for us to show the non-projectivity of the Lefschetz module; we indicate this result in the case of characteristic  $p = 2$ . If  $M$  is projective (in characteristic 2) as a  $(\mathbb{F}_2)G$ -module, then  $M$  splits off from any module in which it appears as a quotient or submodule [13, §77]. Furthermore for  $S$  a Sylow 2-subgroup of  $G$ , and  $M|_S$  the restriction of  $M$  as an  $\mathbb{F}_2 S$  module,  $M|_S$  is a direct sum of the regular module of  $S$ . So for  $M|_S$ , the nontrivial 2-elements vanish, as the identity is the only 2-element with fixed points in the regular module. Hence every projective character induced up from  $S$  to  $G$  must also vanish at the 2-elements. This result of Brauer [17, p.148] is another test for projectivity:

**Lemma 2.5.6 (The vanishing test)** *For the module  $\tilde{L}(\Delta)$ , its Lefschetz character  $\tilde{\Lambda}$  is v-projective if and only if  $\tilde{\Lambda}$  vanishes at the nontrivial 2-elements.*

So if the Lefschetz character of a group  $G$  does not vanish at a 2-element, then the vanishing test tells us its character is not v-projective, which implies the Lefschetz module is non-projective. As with the  $p$ -test, however, passing the vanishing test does not imply that the module itself is projective. But unlike with the  $p$ -test, if a Lefschetz character passes the vanishing test, this conclusively indicates that the character can be expressed as a combination of  $\Phi(\varphi_i)$ . We will explicitly show this combination for each block part that passes the vanishing test.<sup>3</sup>

Of the two tests, the  $p$ -test is a weaker assessment to use to determine non-projectivity: if a Lefschetz character passes the vanishing test, then it will also pass the  $p$ -test (this follows from our discussion above, as the regular  $S$  module has dimension  $|S| = |G|_2$ ). However, the converse is not true in general, and we shall in fact see an example of a Lefschetz character which passes the  $p$ -test, but does not pass the vanishing test.

## 2.6 Non-Contractibility of Fixed Points of 2-Elements

For any  $p$ -element  $a$ , we know that  $\Delta^a$  contractible implies  $\tilde{\chi}(\Delta^a) = 0$  (as all reduced homology groups of a point are zero). Hence a nonzero reduced Euler character of  $\Delta^a$  tells us that  $\Delta^a$  is a non-contractible space. It does not, however, tell us exactly what that space is.

Historically, the Lefschetz character values on the 2-elements were usually among the earliest properties studied for group geometries. Indeed in Webb's [33] contractibility argument (see

<sup>1</sup>This is the test used in [27] and [30] to confirm that the 15 remaining sporadic groups have non-projective Lefschetz modules.

<sup>2</sup>A way to conclusively show that the Lefschetz module is projective is discussed in Section 9.1: if the *vertex* of the module is trivial, then the module is projective.

<sup>3</sup>For each block part in our study that has v-projective character, we are able to express the character as an explicit combination of  $\Phi(\varphi_i)$ . For the seven sporadic groups whose 2-modular irreducibles are still unknown, however, this cannot yet be completed.

Appendix A.1), the aim was to show that for each 2-element  $a$ , the fixed-point subcomplex  $\Delta^a$  was contractible. This hypothesis not only implied that the Lefschetz character value at each 2-element was 0, it also showed that the Lefschetz module was a virtual projective module. This was the method employed by Ryba-Smith-Yoshiara [27] to verify projectivity in their 11 sporadic cases.

In the remaining 15 sporadic cases where  $\tilde{L}(\Delta)$  is not projective, there are often *some* 2-elements  $a$  with  $\Delta^a$  contractible. However, there must also be at least some other 2-element  $b$  for which  $\Delta^b$  is not contractible. Having a non-contractible space does not necessarily imply that  $\tilde{\chi}(\Delta^b) \neq 0$ , but usually (barring a coincidence) the value at  $b$  of  $\tilde{\chi}(\Delta^b)$  ( $= \text{Tr}(b, \tilde{L})$ ) will be nonzero. This nonzero value for the Euler characteristic at 2-element  $b$  often matches the Euler characteristic value for a well-known (and necessarily non-contractible) geometry. While not conclusive, this hints that perhaps our geometric structure  $\Delta^b$  is homotopy equivalent to the other well-known geometry. We will suggest such possible geometries at various later points.

## 2.7 Categorization of the Sporadics Based on their Lefschetz Characters

We classify the sporadics based on three different ways to obtain acyclic Lefschetz modules.

### Class I: The Lefschetz Module is Projective

A Lefschetz module that is projective implies that it is also acyclic.<sup>1</sup> Though the converse is not true in general, this situation is the most naïve possibility for an acyclic module: the entire Lefschetz module is itself projective.<sup>2</sup> This is the case for 11 sporadic groups previously studied in [27]. We will refer to each sporadic group in this class as a “Lefschetz Module Projective.” Since these have already appeared in the literature, we focus our study on the next two classes.

### Class II: The Principal Block Part of $\tilde{\Lambda}$ has V-Projective Character

For groups whose Lefschetz module is known to be non-projective, this is the least complicated situation. The full Lefschetz module is non-projective, but all of the non-projective parts appear outside of the principal block. (Thus groups whose Lefschetz modules are non-projective and whose principal block contributes nothing at all to the Lefschetz module are considered to be a part of this class.) These non-projective parts are necessarily acyclic, as cohomology of an indecomposable  $M$  is defined by  $\text{Ext}^*(M, 1)$ , which can only be nonzero if  $M$  is in the principal block (cf. Appendix A.3). Thus in this case, the Lefschetz module restricted to the principal block has v-projective character (perhaps zero). Note that in this situation we do not know for sure whether the Lefschetz module restricted to the principal block is actually a virtual projective module, we just know that it has v-projective character. This is the case with three groups previously studied in the literature ( $M_{12}$ ,  $J_2$ , and  $HS$ ), and we will show this to be the

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<sup>1</sup>See for example [3, p. 29]. This statement also requires that  $\tilde{L}(\Delta)$  does not contain the projective cover of the trivial module  $P(1)$  (see Appendix A.3). This is automatic for us, since we are working with ample geometries of the sporadics as discussed in Section 2.4.

<sup>2</sup>As mentioned e.g. in [7, Paragraph before Notation 6.9.3], this is essentially equivalent to  $\Delta \sim \mathcal{S}_2(G)$ : this notation means that the natural geometry of the group is homotopy equivalent to the the Brown poset of the group.

case with four more sporadics ( $Suz$ ,  $He$ ,  $Co_3$ ,  $Ru$ ). We will refer to a sporadic group in this class as a “Principal Block Part V-Projective.”

**Class III: The Principal Block Part of  $\tilde{\Lambda}$  Contains a Non-Projective Summand**

This class contains groups with the most subtle acyclicity properties. Lefschetz modules for groups in this class are non-projective, and may have non-projective parts in non-principal blocks. However, they are distinguished from the above class by having a non-projective summand in their principal block parts. Since they must still be acyclic, the non-projective part in the principal block lies in<sup>1</sup> the *representation theoretic nucleus*, as introduced by Benson-Carlson-Robinson [5, p. 68]. Though there are standard constructions in the literature [10] of modules in the nucleus (or “nucleus modules”), this thesis concretely exhibits for the first time an example of such a module naturally occurring in the context of  $L(\Delta)$ : in the O’Nan group. We will refer to a group in this class as a “Principal Block Part Non-Projective.”

As mentioned in the Introduction, there are seven sporadic groups known to have non-projective Lefschetz modules, but whose 2-modular irreducible decompositions are not yet known. These will fall into the latter two classes.

**The Benson-Smith Conjecture**

In section 8.2, Benson-Smith [7] remarks, “For all 15 sporadic groups  $G$ ...for which the 2-local geometry  $\Delta$  is not homotopy equivalent to (the Brown poset)  $\mathcal{S}_2(G)$ , it seems that the reduced Lefschetz module involves an indecomposable in a suitable non-principal 2-block of  $G$  of positive defect. No general explanation for this phenomenon seems known.”

As suggested earlier in this section,  $\Delta \not\approx \mathcal{S}_2(G)$  is essentially equivalent to  $\tilde{L}(\Delta)$  being non-projective [7, Paragraph before Notation 6.9.3]. Also, Benson-Smith had not seen the larger examples we study here, including some unexpected results mentioned above. So for our purposes, we can formulate the conjecture as follows.

**Conjecture 2.7.1 (Refined Benson-Smith Conjecture)** *All 15 sporadic groups affording non-projective Lefschetz modules have a non-projective part in a non-principal block of nonzero<sup>2</sup> defect.*

In view of the  $p$ -test 2.5.5, this conjecture suggests that a non-principal block will contain a part  $\Phi$  that violates the test, i.e. such that  $|G|_p$  does not divide  $\Phi(1)$ .

In addition, Smith [30] observed a pattern: the defect of the largest non-principal block seems to equal the 2-power difference between  $|\tilde{\Lambda}|_2$  and  $|G|_2$ . We seek to verify this pattern for all five sporadic groups we study.<sup>3</sup>

<sup>1</sup>Technically, it is an abuse of terminology to say that a module lies in the nucleus. The definition of representation theoretic nucleus refers to *varieties* for modules, not the modules themselves. We do not study varieties here, however, and instead focus on the acyclic modules themselves. So when we describe a “module in the nucleus,” we actually mean that that the *variety* of the module lies in the nucleus.

<sup>2</sup>Blocks of defect 0 may well occur, and these are not considered relevant to the comparison of the principal block with a block of *positive* defect.

<sup>3</sup>For an approach to this pattern via vertices of modules rather than blocks, see Sawabe [28].

## CHAPTER 3

### KNOWN CASES

We discussed in the Background that there are three sporadic groups with non-projective Lefschetz characters which have already been calculated explicitly:  $M_{12}$ ,  $J_2$ , and  $HS$ . However, the 2-modular block theory of the Lefschetz character has not been uniformly described for all these groups in the literature. In this chapter, we present the Lefschetz characters as previously calculated, and then proceed to display the 2-modular block theory.<sup>1</sup>

#### 3.1 The Mathieu group $M_{12}$

The Atlas [12, p. 33] gives the maximal subgroups of interest:

$$H_1 \cong M_8.S_4 \cong 2_+^{1+4}.S_3, \text{ and}$$

$$H_2 \cong 4^2:D_{12}.$$

From the decomposition matrices website [8] entry for  $M_{12}$ , we see that there are just 2 blocks: the principal block and a block of defect 2. Benson-Wilkerson [6, p. 44] computed the Lefschetz character of  $M_{12}$  in terms of characteristic 0 irreducibles:  $\tilde{\Lambda}_{M_{12}} = [144a] + [176aa]$ . Using the Atlas [12, p. 33], we can express this as  $\tilde{\Lambda}_{M_{12}} = \chi_{14} + 2\chi_{15}$ . We see in Figure 3.3<sup>2</sup> that these characters lie in the non-principal block, meaning that the Lefschetz character comes entirely from this block.

Using GAP [20], we calculate the Lefschetz character as a vector with columns indexed by the conjugacy classes of  $M_{12}$ . Since we are studying projectivity in characteristic  $p = 2$ , we focus our attention on the 2-elements: entries at the conjugacy classes 2A and 2B as given

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<sup>1</sup>Our calculations are summarized briefly here. For further details on this process, see the example of *Suz* in the next chapter.

<sup>2</sup>For aesthetic reasons, we display decomposition matrices with dots (“.”) in place of 0’s.

	$\varphi_3$	$\varphi_4$	$\varphi_6$
$\chi_4$	1	.	.
$\chi_5$	.	1	.
$\chi_{14}$	.	.	1
$\chi_{15}$	1	1	1

Figure 3.3. The decomposition matrix for block 2 of  $M_{12}$ .

	$\varphi_7$	$\varphi_8$	$\varphi_{10}$
$\chi_{12}$	.	.	1
$\chi_{16}$	1	.	1
$\chi_{17}$	.	1	1
$\chi_{19}$	1	1	1

Figure 3.4. The decomposition matrix for block 2 of  $J_2$ .

in the Atlas for  $M_{12}$  [12, p. 33]. These are listed in boldface, along with the 2-part of the dimension:  $\tilde{\Lambda}_{M_{12}} = [ \overbrace{496}^{2^4 \cdot 31}, -\mathbf{4}, \mathbf{0}, -8, -5, 0, 0, 1, -1, 0, 0, 0, 1, 1, 1 ]$ .

### 2-Modular Block Theory

The Lefschetz character fails the  $p$ -test: the 2-part of  $M_{12}$  has order  $2^6$ , which does not divide the dimension of the character  $|\tilde{\Lambda}_{M_{12}}| = 496$ . It also fails the vanishing test, as it does not vanish at the 2A element. Either one of these tests independently tells us that the Lefschetz module is not projective.

Hence we know that  $\Delta^{2A}$  is not contractible. From the viewpoint of Section 2.6, the value -4 in  $\tilde{\Lambda}_{M_{12}}$  matches the degree of the Lefschetz module of the non-contractible building  $\Sigma$  of a subgroup  $L_2(4)$  in the centralizer of a 2A element. This suggests that  $\Delta^{2A}$  may be homotopy equivalent to this particular  $\Sigma$ .

Finally, as we study these non-projective Lefschetz characters, it may be useful to examine how “close” each one is to being projective. Using Figure 3.3, we calculate our “best guess” of the Lefschetz character in terms of its possible projective and non-projective parts as:

$$\tilde{\Lambda}_{M_{12}} = \Phi(\varphi_6) + \chi_{15}.$$

### 3.2 The Janko group $J_2$

The Atlas [12, p. 42] tells us the relevant maximal subgroups:

$$\begin{aligned} H_1 &\cong 2_1^{1+4}.A_5, \text{ and} \\ H_2 &\cong 2^{2+4}:(3 \times S_3). \end{aligned}$$

Benson-Smith [7, after Theorem 8.7.1] observed that the Lefschetz character of  $J_2$  can be expressed as a combination of its irreducible complex characters:  $\tilde{\Lambda}_{J_2} = \chi_{16} + \chi_{17} + \chi_{19}$ , noting that these characters lie in the non-principal block (see Figure 3.4). Now using GAP [20], we

calculate  $\tilde{\Lambda}_{J_2} = [ \overbrace{736}^{2^5 \cdot 23}, \mathbf{0}, -\mathbf{4}, 16, -5, 0, 1, 1, -4, -4, 0, -1, 1, 0, 1, 1, 0, 0, 0, 1, 1 ]$ .

Again from the viewpoint of Section 2.6, the value -4 at class 2B matches the degree of  $\tilde{\Lambda}(\Sigma)$  for building  $\Sigma$  of an  $L_2(4)$  in  $C(2B)$ ,<sup>1</sup> which suggests that  $\Delta^{2B}$  may be homotopy equivalent to this  $\Sigma$ .

### Block Theory

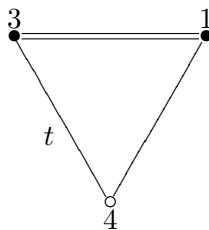
The 2-part of  $J_2$  has order  $2^7$ , which does not divide the dimension of the character  $|\tilde{\Lambda}_{J_2}|$ . So the  $p$ -test tells us right away that the Lefschetz module is non-projective. We can express our “best guess” of the Lefschetz character in terms of possible projective and non-projective parts as:  $\tilde{\Lambda}_{J_2} = \Phi(\varphi_8) + \chi_{16}$ .

### 3.3 The Higman-Sims group $HS$

The maximal 2-local subgroups of  $HS$  are found in the Atlas [12, p. 80]:

$$\begin{aligned} H_1 &\cong 4 \cdot 2^4 : S_5 \\ H_3 &\cong 4^3 : L_3(2), \text{ and} \\ H_4 &\cong 2^4 : S_6. \end{aligned}$$

The 2-local geometry has diagram [7, §8.10]:



where  $t$  indicates a complete triple system with 5 points and 10 triples [27, p. 286].

Klaus Lux first calculated the Lefschetz character for  $HS$  in a private communication to Stephen D. Smith. The values on the conjugacy classes are:

$$\tilde{\Lambda}_{HS} = \left[ \overbrace{42624}^{27 \cdot 333}, \mathbf{0}, -\mathbf{16}, -36, 0, 0, 0, 24, -1, 4, 2, 0, 1, 0, 0, 0, 0, -1, -1, -1, 0, -1, 0, 0 \right].$$

In particular, we note that the Lefschetz module is non-projective since the Lefschetz character does not vanish at element 2B. This value of -16 is the degree of  $\tilde{L}(\Sigma)$  for  $\Sigma$  the building of  $Sp_4(2)$  in  $C(2B)$ . In the viewpoint of Section 2.6, we guess that possibly  $\Delta^{2B} \sim \Sigma$ .

Note that  $|HS| = 2^9$  does not divide the dimension of  $|\tilde{\Lambda}_{HS}|$ . This also tells us that the Lefschetz module is non-projective.

As a combination of irreducible complex characters of  $HS$ , Lux computed:

$$\tilde{\Lambda}_{HS} = \chi_{14} + \chi_{15} + 2\chi_{16} + 2\chi_{17} + \chi_{18} + 2\chi_{19} + 2\chi_{20} + 2\chi_{21} + 2\chi_{22} + 2\chi_{23} + 4\chi_{24}.$$

For convenience in later calculations, we can express this as a “scalar product” vector with entries given by the coefficients of the above expression:

$$\tilde{\Lambda}_{HS} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 2, 1, 2, 2, 2, 2, 2, 4].$$

We are now ready to examine the contributions from each block.

	$\varphi_1$	$\varphi_2$	$\varphi_3$	$\varphi_4$	$\varphi_5$	$\varphi_8$
$\chi_1$	1	.	.	.	.	.
$\chi_2$	2	1	.	.	.	.
$\chi_3$	1	1	1	.	.	.
$\chi_4$	2	1	.	1	.	.
$\chi_5$	2	1	.	1	.	.
$\chi_6$	2	1	.	1	.	.
$\chi_7$	3	2	.	1	.	.
$\chi_8$	3	2	1	1	.	.
$\chi_9$	3	2	.	1	1	.
$\chi_{10}$	4	3	1	1	1	.
$\chi_{11}$	4	3	1	1	1	.
$\chi_{12}$	4	3	1	1	1	.
$\chi_{13}$	3	3	2	1	1	.
$\chi_{16}$	.	.	1	.	.	1
$\chi_{17}$	6	3	1	2	.	1
$\chi_{19}$	4	2	1	1	1	1
$\chi_{20}$	7	4	1	2	1	1
$\chi_{21}$	7	4	1	2	1	1
$\chi_{22}$	8	5	2	2	2	1
$\chi_{23}$	10	7	3	3	2	1

Figure 3.5. The decomposition matrix for the principal block of  $HS$ .

### Block Theory

The decomposition matrix for the principal block of  $HS$  is in Figure 3.5. In comparing it to the characters in  $\tilde{\Lambda}_{HS}$ , we see that this time there is some contribution to the Lefschetz character from the principal block. By inspection of Figure 3.5, observe that  $2\Phi(\varphi_8)$  matches the corresponding coefficients of  $\tilde{\Lambda}_{pr}^{HS}$  exactly. So we can express this as:  $\tilde{\Lambda}_{pr}^{HS} = 2\Phi(\varphi_8)$ . Hence the principal block part of the Lefschetz module of  $HS$  has v-projective character.

A few basic computations using Figure 3.6 show that there is no possible way to express  $\tilde{\Lambda}_{b_2}^{HS}$  as a combination of projective covers of 2-modular characters. The “closest” we could come is  $\tilde{\Lambda}_{b_2}^{HS} = \Phi(\varphi_6) + \Phi(\varphi_7) + \Phi(\varphi_9) + \chi_{24}$ . Hence despite “almost” having v-projective character, the block 2 part of the Lefschetz module of  $HS$  is non-projective.

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<sup>1</sup>C(2B) is shorthand for the centralizer of conjugacy class 2B.

	$\varphi_6$	$\varphi_7$	$\varphi_9$
$\chi_{14}$	1	.	.
$\chi_{15}$	.	1	.
$\chi_{18}$	.	.	1
$\chi_{24}$	1	1	1

Figure 3.6. The decomposition matrix for block 2 of  $HS$ .

### The Lefschetz Character Values Separated into Blocks

Our calculations in the previous subsection showed that  $\tilde{\Lambda}_{pr}^{HS}$  is v-projective and  $\tilde{\Lambda}_{b2}^{HS}$  is non-projective. In this subsection, we are not finding anything new about  $\tilde{\Lambda}_{HS}$ . Instead, we present a potentially useful tool to analyze Lefschetz characters which has perhaps not been pursued in the literature: separating the Lefschetz character values according to the 2-modular blocks of the group.

Earlier we found the Lefschetz character indexed by the conjugacy classes of  $HS$ :

$$\tilde{\Lambda}_{HS} = [42624, 0, -16, -36, 0, 0, 0, 24, -1, 4, 2, 0, 1, 0, 0, 0, 0, -1, -1, -1, 0, -1, 0, 0].$$

First we use GAP [20] to find the vector of Lefschetz character values given just by characters in the principal block:

$$\tilde{\Lambda}_{pr}^{HS} = [ \overbrace{26624}^{2^{11} \cdot 13}, \mathbf{0}, \mathbf{0}, -16, 0, 0, 0, 24, 24, 4, 0, 0, -4, 0, 0, 0, 0, 4, 4, 0, -6, 0, 0 ]$$

We see that  $|HS|_2 = 2^9$  divides the degree above, and that the principal block has zeros at both 2-elements. The latter test tells us that  $\tilde{\Lambda}_{pr}^{HS}$  is v-projective, independent of the above decomposition of  $\tilde{\Lambda}_{pr}^{HS}$  as a combination of projectives.

For the non-principal block, we obtain:

$$\tilde{\Lambda}_{b2}^{HS} = [ \overbrace{16000}^{2^7 \cdot 125}, \mathbf{0}, -\mathbf{16}, -20, 0, 0, 0, 0, -25, 0, 2, 0, 5, 0, 0, 0, 0, -1, -5, -5, 0, 5, 0, 0 ].$$

The  $p$ -test proves that this block is not a projective module, since  $|HS|_2 = 2^9$  does not divide the degree above.

### 3.4 Discussion

These groups were the first sporadics with non-projective Lefschetz modules to have their Lefschetz characters calculated. In  $M_{12}$  and  $J_2$ , there was little more to discover, since the entire Lefschetz character came from the non-principal block.  $HS$  was a bit more interesting, with contributions from both blocks.

Note the pattern observed by Smith:  $|M_{12}|_2 = 2^6$  and  $|\tilde{\Lambda}_{M_{12}}|_2 = 2^4$  and the non-principal block of  $M_{12}$  has defect 2. Next  $|J_2|_2 = 2^7$  and  $|\tilde{\Lambda}_{J_2}|_2 = 2^5$  and the non-principal block of  $J_2$

has defect 2. Finally  $|HS|_2 = 2^9$  and  $|\tilde{\Lambda}_{HS}|_2 = 2^7$  and the non-principal block of  $HS$  has defect 2. In each case, the defect of the non-principal block matches the 2-part difference between the 2-part of the group order and the 2-part of the dimension of the Lefschetz character. We will continue to check this pattern throughout our study.

The original Benson-Smith conjecture [7, §8.2] was casually formulated after seeing these examples. In fact, the way it is stated seems to reflect the less interesting structure of  $M_{12}$  and  $J_2$ , where the only contribution to the Lefschetz module was from the non-principal block, let alone the only non-projective contribution. The groups we study will provide richer Lefschetz characters than these cases previously known. Thus we will examine each group from the view of our refined statement of the conjecture given in 2.7.1.

## CHAPTER 4

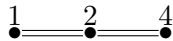
### THE SUZUKI GROUP

The first sporadic group we examine is the group  $Suz$ , discovered by Suzuki. Since we are investigating the 2-local geometry, we study the maximal subgroups which contain a fixed Sylow 2-group. We choose a representative for each set of conjugates, and number them by the ranks of the centers of their largest normal 2-subgroups [12, p. 131]:

$$\begin{aligned} H_1 &\cong 2^{1+6}.U_4(2), \\ H_2 &\cong 2^{2+8} : (A_5 \times S_3), \text{ and} \\ H_4 &\cong 2^{4+6} : 3A_6. \end{aligned}$$

#### 4.1 Geometry

The diagram for the natural geometry of  $Suz$  is given in Ronan-Smith [26, p. 288] as:



We proceed as in Benson-Smith [7, Def. 5.3.2] and give additional background information.<sup>1</sup> The double bonds each represent generalized quadrangles, but they are different types. The left bond is type  $Sp_4(2)$ , while the right bond is type  $U_4(2)$ .<sup>2</sup> We use the set  $I := \{1, 2, 4\}$  to index the simplex  $\mathcal{H}_I$  of subgroups of  $Suz$ . This simplex is determined by the  $H_i$ , with  $H_J := \bigcap_{i \in J} H_i$  for all nonempty subsets  $J \subseteq I$ . For simplicity, we will use the notation  $H_{ij}$  (without braces) to represent the subgroup  $H_{\{i,j\}}$ , etc.

The Lefschetz character in this geometry is an alternating sum of the permutation representations on vertices (with stabilizers  $H_1$ ,  $H_2$ , and  $H_4$ ), edges (with stabilizers  $H_{12}$ ,  $H_{14}$ ,  $H_{24}$ ), and 2-simplices with stabilizer  $H_{124}$ .

In order to examine the 2-local geometry of  $Suz$ , we introduce some notation related to the largest normal 2-groups of each maximal subgroup. The group actions we will study have the normal 2-groups in the kernel, so the action is given by the quotient modulo the 2-groups. Furthermore, we consider actions on the centers and also on the quotients of the 2-groups modulo these centers. Therefore, throughout these calculations, we make the following definitions for subgroups of  $H_i$ ,  $i = 1, 2, 4$ :

$$\mathcal{U}_i := O_2(H_i) \text{ (the largest normal 2-group of } H_i),$$

---

<sup>1</sup>Many thanks to Professor Robert A. Wilson of Queen Mary University of London for showing me the step-by-step details of how to do this process. We will explain this process in fuller detail this first time, as we don't expect our readers to be familiar with working with the geometry. These calculations for other groups work similarly, so we will summarize later cases more briefly.

<sup>2</sup>Some parts of the literature distinguish the two double bonds by indicating the "field of definition" in the diagram. We do not adopt this convention here, and instead discuss the different types as we describe the process.



stabilized in this action is  $Z(\mathcal{U}_2) =: Z_2 = 2^2$ , which is itself a 2-space over  $\mathbb{F}_2$ . However, in  $\text{Res}(H_1)$ , the action is on the space  $\widetilde{\mathcal{U}}_1 = \mathcal{U}_1/Z_1$ , so we must view  $Z_2$  within this space. We have  $Z_2 \leq \mathcal{U}_1$ , so  $Y_2 := Z_2 \cap \mathcal{U}_1 = Z_2$ , and hence  $\widetilde{Y}_2 := (Z_2 \cap \mathcal{U}_1)/Z_1 = Z_2/Z_1 = 2^2/2^1$ , a 1-space of  $\widetilde{\mathcal{U}}_1$ . Thus  $H_{12}$  contains a Sylow 2-group and stabilizes a 1-space in  $H_1$ . In the  $\Omega_6^-(2)$  geometry, a 1-space stabilized by a Sylow 2-group is a “singular vector,” or “isotropic point.”<sup>1</sup>

To find what subgroup of  $\mathcal{L}_1 \cong U_4(2) \cong \Omega_6^-(2)$  corresponds to  $H_{12}$  in  $H_1$ , we can remove the node  $H_2$  from the  $\text{Res}(H_1)$  diagram. Since  $H_{12}$  stabilizes an isotropic point, this is equivalent to fixing an isotropic point  $\widetilde{Y}_2$  from the  $\Omega_6^-(2)$  geometry. Then  $\widetilde{Y}_2^\perp/\widetilde{Y}_2$  gives a 4-space for the orthogonal group  $\Omega_4^-(2)$ , which is isomorphic to the alternating group  $A_5$  [12, p. 2].

From here we can use the Atlas. Looking in the “Maximal subgroups” section of the  $U_4(2)$  entry [12, p. 26], we see that  $A_5$  appears only in the first maximal subgroup,  $2^4:A_5$ . As a double-check, this subgroup has index 27, which matches the number of isotropic vectors in the 6-space  $\widetilde{\mathcal{U}}_1$  for  $\Omega_6^-(2)$ .

In fact, experience with the Atlas could save us even more work here. In the  $U_4(2)$  entry, we can look under the “Orthogonal (2)” column of the Maximal subgroups section, and see that “isotropic point” corresponds to  $2^4:A_5$ . This is the quotient  $\mathcal{L}_{12} := H_{12}/\mathcal{U}_{12}$  that induces the action inside  $H_{12}$ . We can combine this with the normal subgroup  $\mathcal{U}_1$  of  $H_1$  to give the structure of the intersection<sup>2</sup>:  $H_{12} = 2_-^{1+6}2^4A_5$ .

So we induce  $2^4:A_5$  up to  $U_4(2) = H_1/\mathcal{U}_1$ . Then we “inflate”<sup>3</sup> that result to a character of the full group  $H_1$ . This now gives us the induction of  $H_{12}$  up to  $H_1$ . The Atlas gives this character to be  $1a + 6a + 20a$ . The sum of the degrees of this character is 27, which of course agrees with our index. So we have

$$[1a]_{H_{12}} \uparrow^{Suz} = ([1a]_{H_{12}} \uparrow^{H_1})_{H_1} \uparrow^{Suz} = [1a + 6a + 20a]_{H_1} \uparrow^{Suz} .$$

---

<sup>1</sup>Subspaces are singular or non-singular according to whether or not they are stabilized by a Sylow 2-group. A Sylow 2-subgroup of  $H_1$  stabilizes totally singular subspaces of dimensions 1 and 2, whereas non-singular 1- and 2-spaces are not stabilized by a Sylow 2-group. There are no singular spaces of 3 or more dimensions in the  $\Omega_6^-(2)$  geometry.

In the classical literature, the terms “singular” and “totally singular” are used to define vectors and spaces (respectively) on which the quadratic form vanishes, while the terms “isotropic” and “totally isotropic” describe vectors and spaces (respectively) on which the bilinear form vanishes (see Tits [32, ch. 8]). In characteristic 2, all vectors are isotropic. Thus the term “isotropic point” seems less descriptive than “singular vector.” However, the Atlas [12] uses the “isotropic” language, so we also will adopt this terminology.

<sup>2</sup>It can be convenient to write this in “stack” form with the normal subgroup  $\mathcal{U}_1$  on the bottom, i.e.

$$H_{12} = \frac{2^4:A_5}{2_-^{1+6}} .$$

<sup>3</sup>That is, we extend it to the larger group  $H_1$  by declaring it to be trivial on the subgroup  $\mathcal{U}_1$ .

### Inducing $H_{14}$ up to $H_1$

Similarly as with  $H_{12}$ , we examine what is stabilized by  $H_{14}$  in  $H_1$ . The center of  $\mathcal{U}_4$  is  $Z_4 = 2^4$ , which is itself a 4-space over  $\mathbb{F}_2$ . Just as we did with  $Z_2$ , we view  $Z_4$  within  $\widetilde{\mathcal{U}}_1$ . First, we look at  $Z_4 \cap \mathcal{U}_1$ . This is the set of points of  $Z_4$  on isotropic lines in  $Z_4$  containing  $Z_1$ , i.e.  $Z_1^\perp$ , a 3-space of  $Z_4$ . Hence  $\widetilde{Y}_4 := (Z_4 \cap \mathcal{U}_1)/Z_1 = 2^3/2^1$ , which is a 2-space<sup>1</sup> of  $\widetilde{\mathcal{U}}_1$ . Thus  $H_{14}$  contains a Sylow 2-group and stabilizes a 2-space, or “isotropic line,” in  $H_1$ .

In the  $U_4(2)$  entry of the Atlas, we look under the Orthogonal (2) column of the Maximal subgroups section, and see that “isotropic line” corresponds to  $2(A_4 \times A_4).2$ . The Atlas gives this permutation character to be  $1a + 20a + 24a$ . As a double-check, the degree of this character is 45, matching the index of the subgroup  $H_{14}$  in  $H_1$ . We can realize the structure of this subgroup as  $H_{14} = 2_-^{1+6}2(A_4 \times A_4).2$ . We have

$$[1a]_{H_{14}} \uparrow^{Suz} = ([1a]_{H_{14}} \uparrow^{H_1})_{H_1} \uparrow^{Suz} = [1a + 20a + 24a]_{H_1} \uparrow^{Suz} .$$

### Inducing $H_{124}$ up to $H_1$

We could go through any chain of intermediate subgroups  $H_J$  to induce from  $H_{124}$  to Suzuki, but it seems easiest to go via  $H_{12}$  and  $H_1$ . We plan to use the  $H_{12}$  subgroup structure described above to describe  $H_{124}$  in terms of a subspace stabilized in the action of  $H_{12}$  on  $\mathcal{U}_4$ .

In  $H_{12}$ , we know that  $\mathcal{L}_{12} := H_{12}/\mathcal{U}_{12}$  has the structure of  $A_5$  acting on the 5 isotropic vectors of the 4-space  $\widetilde{Y}_2^\perp/\widetilde{Y}_2$ . Now in  $H_{124}$ , we have the additional constraint of stabilizing  $\widetilde{Y}_4 := (Z_4 \cap \mathcal{U}_1)/Z_1$ . We know from above that  $\widetilde{Y}_4$  is a 2-space in  $\mathcal{U}_1 = \Omega_6^-(2)$ . Now looking at  $\widetilde{Y}_4$  in  $H_{12}$ , we see that  $\widetilde{Y}_4$  is still a 2-space in the 5-space  $\widetilde{Y}_2^\perp$ . But the geometry of  $H_{12}$  is the 4-space  $\widetilde{Y}_2^\perp/\widetilde{Y}_2$ . Since the 2-space  $\widetilde{Y}_4$  contains the 1-space  $\widetilde{Y}_2$ , the space that is stabilized in this action is  $\widetilde{Y}_4/\widetilde{Y}_2$ , which is a 1-space.

Viewing this within  $A_5$ , we know that  $A_4$  is the subgroup that stabilizes a 1-space (or “isotropic point”) in  $A_5$ . The Atlas corroborates this, as isotropic point corresponds to  $A_4$ .

Therefore,  $H_{124}$  induced up to  $H_{12}$  will be the inflation of the permutation character of  $A_4$  (index 5) induced up to  $A_5$ . The structure of this subgroup can be written  $H_{124} = 2_-^{1+6}2^4.A_4$ . The induced character must be  $1a + 4a$ , as indicated in the Atlas entry for  $A_5$  [12, p. 2]. Thus in particular,  $H_{124}$  has index 5 in  $H_{12}$ , and so  $|\mathcal{L}_1 : H_{124}| = 135$ . Our approach can be summarized as:

$$[1a]_{H_{124}} \uparrow^{Suz} = (([1a]_{H_{124}} \uparrow^{H_{12}})_{H_{12}} \uparrow^{H_1})_{H_1} \uparrow^{Suz} = ([1a + 4a]_{H_{12}} \uparrow^{H_1})_{H_1} \uparrow^{Suz} .$$

We know GAP can induce any characters from  $H_1$  up to  $Suz$ , so all that remains is to find  $[1a + 4a]_{H_{12}} \uparrow^{H_1}$ . We translate the calculation in  $H_{12}$  to the quotient  $\mathcal{L}_{12} = 2^4:A_5$ . This is the maximal subgroup of  $U_4(2)$  first listed in the Atlas [12, p. 26]. We need to find the appropriate character table for  $2^4:A_5$  (as opposed to just  $A_5$ ) to induce up to  $H_1$ . Since GAP contains the character tables for maximal subgroups of various groups, we might hope it could

---

<sup>1</sup>Since  $\Omega_6^-(2)$  is of minus type, its maximal Witt index is 2 [31, ch. 2]. Hence this is the maximum possible dimension of  $\widetilde{Y}_4$ .

do this for us with the command “CharacterTable(“U4(2)m1”).” Then GAP could make the  $[1a + 4a]_{H_{12}} \uparrow^{H_1}$  calculation and we would be all set. Unfortunately, GAP does not have the table for this particular maximal subgroup, so we need to do some more work. Here there are many different approaches to find out the necessary character.

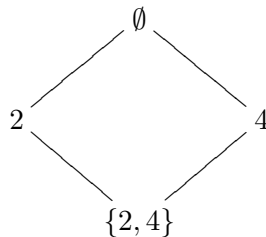
We use the viewpoint of Smith [29].<sup>1</sup> As defined generally in 4.1, we have  $\mathcal{L}_1 = H_1/\mathcal{U}_1 = U_4(2)$ , which is Lie type of rank 2. Our goal of inducing  $H_{124}$  up to  $H_1$  can be achieved by inducing  $\mathcal{L}_{124}$  up to  $\mathcal{L}_1$ . This way we will not have to find the induced character for a specific intermediate step, such as we tried above with  $\mathcal{L}_{12}$ .

For a Lie type group  $G$ , we have that for  $B$  a Borel subgroup,

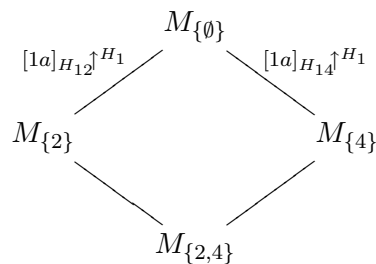
$$\text{Ind}_B^G(1a) = \bigoplus_{J \subseteq I} M_J$$

where the  $M_J$  are component modules corresponding to the types of simplices in the building and the index set  $I$  is the set of maximal parabolics over  $B$  [29, p. 598].

As we are working in  $\text{Res}(H_1)$ , we consider  $\mathcal{L}_1 = H_1/\mathcal{U}_1 = U_4(2)$ —which is of Lie type—to be our full group  $G$  and let  $I := \{2, 4\}$ . We have that  $\mathcal{L}_{12}$  and  $\mathcal{L}_{14}$  are minimal parabolics (also maximal parabolics in this case), and  $\mathcal{L}_{124}$  is Borel. In  $\text{Res}(H_1)$ , we know that  $M_2 \oplus M_{\{\emptyset\}} = [1a]_{H_{12}} \uparrow^{H_1}$  and  $M_4 \oplus M_{\{\emptyset\}} = [1a]_{H_{14}} \uparrow^{H_1}$ , so the subset lattice



corresponds to



as a lattice of modules. To make the calculations, we need only the characters of the modules, and so in particular we are interested in their degrees. We know that  $M_{\{\emptyset\}}$  is the trivial module

---

<sup>1</sup>For our purposes, it will suffice to consider the easier decomposition of the complex (characteristic 0) character. We do not need the stronger result of a direct-sum decomposition for the character in the natural characteristic  $p$  of  $G$ . The decomposition we use is given in Corollary 1.6 [29, p. 603].

and that  $M_{\{2,4\}} = M_I$ , the Steinberg module for  $G = U_4(2)$ . The trivial module obviously has degree 1, and the Steinberg module has degree  $2^N$ , where  $N = 6$  for  $U_4(2)$  [11, p. 262].<sup>1</sup> Furthermore, we already calculated that  $[1a]_{H_{12}} \uparrow^{H_1} = 1a + 6a + 20a$  and  $[1a]_{H_{14}} \uparrow^{H_1} = 1a + 20a + 24a$ , so our lattice can be shown as

$$\begin{array}{ccc}
 & 1a & \\
 & / \quad \backslash & \\
 6a + 20a & & 20a + 24a \\
 & \backslash \quad / & \\
 & 64a &
 \end{array}$$

in terms of degrees. The decomposition in [29] tells us that

$$1_{\mathcal{L}_{124}}^{\mathcal{L}_1} = \bigoplus_{J \subseteq \{2,4\}} M_J = M_{\{\emptyset\}} \oplus M_{\{2\}} \oplus M_{\{4\}} \oplus M_{\{2,4\}} = 1a + (6a + 20a) + (20a + 24a) + 64a.$$

Again as a double-check, the index 135 of  $H_{124}$  in  $H_1$  matches the sum of the above degrees. We have found the character we need in order to induce  $H_{124}$  up to  $H_1$ . We display it in the style of the Atlas [12]:

$$[1a]_{H_{124}} \uparrow^{Suz} = ([1a]_{H_{124}} \uparrow^{H_1})_{H_1} \uparrow^{Suz} = [1a + 6a + 20a + 24a + 64a]_{H_1} \uparrow^{Suz}.$$

#### 4.4 Inducing $H_{24}$ up to $H_2$

$\text{Res}(H_1)$  has taken care of all but one of the intersections of maximal subgroups we need to induce up to Suzuki. The last one needed is  $H_{24}$ , which we must examine in  $\text{Res}(H_2)$  or  $\text{Res}(H_4)$ . We will use  $\text{Res}(H_2)$ .

The diagram of  $\text{Res}(H_2)$  is disconnected:

$$\circ \quad \text{X} \quad \circ$$

Since  $H_2 \cong 2^{2+8} : (S_3 \times A_5)$ , this tells us that  $\text{Res}(H_2)$  will have the structure of a complete bipartite graph of a 5-set for  $A_5$  (with point stabilizer  $H_{24}$ ) with a 3-set for  $S_3$  (with point stabilizer<sup>2</sup>  $H_{12}$ ). Hence in particular this gives that  $H_{24}$  is of index 5 inside  $H_2$ . So looking in the Atlas entry for  $A_5$  [12, p. 2], we find only one possible subgroup:  $A_4$ . Thus  $H_{24} \cong 2^{2+8} : (S_3 \times A_4)$

<sup>1</sup>To use the formula in Theorem 14.3.2, note that  $U_4(2)$  is  ${}^2A_3(2)$  [12, p. 26]. Then  $l = 3$ , so  $N = \frac{l(l+1)}{2} = 6$ . Alternatively, we could simply use the 2-part of the group order:  $|U_4(2)|_2 = 2^6$ .

<sup>2</sup>As viewed in  $\text{Res}(H_1)$  earlier,  $H_{12} = 2^{1+6} : A_5$ . Here in  $\text{Res}(H_2)$ , we can view it with  $\mathcal{U}_2$  and  $\mathcal{L}_2$ , instead of with  $\mathcal{U}_1$  and  $\mathcal{L}_1$ :  $H_{12} = 2^{2+8} : (S_2 \times A_5)$ . These two descriptions of  $H_{12}$  are of course isomorphic.

inside  $H_2$  will be the inflation of the permutation character of  $A_4$  inside  $A_5$ , which is  $1a + 4a$ . So we have

$$[1a]_{H_{24}} \uparrow^{Suz} = ([1a]_{H_{24}} \uparrow^{H_2})_{H_2} \uparrow^{Suz} = [1a + 4a]_{H_2} \uparrow^{Suz} .$$

#### 4.5 The GAP Calculation

We are ready to calculate the Lefschetz character of  $Suz$ . Recall that the definition of the Lefschetz character is an alternating sum. Hence induced representations of vertex stabilizers will be positive, those of edge stabilizers will be negative, and of the 2-simplex stabilizer will be positive in the alternating sum. We summarize our findings:

$$\begin{array}{r}
 [1a]_{H_{124}} \uparrow^{Suz} = [1a + 6a + 20aa + 24a + 64a]_{H_1} \uparrow^{Suz} \\
 -[1a]_{H_{12}} \uparrow^{Suz} = -[1a + 6a + 20a]_{H_1} \uparrow^{Suz} \\
 -[1a]_{H_{14}} \uparrow^{Suz} = -[1a + 20a + 24a]_{H_1} \uparrow^{Suz} \\
 [1a]_{H_1} \uparrow^{Suz} = [1a]_{H_1} \uparrow^{Suz} \\
 \hline
 -[1a]_{H_{24}} \uparrow^{Suz} = -[1a + 4a]_{H_2} \uparrow^{Suz} \\
 [1a]_{H_2} \uparrow^{Suz} = [1a]_{H_2} \uparrow^{Suz} \\
 \hline
 [1a]_{H_4} \uparrow^{Suz} = [1a]_{H_4} \uparrow^{Suz}
 \end{array}$$

We compute the Lefschetz character by taking the above characters induced to  $H_i$ , and inducing these characters from  $H_i$  up to  $Suz$  for  $i = \{1, 2, 4\}$ . We can do some quick cancellation in the above equations and simplify the final calculation (remembering to subtract 1 so that it is *reduced*):

$$\tilde{\Lambda}_{Suz} = [64a]_{H_1} \uparrow^{Suz} - [4a]_{H_2} \uparrow^{Suz} + [1a]_{H_4} \uparrow^{Suz} - 1. \quad (4.1)$$

Using the Atlas [12, p. 26], we find that the only character of  $U_4(2)$  of degree 64 is  $\chi_{19}$ . Similarly, looking in the entry of  $A_5$  [12, p. 2], we see that  $4a$  corresponds to  $\chi_4$ . And of course,  $1a$  is  $\chi_1$  for the  $H_4$  calculation. Thus we will induce character  $\chi_{19}$  from  $H_1$  up to  $Suz$ , character  $\chi_4$  from  $H_2$  up to  $Suz$ , and character  $\chi_1$  from  $H_4$  up to  $Suz$ . We are all set up to let GAP perform the computation.

As the Lefschetz character is a class function, it will have 43 entries, matching the number of conjugacy classes of  $Suz$ . Here are the GAP commands which implement the calculation in Equation 4.1. We have suppressed the outputs of most commands with double semicolons. We use a single semicolon to display the output on the final step.

```

gap> S := CharacterTable("Suz"); This gets the characters of Suz.
gap> h1 := CharacterTable("Suzm4"); Gets the characters of H1.
gap> h1indS := InducedClassFunctions(Irr(h1),S); Gets all the induced characters H1 up to Suz.
gap> h1c19 := h1indS[19]; Picks out [64a]H1 up to Suz.
gap> h2 := CharacterTable("Suzm9"); Gets the characters of H2.
gap> h2indS := InducedClassFunctions(Irr(h2),S); Gets the induced characters H2 up to Suz.
gap> h2c4 := h2indS[4]; Picks out [4a]H2 up to Suz.
gap> h4 := CharacterTable("Suzm7"); Gets the characters of H4.

```

```

gap> h4indS := InducedClassFunctions(Irr(h4),S);    Gets the induced characters  $H_4 \uparrow^{Suz}$ .
gap> h4c1 := h4indS[1];                            Picks out  $[1a]_{H_4} \uparrow^{Suz}$ .
gap> LefSuz := h1c19-h2c4+h4c1-1;                  Computes (Equation 4.1).
[ 4189184, 0, -64, 3968, -352, -73, 0, 0, 0, 8, -16, 9, 0, 0, 0, 0, -1, -1, 0, 0, 0, 2, 2, 0, 1, -1, 0,
0, 0, -1, 0, -1, -1, -1, 2, 2, 3, 0, 0, 0, -1, -1, 0 ]

```

This is  $\tilde{\Lambda}_{Suz}$  in vector form, i.e., the columns are indexed by the conjugacy classes of  $Suz$ .

Note that the degree  $\tilde{\chi}$  of the Lefschetz character is  $4189184 = 2^{10} \cdot 4091$ . Thus the 2-part of the group order  $|Suz|_2 = 2^{13}$  does not divide  $|\tilde{\Lambda}_{Suz}|$ , which by the  $p$ -test 2.5.5 tells us immediately that  $\tilde{L}_{Suz}(\Delta)$  is non-projective.

#### 4.6 Block Theory

The decomposition matrices website [8] has the 2-modular decomposition matrices for  $Suz$ . There are just two blocks for  $Suz$ , the principal block and a much smaller block. We can get some more information about the blocks of  $Suz$  using the “PrimeBlocks” command (and continuing with the definitions already assigned) in GAP [20]:

```
gap> PrimeBlocks(S,2);
```

In particular, part of GAP’s output is

```
defect := [ 13, 3 ],
```

which tells us that block 2 is of defect 3. We also could have calculated this from the given degrees of the  $\chi$ ’s in block 2 of [8].

From 4.5, we have the Lefschetz character as a set of values corresponding to the conjugacy classes of  $Suz$ . We want to see it expressed as a combination of irreducible complex characters of  $Suz$ . We use the “MatScalarProducts” command to do this.

```

gap> SPsuz := MatScalarProducts(Irr(S),[LefSuz]);
[[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 2,
2, 1, 2, 2, 2, 2, 2, 3 ]]

```

This is  $\tilde{\Lambda}_{Suz}$  in scalar product form, i.e., the columns are indexed by the irreducible complex characters of  $Suz$ .

We use these coefficients of irreducible characters along with the decomposition matrices website [8] to break down  $\tilde{\Lambda}(\Delta)$  into projective and non-projective parts. In the case of  $Suz$ , we can actually do much of this process with the “naked eye.”

#### The Principal Block

Let  $\tilde{\Lambda}_{pr}^{Suz}$  denote the Lefschetz character of  $Suz$  restricted to the principal block. Observe that the first 23 entries of SPsuz are all zero. In looking at the decomposition matrix for block 1 of  $Suz$  (See Figure 4.7<sup>1</sup>), we note that the projective covers of characteristic 2 characters  $\varphi_1$  through  $\varphi_{12}$  all have at least one positive coefficient for a complex character  $\chi_i$ , with  $i \leq 23$ . Therefore, the projective part of  $\tilde{\Lambda}_{pr}^{Suz}$  can contain the projective covers of at most  $\varphi_{13}$

---

<sup>1</sup>This table appears in the decomposition matrices website [8], and the LaTeX code was provided by GAP [20], retaining our previous commands and using the commands:

```

gap> b:=BrauerTable(S,2);
gap> Print(LaTeXStringDecompositionMatrix(b,1));

```

	$\varphi_1$	$\varphi_2$	$\varphi_3$	$\varphi_4$	$\varphi_5$	$\varphi_6$	$\varphi_7$	$\varphi_8$	$\varphi_9$	$\varphi_{10}$	$\varphi_{11}$	$\varphi_{12}$	$\varphi_{13}$	$\varphi_{14}$
$\chi_1$	1	.	.	.	.	.	.	.	.	.	.	.	.	.
$\chi_2$	1	.	.	1	.	.	.	.	.	.	.	.	.	.
$\chi_3$	2	1	1	1	.	.	.	.	.	.	.	.	.	.
$\chi_4$	.	.	.	1	.	.	1	.	.	.	.	.	.	.
$\chi_5$	1	1	1	1	.	.	1	.	.	.	.	.	.	.
$\chi_6$	.	.	.	.	.	.	.	1	.	.	.	.	.	.
$\chi_7$	1	1	1	1	1	.	1	1	.	.	.	.	.	.
$\chi_8$	1	1	1	1	.	1	1	1	.	.	.	.	.	.
$\chi_9$	2	2	2	2	1	1	1	1	.	.	.	.	.	.
$\chi_{10}$	1	1	1	1	1	1	2	1	1	.	.	.	.	.
$\chi_{11}$	4	3	3	4	1	1	2	1	.	1	.	.	.	.
$\chi_{12}$	2	2	2	2	1	1	1	2	.	1	.	.	.	.
$\chi_{13}$	1	1	2	1	1	1	1	1	.	.	.	1	.	.
$\chi_{14}$	1	2	1	1	1	1	1	1	.	.	1	.	.	.
$\chi_{15}$	1	1	1	2	1	1	2	1	1	1	.	.	.	.
$\chi_{16}$	2	2	2	1	2	2	2	3	1	.	.	.	.	.
$\chi_{17}$	3	3	3	3	2	2	3	3	1	1	.	.	.	.
$\chi_{18}$	3	3	3	3	2	2	3	3	1	1	.	.	.	.
$\chi_{19}$	3	3	3	3	2	2	3	3	1	1	.	.	.	.
$\chi_{20}$	4	4	4	4	3	3	2	3	.	1	1	1	.	.
$\chi_{21}$	4	5	5	4	3	4	4	4	1	1	1	1	.	.
$\chi_{22}$	4	5	5	4	4	3	4	4	1	1	1	1	.	.
$\chi_{23}$	4	5	5	4	4	4	4	5	1	1	1	1	.	.
$\chi_{24}$	4	5	5	2	4	4	4	6	2	1	.	.	1	1
$\chi_{25}$	2	4	5	1	4	4	3	4	1	.	1	2	1	.
$\chi_{26}$	2	5	4	1	4	4	3	4	1	.	2	1	.	1
$\chi_{28}$	5	7	7	5	5	5	5	5	1	1	2	2	.	.
$\chi_{30}$	6	8	8	5	6	6	6	7	2	1	2	2	.	.
$\chi_{31}$	5	7	8	4	6	6	5	7	1	1	2	2	1	.
$\chi_{32}$	5	8	7	4	6	6	5	7	1	1	2	2	.	1
$\chi_{33}$	8	10	10	8	7	7	8	8	2	2	2	2	.	.
$\chi_{34}$	8	11	11	8	9	9	8	10	2	3	3	3	.	.
$\chi_{36}$	6	12	12	5	10	10	8	10	2	1	4	4	1	1
$\chi_{38}$	12	16	16	11	13	13	13	15	4	3	3	3	1	1
$\chi_{39}$	8	15	15	6	12	12	9	12	2	1	5	5	1	1
$\chi_{40}$	12	16	16	10	13	13	12	16	4	4	3	3	1	1
$\chi_{41}$	12	18	18	10	13	13	11	14	2	2	5	5	1	1
$\chi_{42}$	15	21	21	13	16	16	15	18	4	3	5	5	1	1

Figure 4.7. The decomposition matrix for the principal block of  $Suz$ .

	$\varphi_{15}$	$\varphi_{16}$	$\varphi_{17}$
$\chi_{27}$	1	.	.
$\chi_{29}$	.	1	.
$\chi_{35}$	1	1	.
$\chi_{37}$	1	.	1
$\chi_{43}$	1	1	1

Figure 4.8. The decomposition matrix for Block 2 of  $Suz$ .

and  $\varphi_{14}$ . This greatly limits the number of possible combinations, and in fact we can see by inspection that  $\Phi(\varphi_{13}) + \Phi(\varphi_{14})$  matches  $SP_{Suz}$  exactly for the entries in block 1. So  $\tilde{\Lambda}_{pr}^{Suz} = \Phi(\varphi_{13}) + \Phi(\varphi_{14})$ . Hence the principal block part of the Lefschetz module of  $Suz$  has v-projective character.<sup>1</sup>

### Block 2

Let  $\tilde{\Lambda}_{b2}^{Suz}$  denote the Lefschetz character of  $Suz$  restricted to block 2. Now block 2 has only five complex characters:  $\chi_{27}$ ,  $\chi_{29}$ ,  $\chi_{35}$ ,  $\chi_{37}$ , and  $\chi_{43}$ . We can simplify our calculation in block 2 of  $\tilde{\Lambda}_{Suz}$  by indexing the columns with just these characters from  $SP_{Suz}$ , thus defining  $SP_{Suzb2} := [1, 1, 2, 1, 3]$ . Since  $\Phi(\varphi_{15}) = [1, 0, 1, 1, 1]$  and  $\Phi(\varphi_{16}) = [0, 1, 1, 0, 1]$  (see Figure 4.8), we calculate:

$$SP_{Suzb2} - \Phi(\varphi_{15}) - \Phi(\varphi_{16}) = [0, 0, 0, 0, 1].$$

Since the only projective cover left is  $\Phi(\varphi_{17}) = [0, 0, 0, 1, 1]$ , clearly there is no possible combination of projective covers of characteristic 2 characters in block 2 that will comprise  $SP_{Suzb2}$ . Our best guess as to the “closest” we can come to decomposing this as a v-projective character

$$\text{is: } \tilde{\Lambda}_{b2}^{Suz} = \overbrace{\Phi(\varphi_{15}) + \Phi(\varphi_{16})}^{\text{v-projective part}} + \chi_{43}.$$

Hence despite “almost” having v-projective character, we see that the block 2 part of the Lefschetz character of  $Suz$  is non-projective.

We have now verified the Benson-Smith conjecture [7, footnote after Theorem 8.2.1] that the non-projective part appears in a non-principal block positive defect. Thus  $Suz$  is categorized into Class II—it is a “Principal Block Part V-Projective” group, joining  $M_{12}$ ,  $J_2$ , and

---

<sup>1</sup>As a reminder, that this does not imply that the principal block part of the actual Lefschetz module itself is projective, though it does give us strong suspicions. It is possible that methods of Webb [33] and Grodal [19] could force the *vertex* of this block part to be trivial in order to conclude that this principal block part is in fact a projective module. See Appendix A.4 for brief background on vertices and sources.

*HS*. Furthermore, the non-principal block has defect 3, which precisely matches the 2-power difference between  $|G|_2 = 2^{13}$  and  $\tilde{\chi}_2 = |\tilde{\Lambda}_{Suz}|_2 = 2^{10}$ , continuing the pattern noticed by Smith.

#### 4.7 The Lefschetz Character Values Separated into Blocks

For future study of block parts of the Lefschetz module, it might be useful to have the Lefschetz character separated according to its blocks. Earlier we found the Lefschetz character indexed by the conjugacy classes of *Suz*:

LefSuz = [ 4189184, **0**, **-64**, 3968, -352, -73, 0, 0, 0, 8, -16, 9, 0, 0, 0, 0, -1, -1, 0, 0, 0, 2, 2, 0, 1, -1, 0, 0, 0, -1, 0, -1, -1, -1, 2, 2, 3, 0, 0, 0, -1, -1, 0 ].

Since we are studying projectivity in characteristic  $p = 2$ , we focus our attention on the 2-elements: entries at the conjugacy classes 2A and 2B as given in the Atlas for *Suz* [12, p. 128]. These are listed in boldface.

The -64 value in the 2B entry tells us right away that the Lefschetz module for *Suz* is not projective, since the character fails the vanishing test 2.5.6. From the viewpoint of Section 2.6, this value of -64 is also the degree of  $\tilde{L}(\Sigma)$  for the building  $\Sigma$  of  $L_3(4)$  in  $C(2B)$ . We propose that possibly  $\Delta^{2B} \sim \Sigma$ .

##### The Principal Block

The principal block has all of the characters but five:  $\chi_{27}$ ,  $\chi_{29}$ ,  $\chi_{35}$ ,  $\chi_{37}$ , and  $\chi_{43}$ . We take SPsuz and put italic  $\theta$ 's as placeholders for the characters not in the principal block:

SPsuzPr := [ 0, 2, 1, 1,  $\theta$ , 0,  $\theta$ , 0, 1, 1, 0, 0,  $\theta$ , 2,  $\theta$ , 2, 2, 2, 2, 2,  $\theta$  ]

Now using the “LinearCombination” command in GAP [20], we go through all the characters of *Suz* and multiply each by its corresponding coefficient in SPsuzPr. Then we sum the weighted characters to give us a single vector of Lefschetz character values given just by characters in the principal block:

LsuzPr := [ 2834432, **0**, **0**, 512, -352, -64, 0, 0, 0, 0, 32, 32, 0, 0, 0, 0, 0, -8, 0, 0, 0, -4, -4, 0, 0, -4, 0, 0, 0, 0, 0, 3, 3, 0, -4, -4, 2, 0, 0, 0, 1, 1, 0 ].<sup>1</sup>

**Remark 4.7.1 (Confirming Calculations)** Though it is not necessary, we mention how one can—as a double-check—confirm that these values meet the requirements we discussed earlier about projectivity.

The Lefschetz character of the principal block  $\tilde{\Lambda}_{pr}^{Suz}$  has degree  $2834432 = 2^{14} \cdot 173$ . Therefore,  $|Suz|_2 = 2^{13}$  does divide  $|\tilde{\Lambda}_{pr}^{Suz}|$ , so it passes the  $p$ -test 2.5.5. Observe also that this character has zeros at both 2-elements, so it passes the vanishing test and hence is a v-projective character.

---

<sup>1</sup>Having kept the previous definitions and defining LsuzPr, we can check the accuracy of these values with the following GAP commands:

```
gap> IrrSuz := Irr(S);
gap> MatScalarProducts(IrrSuz,[LsuzPr]);
[[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 2, 0, 2, 2, 2, 2, 2, 0 ]]
```

This matches SPsuzPr.



## CHAPTER 5

### THE O'NAN GROUP

The next group we look at is  $O'N$ , discovered by O'Nan. Much of this process is the same as with  $Suz$ , so we present a somewhat more abbreviated study with fewer details. The maximal subgroups of interest are [12, p. 132]:

$$\begin{aligned} H_1 &\cong 4_2 \cdot L_3(4):2_1, \text{ and} \\ H_3 &\cong 4^3 \cdot L_3(2). \end{aligned}$$

With only two maximal subgroups, the process of obtaining the Lefschetz character is much simpler. The simplicial complex is just dimension 1, i.e. a bipartite graph. So the convention in the literature is to not write down a diagram. We have only one intersection to find:  $H_{13}$ .

#### 5.1 $H_{13}$

We view  $H_{13}$  inside  $H_3$ , since the structure of  $L_3(2)$  is easier to work with than  $L_3(4)$ .  $H_{13}$  is the stabilizer in  $H_3$  of  $Z_1 := Z(\mathcal{U}_1)$ , a single involution. We have  $\mathcal{L}_3 = H_3/\mathcal{U}_3 \cong L_3(2)$ . It is a standard fact<sup>1</sup> that the stabilizer in  $L_3(2)$  of an involution is  $2^2 L_2(2) \cong 2^2 S_3 \cong S_4$ . Thus the structure of this subgroup is  $H_{13} = \frac{S_4}{4^3}$ .

So inducing  $H_{13}$  up to  $H_1$  will be the inflation of the permutation character of  $S_4$  up to  $L_3(2)$ . Using the Atlas  $L_3(2)$  entry [12, p. 3], this permutation character is  $1a+6a$ , or  $\chi_1 + \chi_4$ . So we have

$$[1a]_{H_{13}} \uparrow^{O'N} = ([1a]_{H_{13}} \uparrow^{H_3})_{H_3} \uparrow^{O'N} = [1a + 6a]_{H_3} \uparrow^{O'N}.$$

So we induce the following characters:

$$\begin{aligned} [1a]_{H_1} \uparrow^{O'N} &= [1a]_{H_1} \uparrow^{O'N} \\ -[1a]_{H_{13}} \uparrow^{O'N} &= -[1a + 6a]_{H_3} \uparrow^{O'N} \\ [1a]_{H_3} \uparrow^{O'N} &= [1a]_{H_3} \uparrow^{O'N} \end{aligned}$$

One easy cancellation leads to the final calculation.

$$\tilde{\Lambda}_{O'N} = [1a]_{H_1} \uparrow^{O'N} - [6a]_{H_3} \uparrow^{O'N} - 1.$$

Using GAP [20], we obtain the Lefschetz character to be  
 $[-254294272, -8960, 44, 0, 0, 8, 4, 48, 13, 0, 0, 0, -1, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, -1, 0, 0, 0, 0, -1, -1]$

We negate this virtual character (as we did in Section 2.3) so that its dimension is positive, giving us  $\tilde{\Lambda}_{O'N} = \overbrace{[254294272]}^{2^8 \cdot 993337}, \mathbf{8960}, -44, 0, 0, -8, -4, -48, -13, 0, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1,$

<sup>1</sup>See for example ‘‘Building for Lie type groups’’ in [31].



	$\varphi_1$	$\varphi_3$	$\varphi_6$	$\varphi_7$	$\varphi_8$
$\chi_1$	1	.	.	.	.
$\chi_5$	.	.	.	1	.
$\chi_6$	.	.	.	1	.
$\chi_8$	1	1	1	.	.
$\chi_9$	1	1	1	.	.
$\chi_{11}$	2	.	1	.	1
$\chi_{12}$	1	1	1	1	.
$\chi_{13}$	1	1	1	1	.
$\chi_{14}$	1	1	1	1	.
$\chi_{15}$	1	.	.	1	1
$\chi_{16}$	2	2	2	.	.
$\chi_{17}$	2	2	2	.	.
$\chi_{18}$	4	1	2	.	1
$\chi_{19}$	1	1	1	2	1
$\chi_{20}$	4	2	3	1	1
$\chi_{21}$	4	2	3	2	1
$\chi_{22}$	4	2	3	2	1
$\chi_{25}$	6	3	4	1	1
$\chi_{29}$	6	4	5	2	1
$\chi_{30}$	6	4	5	2	1

Figure 5.9. The decomposition matrix for the principal block of  $O'N$ .

We see that  $|O'N|_2$  does not divide  $|\tilde{\Lambda}_{b_2}^{O'N}|$ , which tells us conclusively that this block part is not projective.

Using the scalar product form of  $\tilde{\Lambda}_{O'N}$  and the decomposition matrix for block 2 in Figure 5.10, we calculate  $\tilde{\Lambda}_{b_2}^{O'N} = \underbrace{8\Phi(\varphi_2) + 4\Phi(\varphi_4) + 8\Phi(\varphi_5)}_{\text{v-projective part}} + 14\chi_7$ .

### The Defect 0 blocks

Unlike in *Suz*, the Lefschetz character of  $O'N$  has several blocks of defect 0. We already know that defect 0 blocks must be projective (Appendix A.4). Using the full decomposition matrix of  $O'N$  in [8], we can see that blocks 3 through 7 each consist of a multiple of a projective cover of a single character:

$$\tilde{\Lambda}_{b_3}^{O'N} = 97\Phi(\varphi_9), \tilde{\Lambda}_{b_4}^{O'N} = 97\Phi(\varphi_{10}), \tilde{\Lambda}_{b_5}^{O'N} = 115\Phi(\varphi_{11}), \tilde{\Lambda}_{b_6}^{O'N} = 115\Phi(\varphi_{12}), \tilde{\Lambda}_{b_7}^{O'N} = 115\Phi(\varphi_{13}).$$

	$\varphi_2$	$\varphi_4$	$\varphi_5$
$\chi_2$	1	.	.
$\chi_3$	.	1	.
$\chi_4$	.	.	1
$\chi_7$	.	1	1
$\chi_{10}$	1	1	1

Figure 5.10. The decomposition matrix for block 2 of  $O'N$ .

Viewing the Lefschetz character values restricted to each of the five defect 0 blocks is not very suggestive, so we do not include them.

### 5.3 Discussion

The refined Benson-Smith conjecture 2.7.1 states that for all 15 sporadics with non-projective Lefschetz modules, the non-projectivity of  $\tilde{L}$  involves an indecomposable in a non-principal 2-block of positive defect. This is true for  $O'N$ , as the block of defect 3 is non-projective. However, we have found a surprising result for  $O'N$  that Benson-Smith did not seem to predict: the principal block part of its Lefschetz character contains a non-projective summand. Thus  $O'N$  is categorized into Class III—it is a “Principal Block Part Non-projective” sporadic group. This discovery makes  $O'N$  the charter member of this class.

The variety<sup>1</sup> of the non-projective part in the principal block lies in what Benson-Carlson-Robinson call the *representation theoretic nucleus* [5, p. 68]. This idea was further explored by Benson [4, prop. 1.1]. This discovery for O’Nan is very interesting, as it is the first example found “in nature” with this property, i.e. in the sense of occurring in a natural geometry. Furthermore, the principal block part is the only one we find in our study to pass the  $p$ -test, yet fail the vanishing test. Further exploration is anticipated on this block.

Recall from the Background that Smith [30] observed a pattern for several sporadics: the defect of the largest non-principal block equals the 2-power difference between  $|\tilde{\Lambda}|_2$  and  $|G|_2$ . As stated, this pattern does not hold for  $O'N$ , since its largest non-principal block has defect 3 but the 2-power difference between  $|\tilde{\Lambda}|_2$  and  $|G|_2$  is only 1. We discuss possible further investigations into this behavior in Section 9.3.

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<sup>1</sup>We do not study varieties in this work. See Benson-Carlson-Robinson [5] for the basic notions we refer to here on this topic.

## CHAPTER 6

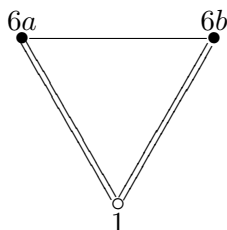
### THE HELD GROUP

Our next group is the Held group  $He$ . We omit most details and summarize our calculations briefly, as the process is similar to that of  $Suz$  and  $ON$ .

The maximal 2-local subgroups are [12, p. 131]:

$$\begin{aligned} H_1 &\cong 2_+^{1+6}.L_3(2), \\ H_{6a} &\cong 2^6 : 3 S_6, \text{ and} \\ H_{6b} &\cong 2^6 : 3 S_6. \end{aligned}$$

There is an outer automorphism of  $He$  that switches  $H_{6a}$  and  $H_{6b}$ , and fixes  $H_1$ . The diagram for the natural geometry of  $He$  is given in Ronan-Smith [26, p. 288] as:



We obtain the subgroups

$$\begin{aligned} H_{16a} &= H_{16b} = S_3 2^2 2_+^{1+6}, \\ H_{16a6b} &= \frac{2^2}{2_+^{1+6}}, \text{ and} \\ H_{6a6b} &= \frac{3(S_4 \times 2)}{2^6}. \end{aligned}$$

Using the diagram, the Atlas [12], and the techniques described thoroughly in previous chapters, we arrive at the following induced characters:<sup>1</sup>

$$\begin{array}{r} [1a]_{H_{16a6b}} \uparrow^{He} = [1a + 6aa + 8a]_{H_1} \uparrow^{He} \\ -[1a]_{H_{16a}} \uparrow^{He} = -[1a + 6a]_{H_1} \uparrow^{He} \\ -[1a]_{H_{16b}} \uparrow^{He} = -[1a + 6a]_{H_1} \uparrow^{He} \\ [1a]_{H_1} \uparrow^{He} = [1a]_{H_1} \uparrow^{He} \\ \hline -[1a]_{H_{6a6b}} \uparrow^{He} = -[1a + 5d + 9b]_{H_{6a}} \uparrow^{He} \\ [1a]_{H_{6a}} \uparrow^{He} = [1a]_{H_{6a}} \uparrow^{He} \\ \hline [1a]_{H_{6b}} \uparrow^{He} = [1a]_{H_{6b}} \uparrow^{He} \end{array}$$

---

<sup>1</sup>I am grateful to Silvia Onofrei and John Maginnis for pointing out an error in my original calculation. They were the first to *correctly* complete the remaining work in this chapter.

Using cancellation, we obtain:

$$\tilde{\Lambda}_{He} = [8a]_{H_1} \uparrow^{He} - [5d + 9b]_{H_{6a}} \uparrow^{He} + [1a]_{H_{6b}} \uparrow^{He} - 1.$$

GAP makes the calculation from here, and we get the Lefschetz character with columns indexed by the conjugacy classes of  $He$ :  $\tilde{\Lambda}_{He} = [ \overbrace{1120384}^{2^7 \cdot 8753}, -\mathbf{64}, \mathbf{0}, -197, -8, -8, 0, 0, 9, -1, 0, -1, -1, -1, 6, 6, 0, 1, 1, 0, -1, -1, 0, 0, 3, -1, -1, -1, -1, -1, -1, -1, -1 ]$ .

The 2-part of  $He$  has order  $2^{10}$ , which does not divide the dimension of the character  $|\tilde{\Lambda}_{He}| = 1120384 = 2^7 \cdot 8753$ . So by the  $p$ -test, we immediately see that the Lefschetz module is non-projective.

### 6.1 Block Theory

There are four 2-modular blocks for  $He$  [8]: the principal block, again a block of defect 3, and two blocks of defect 0. As before, we calculate the scalar product form of the Lefschetz character, with columns indexed by irreducible complex characters of  $He$ :  $\tilde{\Lambda}_{He} = [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 3, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 7, 7, 6, 6 ]$ .

We now examine the projectivity of each block part.

#### The Principal Block

As with the previous groups, we take the Lefschetz character in terms of its complex characters and separate it into blocks. For the principal block, we get:

$$\tilde{\Lambda}_{pr}^{He} = [ \overbrace{760832}^{2^{10} \cdot 743}, \mathbf{0}, \mathbf{0}, 152, -40, 0, 0, 0, -68, 0, 0, 16, 16, 58, -12, -12, 0, 0, 0, 0, 0, 0, 0, 0, -8, 14, 14, -2, -2, -5, -5, 0, 0 ]$$

We see that this character passes both the  $p$ -test and the vanishing test. Hence  $He$  is the fifth ‘‘Principal Block Part V-Projective’’ to be discovered. Using the 2-modular table, we can express this as  $\tilde{\Lambda}_{pr}^{He} = \Phi(\varphi_{10}) + \Phi(\varphi_{11}) + \Phi(\varphi_{12})$ .

#### Block 2

Let  $\tilde{\Lambda}_{b_2}^{He}$  denote the Lefschetz character of  $He$  restricted to block 2, the block of defect 3.

$$\tilde{\Lambda}_{b_2}^{He} = [ \overbrace{58496}^{2^7 \cdot 457}, -\mathbf{64}, \mathbf{0}, -13, 32, -8, 0, 0, 21, -1, 0, -17, -17, 39, 18, 18, 0, 1, 1, 0, -1, -1, 0, 0, -3, -1, -1, -6, -6, 4, 4, -1, -1 ]$$

Either test tells us that the Lefschetz module restricted to this block is non-projective. Using Figure 6.11, we see how ‘‘close’’ this block is to being  $v$ -projective:

$$\tilde{\Lambda}_{b_2}^{He} = \overbrace{3\Phi(\varphi_{14})}^{v\text{-projective part}} + \chi_{15}$$

	$\varphi_9$	$\varphi_{13}$	$\varphi_{14}$
$\chi_{12}$	1	.	.
$\chi_{14}$	.	1	.
$\chi_{15}$	1	1	.
$\chi_{16}$	1	.	1
$\chi_{22}$	1	1	1

Figure 6.11. The decomposition matrix for block 2 of  $He$ .

### Blocks 3 and 4—Defect 0 Blocks

Blocks 3 and 4 must be projective, since they have defect 0 (Appendix A.4). Using the full decomposition matrix of  $He$  in [8], we can see that each block part is a multiple of a single projective character:  $\tilde{\Lambda}_{b_3}^{He} = 7\Phi(\varphi_{15})$ , and  $\tilde{\Lambda}_{b_4}^{He} = 7\Phi(\varphi_{16})$ .

### 6.2 Discussion

Unlike with  $O'N$ , the Lefschetz character of  $He$  is v-projective. Thus  $He$  joins  $M_{12}$ ,  $J_2$ ,  $HS$ , and  $Suz$  in Class II. Also unlike with  $O'N$ , Smith's observation [30] does indeed hold for  $He$ : block 2 has defect 3, matching the 2-power difference between  $|\tilde{\Lambda}_{He}|_2 = 2^7$  and  $|He|_2 = 2^{10}$ .

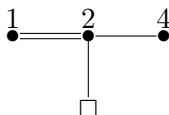
## CHAPTER 7

### THE CONWAY GROUP $CO_3$

The next sporadic we study is the Conway group  $Co_3$ . We once again present this tersely. The maximal subgroups we examine are [12, p. 134]:

$$\begin{aligned} H_1 &\cong 2 \cdot Sp_6(2), \\ H_2 &\cong 2^2 \cdot [2^7 \cdot 3^2] \cdot S_3, \text{ and} \\ H_4 &\cong 2^4 \cdot A_8. \end{aligned}$$

The diagram for the natural geometry of  $Co_3$  is given in Benson-Smith [7, §8.13] as:



This gives us the needed intersections:

$$\begin{aligned} H_{12} &= 2^2 \cdot [2^6] : (S_3 \times S_3), \\ H_{14} &= 2 \cdot 2^6 : L_3(2), \\ H_{124} &= 2 \cdot 2^6 : S_4, \text{ and} \\ H_{24} &= 2^4 \cdot 2^4 : (S_3 \times S_3). \end{aligned}$$

These lead us to induce the following characters:

$$\begin{array}{r} [1a]_{H_{124}} \uparrow^{Co_3} = [1a + 15a + 27a + 35bb + 84aa + 168a + 216a + 280b]_{H_1} \uparrow^{Co_3} \\ -[1a]_{H_{12}} \uparrow^{Co_3} = -[1a + 27a + 35b + 84a + 168a]_{H_1} \uparrow^{Co_3} \\ -[1a]_{H_{14}} \uparrow^{Co_3} = -[1a + 15a + 35b + 84a]_{H_1} \uparrow^{Co_3} \\ \hline [1a]_{H_1} \uparrow^{Co_3} = [1a]_{H_1} \uparrow^{Co_3} \\ -[1a]_{H_{24}} \uparrow^{Co_3} = -[1a + 14a + 20a]_{H_4} \uparrow^{Co_3} \\ \hline [1a]_{H_4} \uparrow^{Co_3} = [1a]_{H_4} \uparrow^{Co_3} \\ [1a]_{H_4} \uparrow^{Co_3} = [1a]_{H_2} \uparrow^{Co_3} \end{array}$$

These equations simplify to

$$\tilde{\Lambda}_{Co_3} = [216a + 280b]_{H_1} \uparrow^{Co_3} - [14a + 20a]_{H_4} \uparrow^{Co_3} + [1a]_{H_2} \uparrow^{Co_3} - 1.$$

Now we use GAP [20] to obtain the Lefschetz character with columns indexed by the conjugacy classes of  $Co_3$ :  $\tilde{\Lambda}_{Co_3} = [ \overbrace{50378624}^{2^7 \cdot 393583}, \mathbf{0}, -496, -2080, -784, 125, 0, 0, 24, 19, 0, 0, 0, 8, 5, 2, 0, 0, 0, 8, -1, 0, -1, -1, -1, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, -1, -1, -1, -1, 0, 0, 0 ]$ .

	$\varphi_4$	$\varphi_5$	$\varphi_{11}$	$\varphi_{13}$	$\varphi_{16}$
$\chi_6$	1	.	.	.	.
$\chi_7$	.	1	.	.	.
$\chi_{18}$	1	.	1	.	.
$\chi_{19}$	.	1	1	.	.
$\chi_{29}$	.	.	.	1	.
$\chi_{32}$	.	.	1	1	.
$\chi_{38}$	1	1	1	1	1
$\chi_{39}$	1	1	2	1	1

Figure 7.12. The decomposition matrix for block 2 of  $C_{O_3}$ .

Clearly this character does not vanish at element 2B, telling us that the Lefschetz module is not projective. From the viewpoint of Section 2.6, we note that -496 is the degree of  $\tilde{L}(\Sigma)$  for the 2-local geometry  $\Sigma$  of an  $M_{12}$  in  $C(2B)$ . So we predict that perhaps  $\Delta^{2B} \sim \Sigma$ .

### 7.1 Block Theory

We compute  $\tilde{\Lambda}_{C_{O_3}}$  as a combination of complex characters of  $C_{O_3}$ :  $\tilde{\Lambda}_{C_{O_3}} = [ [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 4, 4, 2, 2, 2, 2, 2, 2, 2, 6, 4, 8, 6, 6, 10, 11, 12, 12, 18, 18, 24, 23, 25, 26, 26, 26 ] ]$ .

There are three blocks [8]: the principal block, again a block of defect 3, and a block of defect 1. We study the possible projectivity of each block part.

#### The Principal Block

As with the previous groups, we take the Lefschetz character in terms of its complex characters and separate it into blocks. For the principal block, we get:

$$\tilde{\Lambda}_{pr}^{C_{O_3}} = [ \overbrace{34263040}^{2^{12} \cdot 8365}, \mathbf{0}, \mathbf{0}, -11840, 544, -512, 0, 0, 440, -40, 0, 0, 0, 0, 0, -56, 0, 0, 0, 28, 106, 0, 0, 20, 20, 0, 0, 0, 0, -40, 44, 0, 0, 0, -8, 0, 0, -37, -37, 0, 0, 0 ]$$

This character passes both the  $p$ -test and the vanishing test, telling us that it is  $v$ -projective. Using [8], we can express the Lefschetz character of this block completely in terms of projective covers:  $\tilde{\Lambda}_{pr}^{C_{O_3}} = \Phi(\varphi_9) + \Phi(\varphi_{10}) + 6\Phi(\varphi_{12}) + 8\Phi(\varphi_{14})$ . Thus  $C_{O_3}$  is a ‘‘Principal Block Part V-Projective’’ and joins *Suz* and *He* from our study in Class II.

#### Block 2

$$\tilde{\Lambda}_{b_2}^{C_{O_3}} = [ \overbrace{13006720}^{2^7 \cdot 101615}, \mathbf{0}, -\mathbf{496}, 11296, -2384, 445, 0, 0, -80, 155, 0, 0, 0, 8, 5, 34, 0, 0, 0, 4, -83, 0, -1, -21, -21, 0, 0, 0, 0, 16, -19, 0, 0, 0, -17, -1, -1, 36, 36, 0, 0, 0 ]$$

We see that this character fails the  $p$ -test, and so the Lefschetz module restricted to this block part is non-projective. Attempting to minimize the number of non-projective characters, we can express the part of the Lefschetz character in this block as:

$$\tilde{\Lambda}_{b_2}^{Co_3} = \overbrace{4\Phi(\varphi_{11}) + 6\Phi(\varphi_{13}) + 11\Phi(\varphi_{16})}^{\text{v-projective part}} + \chi_{32} + 2\chi_{38}.$$

### Block 3

This block has defect 1.  $\tilde{\Lambda}_{b_3}^{Co_3} = [ \overbrace{3108864}^{2^{12} \cdot 759}, \mathbf{0}, \mathbf{0}, -1536, 1056, 192, 0, 0, -336, -96, 0, 0, 0, 0, 0, 24, 0, 0, 0, -24, -24, 0, 0, 0, 0, 0, 0, 0, 0, 24, -24, 0, 0, 0, 24, 0, 0, 0, 0, 0, 0, 0 ]$ .

	$\varphi_{15}$
$\chi_{33}$	1
$\chi_{34}$	1

Figure 7.13. The decomposition matrix for block 3 of  $Co_3$ .

We see that this block part is v-projective, as it passes the vanishing test. Using the complex character decomposition of  $\tilde{\Lambda}_{Co_3}$  and Figure 7.13, we see that the character restricted to this block can be expressed as  $\tilde{\Lambda}_{b_3}^{Co_3} = 12\Phi(\varphi_{15})$ .

### 7.2 Discussion

We have found that the Lefschetz character of  $Co_3$  behaves as the refined Benson-Smith conjecture anticipated 2.7.1, as its non-projective part is completely contained in the block of defect 3. So  $Co_3$  is classified into Class II as a ‘‘Principal Block Part V-Projective.’’ Also, this defect value of 3 matches the 2-power difference between  $|Co_3|_2$  and  $|\tilde{\Lambda}_{Co_3}|_2$ , following Smith’s observation.

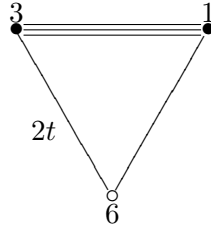
## CHAPTER 8

### THE RUDVALIS GROUP

The final sporadic group we consider (since its 2-modular decomposition matrix is known) is  $Ru$ , discovered by Rudvalis. Our presentation will again skip most details along the way. The maximal 2-local subgroups are [12, p. 126]:

$$\begin{aligned} H_1 &\cong 2 \cdot 2^{4+6} : S_5, \\ H_3 &\cong 2^{3+8} : L_3(2), \text{ and} \\ H_6 &\cong (2^6 : U_3(3)) : 2 \cong 2^6 G_2(2). \end{aligned}$$

The diagram for the natural geometry of  $Ru$  is [26, p. 288]:



where  $2t$  is a 2-split cover of the complete triple system on 5 points [26, p. 288]. The diagram gives us the needed intersections, which we show in “stack” form to make the subsequent inductions clearer:

$$H_{16} = \frac{S_3}{2^{2+1+2} \cdot 2^6}, \quad H_{36} = \frac{S_3}{2^{1+4} \cdot 2^6}, \quad H_{136} = \frac{2}{2^{1+4} \cdot 2^6}, \quad \text{and} \quad H_{13} = \frac{S_3}{2^2 \cdot 2^{3+8}}.$$

We obtain the following induced characters:

$$\begin{array}{l} [1a]_{H_{136}} \uparrow^{Ru} = [1a + 7bc + 14a + 21aa + 27aa + 32ab]_{H_6} \uparrow^{Ru} \\ -[1a]_{H_{16}} \uparrow^{Ru} = -[1a + 7bc + 21a + 27a]_{H_6} \uparrow^{Ru} \\ -[1a]_{H_{36}} \uparrow^{Ru} = -[1a + 14a + 21a + 27a]_{H_6} \uparrow^{Ru} \\ \hline [1a]_{H_6} \uparrow^{Ru} = [1a]_{H_6} \uparrow^{Ru} \\ -[1a]_{H_{13}} \uparrow^{Ru} = -[1a + 6a]_{H_3} \uparrow^{Ru} \\ \hline [1a]_{H_3} \uparrow^{Ru} = [1a]_{H_3} \uparrow^{Ru} \\ \hline [1a]_{H_1} \uparrow^{Ru} = [1a]_{H_1} \uparrow^{Ru} \end{array}$$

These equations simplify to

$$\tilde{\Lambda}_{Ru} = [32ab]_{H_6} \uparrow^{Ru} - [6a]_{H_3} \uparrow^{Ru} + [1a]_{H_1} \uparrow^{Ru} - 1.$$

We obtain [20] the Lefschetz character with columns indexed by the conjugacy classes of  $Ru$ :

$\tilde{\Lambda}_{Ru} = [ \overbrace{10113024}^{2^{12} \cdot 2469}, \mathbf{0}, \mathbf{64}, -96, 0, 0, 0, 0, 24, -1, 0, 5, 0, 0, 0, 0, -1, 0, 0, -1, 1, 1, 1, -1, 0, 0, 0, 0, 0, 0, -1, -1, -1, -1, -1 ]$ .

This character does not vanish at the 2B element, telling us that the Lefschetz module is non-projective. From the perspective of Section 2.6, we see that 64 matches the degree of  $\tilde{L}(\Sigma)$ , where  $\Sigma$  is the building of  $Sz(8)$  in  $C(2B)$ . So we conjecture that maybe  $\Delta^{2B} \sim \Sigma$ .

### 8.1 Block Theory

We compute the “scalar product” form of  $\tilde{\Lambda}_{Ru}$  as a combination of complex characters:  
 $\tilde{\Lambda}_{Ru} = [ 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4, 5, 4, 5, 5, 6, 6, 6, 7, 7, 7, 6, 8, 7, 7, 9 ]$ .  $Ru$  has just two blocks [8]: the principal block and a block of defect 2.

#### The Principal Block

The decomposition matrix for the principal block of  $Ru$  is in Figure 8.14. Again, we take the Lefschetz character in terms of its complex characters and separate it into blocks. For the principal block, we get  $\tilde{\Lambda}_{pr}^{Ru} = \overbrace{6881280}^{2^{16} \cdot 105}, \mathbf{0}, \mathbf{0}, -48, 0, 0, 0, 0, 280, 80, 0, 28, 0, 0, 0, 0, 0, 0, 0, -36, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -14, -14 ]$ .

We see that  $\tilde{\Lambda}_{pr}^{Ru}$  passes both projectivity tests, so it is v-projective. Thus  $Ru$  is a “Principal Block Part V-Projective” and joins  $Suz$ ,  $He$ , and  $Co_3$  from our study in Class II. We can express the Lefschetz character of this block in terms of projective covers:  $\tilde{\Lambda}_{pr}^{Ru} = \Phi(\varphi_5) + \Phi(\varphi_8)$ .

#### Block 2

$\tilde{\Lambda}_{b2}^{Ru} = [ \overbrace{3231744}^{2^{12} \cdot 789}, \mathbf{0}, \mathbf{64}, -48, 0, 0, 0, 0, -256, -81, 0, -23, 0, 0, 0, 0, -1, 0, 0, 35, 1, 1, 1, -3, 0, 0, 0, 0, 0, 0, -1, -1, -1, 13, 13 ]$ . We see that this does not vanish at element 2B, so  $\tilde{\Lambda}_{b2}^{Ru}$  is non-projective. We minimize the number of non-projective characters and express the Lefschetz character of this block as:  $\tilde{\Lambda}_{b2}^{Ru} = \underbrace{\Phi(\varphi_6) + \Phi(\varphi_7) + 6\Phi(\varphi_9)}_{\text{v-projective part}} + \chi_{36}$ .

### 8.2 Discussion

The Lefschetz character of  $Ru$  has its non-projective part contained in the block of defect 2 as predicted. This defect value of 2 matches the 2-power difference between  $|Ru|_2$  and  $|\tilde{\Lambda}_{Ru}|_2$ , following Smith’s observation.

	$\varphi_1$	$\varphi_2$	$\varphi_3$	$\varphi_4$	$\varphi_5$	$\varphi_8$
$\chi_1$	1	.	.	.	.	.
$\chi_2$	2	.	1	.	.	.
$\chi_3$	2	.	1	.	.	.
$\chi_4$	2	1	1	.	.	.
$\chi_5$	3	1	2	.	.	.
$\chi_6$	4	1	2	2	.	.
$\chi_7$	6	1	3	2	.	.
$\chi_8$	11	2	5	2	.	1
$\chi_9$	12	3	5	3	.	1
$\chi_{10}$	3	2	1	.	1	1
$\chi_{11}$	8	2	3	2	1	1
$\chi_{12}$	8	2	3	2	1	1
$\chi_{13}$	8	2	3	2	1	1
$\chi_{14}$	9	3	4	2	1	1
$\chi_{15}$	10	4	4	1	.	2
$\chi_{16}$	10	4	4	1	.	2
$\chi_{17}$	12	4	5	2	1	2
$\chi_{18}$	12	4	5	2	1	2
$\chi_{19}$	12	4	5	2	1	2
$\chi_{20}$	14	5	6	3	1	2
$\chi_{21}$	14	5	6	3	2	2
$\chi_{22}$	18	6	8	5	3	2
$\chi_{23}$	25	7	11	5	1	3
$\chi_{24}$	21	7	9	4	2	3
$\chi_{25}$	26	8	10	7	2	3
$\chi_{26}$	15	6	6	3	3	3
$\chi_{27}$	24	8	10	6	3	3
$\chi_{28}$	30	9	13	6	2	4
$\chi_{29}$	24	9	10	4	3	4
$\chi_{30}$	28	10	12	6	3	4
$\chi_{31}$	28	10	12	6	3	4
$\chi_{33}$	28	10	12	6	4	4

Figure 8.14. The decomposition matrix for the principal block of  $Ru$ .

	$\varphi_6$	$\varphi_7$	$\varphi_9$
$\chi_{32}$	.	.	1
$\chi_{34}$	.	1	1
$\chi_{35}$	1	.	1
$\chi_{36}$	1	1	1

Figure 8.15. The decomposition matrix for block 2 of  $Ru$ .

## CHAPTER 9

### FUTURE RESEARCH

Our study has given us some new information, but it has left many unanswered questions as well as opened some new questions. From this point, there are several different possible directions to pursue.

#### 9.1 Proving Projectivity

At this point in our understanding of the Lefschetz modules in this work, we don't know the indecomposable summands and what their vertices (see Appendix A.4) are within the block parts. We just know the vertex is (somewhere) in the defect group in each case. For the block parts of Lefschetz modules with v-projective character, the method of Steinberg module inversion by Webb [33] as extended by Grodal [19] can probably be used to show that any indecomposable in the projective part has a comparatively small vertex. If we could show the vertex to be trivial, then we would conclusively know that the Lefschetz module restricted to the block is actually a projective module. These methods could prove to be quite useful for future research into the properties of the Lefschetz modules of sporadic groups.

#### 9.2 Looking for Patterns

We now know 8 sporadic groups whose 2-modular Lefschetz modules are non-projective. Closer examination of these 8, in contrast to the 11 sporadics with projective Lefschetz modules, could lead to some understanding of what separates these classes of groups. Patterns could be studied possibly using some methods of Sawabe [28] or of Landrock [21].

Also comparing the Class III group  $O'N$  to the Class II groups could perhaps give a fuller picture of the reasons for the different classes. In particular, the nucleus and the vertex for the non-projective module  $\tilde{L}_{pr}^{O'N}$  could be investigated in more detail.

The pattern observed by Smith does not hold in general, as we found that it is not true for  $O'N$ . It does hold for all the other examples, however. Perhaps more investigation into  $O'N$  (as outlined below) would explain why it is the anomaly. It is possible that altering the statement of Smith could lead to a pattern that is satisfied by all the sporadics.

#### 9.3 Unusual Behavior of $O'N$

Of the sporadics we examined,  $O'N$  clearly has the most unusual behavior. It is the only group so far categorized into Class III, and it seems interesting that  $\tilde{\Lambda}_{pr}^{O'N}$  passes the  $p$ -test 2.5.5 but fails the vanishing test 2.5.6.  $O'N$  is also the only group that did not follow the pattern observed by Smith, since its largest non-principal block has defect 3 but the 2-power difference between  $|\tilde{\Lambda}|_2$  and  $|G|_2$  is only 1.

This brings up some interesting questions about the block of defect 3. Since the part of  $\tilde{\Lambda}$  in this block is non-projective, its vertex must be nontrivial. It looks like it might have vertex of order  $2^1$ , a subgroup of the defect group of order  $2^3$ . Methods of Sawabe [28] could possibly help explain some of this unusual behavior.

#### 9.4 Showing Homotopy Equivalence of Geometries

Several of the Lefschetz modules (and block parts) that were found to be non-projective had Euler characteristic values that matched those of well-known non-contractible geometries. We speculated throughout about the possibility in each case that the fixed points of the natural geometry of the group  $\Delta^a$  could be homotopy equivalent to a well-known geometry  $\Sigma$ . Each of these could be studied further, and it is likely that our hypotheses will be correct in some cases.<sup>1</sup>

#### 9.5 Groups of Different Characteristic

Aschbacher-Smith [2] studies certain small groups of characteristic 3 by using geometries of characteristic 2. Thus the methods we used in this thesis with  $p = 2$  could be used to study non-sporadic groups of characteristic  $p \neq 2$ . There is more work to be done for groups in characteristic 3, and the method could also be expanded beyond the work of Aschbacher-Smith to study groups of characteristic  $p \geq 5$ .

#### 9.6 Different Geometries

Benson-Smith indicates—but doesn’t compute in most cases—a geometry determined by the centric radicals, which is different in some cases. It may be of interest to compute the Lefschetz character for this geometry also. It would be interesting to see if it would demonstrate some of the behavior we see from the usual 2-local geometry, especially for the “Principal Block Part Non-Projective” group  $O'N$ .

#### 9.7 Geometries of Different Characteristic

We have used  $p = 2$  to form the natural geometry for each group. The same sporadic groups could be studied using  $p \neq 2$  to construct the natural geometries in a different characteristic. It would be interesting to find, for example as a starting point, the 3-local geometry of  $M_{12}$ .

#### 9.8 The Remaining Sporadics

As described in the Introduction and Background, there are seven sporadic groups whose irreducible 2-modular decomposition matrices are not yet known [8]. At this time their Lefschetz characters could be calculated, and the central characters could be computed for the complex character table. These characters could be used to divide  $\tilde{\Lambda}_G$  up into block parts. Then the groups could be categorized into Class II or Class III by the vanishing test.<sup>2</sup> (The guess here is that  $O'N$  will not remain the lone member of Class III.) Block parts that are found to be v-projective, however, could not be explicitly written as a combination of  $\Phi(\varphi_i)$ ’s until the group’s 2-modular irreducibles became known.

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<sup>1</sup>It appears that Magginis and Onofrei [22] have just proved our conjectures true in all cases, which would relegate this subsection to “Past Research.”

<sup>2</sup>It appears that Magginis and Onofrei [22] have very recently classified  $Fi_{22}$  into Class II.

## CHAPTER 10

### CONCLUSION

Our study of the projectivity of block parts of Lefschetz characters of the sporadics has found several repeated patterns and also has produced some unexpected results. We have a better grasp of the situation after seeing several more examples, and are now able to classify most of the sporadics according to projectivity properties of their Lefschetz modules.

#### 10.1 Categorization into Classes

Of the 26 sporadic groups, 19 of them are now classified according to projectivity of their Lefschetz modules for their 2-local geometries, as described in Section 2.7.

Class	Sporadic Groups
I: Lefschetz Module Projective	$M_{11}, J_1, M_{22}, M_{23}, J_3, M_{24}, M^cL, Co_2, Ly, J_4, Th$
II: Principal Block Part V-Projective	$M_{12}, J_2, HS, \mathbf{Suz}, \mathbf{He}, \mathbf{Co}_3, \mathbf{Ru}$
III: Principal Block Part Non-Projective	$\mathbf{O}^{\mathbf{N}}$

The groups  $Co_1, Fi_{22}, Fi_{23}, Fi'_{24}, HN, B,$  and  $M$  have yet to be classified.<sup>1</sup> Their 2-modular decomposition matrices [8] are not yet known. We do know by the  $p$ -test that their Lefschetz modules are non-projective, so they will be in Class II or Class III.

#### 10.2 The Benson-Smith Conjecture

The examples we have studied shed more light on this situation than was available at the time of the original Benson-Smith conjecture. The two early examples of  $M_{12}$  and  $J_2$  made it appear that the principal block had no part to play whatsoever in the Lefschetz character. The Lefschetz character of  $HS$  was the first sporadic example<sup>2</sup> that had a nonzero contribution from the principal block. That contribution was v-projective, as was the case for the principal block parts of the Lefschetz characters of  $Suz, He, Co_3,$  and  $Ru$ . The most surprising result was the case of  $O^{\mathbf{N}}$ . Though it did follow the Benson-Smith conjecture, it additionally was found to have a non-projective component in its principal block. It does not appear that this was foreseen by Benson and Smith.

<sup>1</sup>As mentioned in the previous chapter, it appears that Magginis and Onofrei [22] have very recently classified  $Fi_{22}$  into Class II.

<sup>2</sup>The Lefschetz character of  $HS$  given in Chapter 3 is the first time it has appeared in the literature.

### **10.3 Final Thoughts**

We understand the projectivity of block parts of the Lefschetz characters of the sporadics substantially better now than what was known before our study. We have been able to establish a categorization system to see similarities and differences among many of the sporadics based on this specific property. Still, with seven of the most complicated finite simple groups still unclassified, there may be yet more subtle categories needed.

It is hoped that understanding of the Lefschetz modules of the 2-local geometries of sporadic groups will enhance our understanding of the sporadic groups themselves and eventually lead to the discovery of significant and illuminating connections among them.

## APPENDIX

### GENERAL BACKGROUND CONCEPTS

#### A.1 Contractibility

This section is a support of remarks made in Section 2.6. Webb [33] showed that if  $\Delta^x$  is contractible for every element  $x$  of prime order  $p$ , then  $\tilde{L}(\Delta)$  is a virtual projective module, i.e., it is  $\mathbb{Z}$ -linear combination of a projective modules.<sup>1</sup> Ryba-Smith-Yoshiara [27] showed that this held for 11 sporadic groups. Here we study groups whose entire Lefschetz characters fail the  $p$ -test 2.5.5, so we know in advance of our study that Webb's condition fails for these groups:  $\Delta^x$  is not contractible for at least some  $x$  of order  $p$ .

Benson-Smith [7, prop. 6.6.7] translates Webb's condition into the language of Dwyer [16], finding that if  $\Delta^P$  is contractible for all  $p$ -subgroups  $P$  of  $G$ , then  $\Delta$  is ample. They went a step further, however, and found less stringent conditions to obtain ample geometries without requiring  $\Delta^P$  to be contractible for all  $P$ . Thus we are able to work with ample geometries throughout our study, even though the Lefschetz modules we examine are not projective.

#### A.2 Webb's Decomposition Argument

Here we review the standard observation (essentially following the more general argument of Webb [33, p. 141]) that the reduced Lefschetz module of  $G$  on  $\Delta$  is acyclic if and only if  $G$  has a cohomology decomposition in terms of its subgroups stabilizing simplices in the module.

The *reduced* part is the key to this straightforward fact. The reduced Lefschetz module starts the summation at  $i = -1$ . Now we also have by definition that  $\tilde{L}(\Delta)$  acyclic  $\implies H^*(\tilde{L}(\Delta)) = 0$ , so

$$0 = H^*(\tilde{L}(\Delta)) = H^*\left(\sum_{\sigma \in \Delta/G} (-1)^{\dim \sigma} \text{Ind}_{G_\sigma} \uparrow^G (1)\right) \quad (\text{A.1})$$

$$= H^*\left(\sum_{i=-1}^{\dim \Delta} \sum_{\sigma \in \Delta_i/G} (-1)^i \text{Ind}_{G_\sigma} \uparrow^G (1)\right) \quad (\text{A.2})$$

$$= \sum_{i=-1} \sum_{\sigma \in \Delta_i/G} (-1)^i H^*(\text{Ind}_{G_\sigma} \uparrow^G (1)) \quad (\text{A.3})$$

$$= \sum_{i=-1} \sum_{\sigma \in \Delta_i/G} (-1)^i H^*(G_\sigma), \quad (\text{A.4})$$

where  $\Delta_i$  is the set of simplices of dimension  $i$  and  $\Delta_i/G$  is the set of  $i$ -dimensional orbits of  $G$ , so  $\sigma$  is a representative of its orbit with stabilizer  $G_\sigma$ . The symbol  $\Delta_{-1}$  formally represents

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<sup>1</sup>Note that this criteria is stronger than simply the Lefschetz module having v-projective character.

## APPENDIX (Continued)

the “empty” simplex (of dimension  $-1$ ), which is fixed by all of  $G$ , so that  $G_\emptyset$  (also called  $G_{-1}$ )  $= G$ . The first step comes from one definition of the reduced Lefschetz module and the last step comes from the Eckmann-Shapiro Lemma [3, p. 42]. Moving the  $i = -1$  term to the left side of the equation gives

$$H^*(G_{-1}) = \sum_{i=0} \sum_{\sigma \in \Delta_i/G} (-1)^i H^*(G_\sigma),$$

and since  $G_{-1} = G$ , we have

$$H^*(G) = \sum_{i=0} \sum_{\sigma \in \Delta_i/G} (-1)^i H^*(G_\sigma).$$

This is the definition of a decomposition of group cohomology. All these steps are reversible, so we have shown that  $G$  having a cohomology decomposition in terms of stabilizing subgroups is equivalent to the reduced Lefschetz module of  $G$  on  $\Delta$  being acyclic.<sup>1</sup>

### A.3 Elementary Block Theory

We include the following elementary block theory for readers not familiar with these standard facts. For further reference, see for example [13, §65]. Let  $G$  be a group. Then the group algebra of  $G$  can be decomposed into a direct sum of block algebras; so any module can be decomposed into corresponding block parts. In characteristic 0, modules can always be broken up into irreducibles, so each “block” contains just one irreducible. The characteristic  $p$  decomposition is more interesting, as it is not always possible to decompose modules into irreducibles when the characteristic of  $p$  divides  $|G|$ .<sup>2</sup> The *principal block* is defined to be the block that contains the trivial module.

The decomposition matrices website [8] shows the modular character tables for each sporadic group for many different values of  $p$ . We focus our study on the prime  $p = 2$ ,<sup>3</sup> and each table gives the characteristic 0 (also known as “ordinary” or “complex”) character  $\chi$  as a linear combination of the characteristic 2 (or “2-modular”) characters. Note that each complex character  $\chi$  is irreducible, and hence projective in characteristic 0. However, each is not necessarily projective in characteristic 2, and indeed the vast majority are not. Only those of defect 0 are both projective and irreducible.

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<sup>1</sup>In the cases of  $\Delta$  in Benson-Smith [7], the action of  $G$  on  $\Delta$  is flag transitive, so orbits correspond to subsets  $J \subseteq I$  and we get

$$H^*(G) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|} H^*(H_J).$$

<sup>2</sup>Since we study  $p = 2$ , this latter situation is always the case for the groups we examine.

<sup>3</sup>The prime  $p = 2$  provides the most interesting geometries, but is the “hardest” prime for which to compute modular irreducibles and hence decomposition matrices. This is why currently we cannot fully study seven of the sporadic groups.

## APPENDIX (Continued)

**Definition A.3.1 (Defect)** *Let  $G$  be a group, and let  $\chi_i$  be the set of irreducible complex characters in a  $p$ -block of  $G$ . The defect of the block is the difference between the exponent of  $p$  in  $|G|_p$  and the exponent of  $p$  in  $\min(|\chi_i|_p)$  [13, p. 611]. In particular, the principal block has defect equal to the exponent of  $p$  in  $|G|_p$  and thus has full defect.*

A defect 0 block means that every irreducible character in the block has degree divisible by the Sylow order. Brauer showed that if one such character exists in the block, then it is the only one of its type in that block [3, p. 195]. In a Lie type group, the Steinberg module has defect 0, so there is always such a block. In the sporadics, however, there is not always a defect 0 block, as we see in our study.

**Definition A.3.2 (Projective Cover)** *We say that a module  $P$  is a projective cover of an irreducible module  $M$  if  $P$  is projective, indecomposable, and there exists a surjective homomorphism from  $P$  onto  $M$ .*

A general projective module decomposes into a sum of projective indecomposables, each of which is a projective cover of an irreducible module. Hence a projective character can be broken up into a sum of projective covers of irreducible characters. The decomposition matrices website [8] shows us the character of the projective covers of the 2-modular characters  $\varphi$ . Given a 2-modular character  $\varphi_i$ , we can find the character  $\Phi(\varphi_i)$  simply by following the column and forming a linear combination of the corresponding complex characters with coefficients given in the table.

In the full representation theory of a group, the principal block always contains  $P(1)$ , the projective cover of the trivial module. Though  $P(1)$  is a projective module, it is not quite acyclic.<sup>1</sup> The cohomology of  $P(1)$  vanishes in every degree above zero, but its  $0^{\text{th}}$  cohomology group  $\text{Hom}_{\mathbb{F}_2 G}(P(1), 1)$  is nonzero, and hence it is not acyclic. This is the only projective indecomposable which is not acyclic. Benson-Smith found ample geometries  $\Delta$  that do provide acyclic Lefschetz modules for each sporadic group (see Section 2.4). We work with these geometries, and so we know in advance of our study that  $P(1)$  will not appear as a summand in any of our Lefschetz modules.

### A.4 Vertices and Sources

**Definition A.4.1 (Vertex and source)** *Let  $G$  be a finite group,  $k$  be a field of characteristic  $p$  dividing  $|G|$ , and  $M$  be an indecomposable  $kG$ -module. Then  $M$  corresponds to a minimal  $p$ -subgroup  $B$  (unique up to conjugacy) of  $G$ , called the vertex, and an indecomposable  $kB$ -module  $L$ , called the source, such that  $M$  is a component of the induced module  $\text{Ind}_B^G(L)$  [13, p. 435].*

Essentially the vertex  $B$  of  $M$  is a measure for “degree of relative projectivity” of  $M$ .<sup>2</sup> The smaller  $B$  is, the “more projective”  $M$  is. The Green correspondence [13, p. 438] tells us that if the vertex  $B$  is trivial, then  $M$  is a projective module.

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<sup>1</sup>Acyclic is defined as the cohomology vanishing in *all* dimensions [23].

<sup>2</sup>The concept of “relative projectivity” is discussed in [13, §65] and [1, §9].

**APPENDIX (Continued)**

The *defect group*  $D$  of a block is a  $p$ -subgroup (unique up to conjugacy) attached to the block [13, p. 618]. The order of the defect group is  $2^d$ , where  $d$  is the defect. For  $I$  an indecomposable in a block with defect group  $D$ , a vertex of  $I$  is a subgroup of  $D$  (follows from [14, Thm 54.10]). Thus a block of small defect forces a small vertex and thus must correspond to a “largely” projective module with a high  $p$ -power in the degree. In particular, a block of defect 0 forces the vertex to be trivial, telling us immediately that the block is projective.

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