On Lefschetz Characters of 2-local Geometries for Some Sporadic Groups

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Outline

1. Introduction and Background
   - Historical Framework
   - Definitions
   - Example Calculation
   - Methods
2. Description of the process for the Suzuki group
   - Geometry
   - Inducing Subgroups
   - Block Theory
3. Summaries for other sporadic groups
   - $O'N$
   - He
   - Co_3
   - Ru
4. Future Directions

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The classification of finite simple groups was announced in 1983 and its proof was formally finalized in 2004. The theorem states that any finite simple group must belong to one of the following classes.

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- A *virtual module* is a formal $\mathbb{Z}$-linear combination of modules. Observe this can be written as a formal difference of two ordinary modules (i.e., modules with positive coefficients).
- Thus a virtual projective module is a $\pm$ combination of projective indecomposables $P(\varphi)$. In our study, we will refer to a virtual projective module as a "projective module."
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- The (reduced) *Lefschetz module* of a simplicial complex Δ is the virtual module given by the alternating sum of the chain groups:

$$\widetilde{L}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i C_i(\Delta).$$

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Notation (Fixed Point Notation)

Let $g \in G$ and let Δ be a geometry acted upon by G. Then we use the symbol Δ^g to denote the set of objects in Δ fixed by g.

- The (reduced) *Lefschetz character* is the sequence of Euler characteristic values on the geometry fixed by representatives of each conjugacy class of G: $Tr(g, \widetilde{L}) = \widetilde{\chi}(\Delta^g)$.
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Example: $L_3(2)$

- As an introductory example, we calculate the Lefschetz character for the simple Lie type group $G = L_3(2)$ of order 168.
- Its natural geometry is $\mathbb{P}_2(2)$ of seven points (1-spaces), seven lines (2-spaces), and 21 point-in-line pairs (which we will call "flags").
- For simplicity, we rename the vectors:

$$(1,0,0) \leftrightarrow \text{“1”}$$
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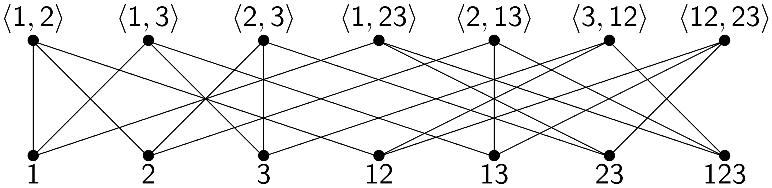

The Lefschetz character will have a value for each conjugacy class of $L_3(2)$, of orders 1, 2, 3, 4, and 7. We choose a representative g of each conjugacy class and calculate the Euler characteristic of the subset of the geometry that is fixed by g.

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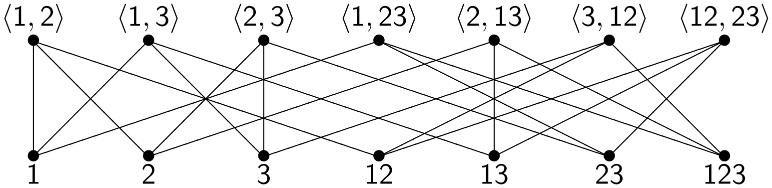

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- We start with the identity element 1, which fixes the entire geometry of 7 points, 7 lines, and 21 flags. So we have

$$\begin{aligned} \tilde{\chi} &= -1 + |C_0| - |C_1| && (1) \\ &= -1 + (7 + 7) - 21 && (2) \\ &= -8 && (3) \end{aligned}$$

- The order 2 class can be represented by the element $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. We multiply on the right, so for example

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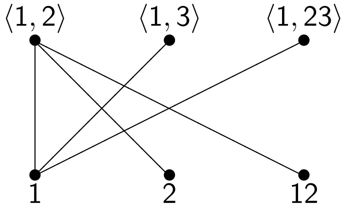

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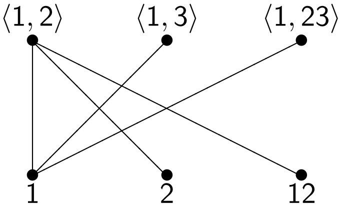

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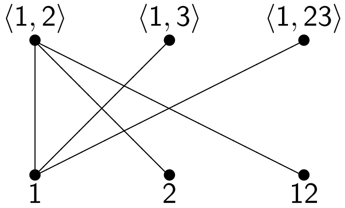

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- Similar calculations with a representative from each conjugacy class give us the Lefschetz character of $L_3(2)$:

Order of g	1	2	3	4	7
$\mathrm{Tr}\left(g, \widetilde{L}\right)$	-8	0	1	0	-1

- Since the Lefschetz module is actually a virtual module, we can negate the Lefschetz character so that its dimension is positive:

Order of g	1	2	3	4	7
$\mathrm{Tr}\left(g, \widetilde{L}\right)$	8	0	-1	0	1

- Experts will recognize this as the *Steinberg* character of $L_3(2)$, which is projective.

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In this work, we will study the 5 sporadic groups Suz, $O'N$, He, Co_3, and Ru.

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The Suzuki Group

The first sporadic group we examine is the group *Suz* of order $\mathbf{2^{13}} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, discovered by Suzuki. We show more details of this case, as later cases follow similarly.

- Since we are investigating the 2-local geometry, we study the maximal subgroups which contain the Sylow 2-group, numbering them by the ranks of the centers of their largest normal 2-subgroups:

$$H_1 \cong 2_-^{1+6} \cdot U_4(2), \; H_2 \cong 2^{2+8}{:}(A_5 \times S_3), \; H_4 \cong 2^{4+6}{:}3A_6.$$

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The set $I := \{1, 2, 4\}$ *indexes the simplex* $\mathcal{H}_I$ *of subgroups of* Suz*.* *For simplicity, we will use the notation* $H_{12} = H_1 \cap H_2$*,* $H_{14} = H_1 \cap H_4$*,* $H_{24} = H_2 \cap H_4$*, and* $H_{124} = H_1 \cap H_2 \cap H_4$*.*

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- Hence we will use selected residues described by the diagram to find which characters of the intersections are needed in the calculation of induced representations.

  By the transitivity of induction, we know that, for example,

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Inducing H_{12} up to H_1

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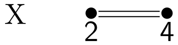

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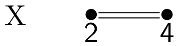

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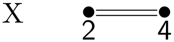

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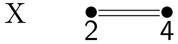

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Similarly, we find the following characters to induce up to *Suz*.

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We now have the Lefschetz character in a form that GAP can compute. With columns **indexed by the conjugacy classes** of *Suz*, we express the Lefschetz character as:

- $\widetilde{L}_{Suz} = [\overbrace{4189184}^{2^{10}\cdot 4091}, \overset{\mathbf{2A}}{\mathbf{0}}, \overset{\mathbf{2B}}{\mathbf{-64}}$, 3968, -352, -73, 0, 0, 0, 8, -16, 9, 0, 0, 0, 0, -1, -1, 0, 0, 0, 2, 2, 0, 1, -1, 0, 0, 0, -1, 0, -1, -1, -1, 2, 2, 3, 0, 0, 0, -1, -1, 0].
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Block Theory

The decomposition matrices website tells us that there are just two 2-modular blocks for Suz: the principal block and a much smaller block of defect 3.

Notation

Let G be a group, and let B be a particular block part of the 2-modular decomposition matrix of G. Then we define $\widetilde{L}_B^G$ to be $\widetilde{L}_G$ restricted to characters in block B.

We use the coefficients of irreducible characters along with the decomposition matrices to break down $\widetilde{L}_{pr}^{Suz}$ and $\widetilde{L}_{b2}^{Suz}$ into projective and non-projective parts.

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We can divide up the original Lefschetz character into its contributions from each block. Indexed by the conjugacy classes of *Suz*, we have:

- $\widetilde{L}^{Suz}_{pr} = [\, \overbrace{2834432}^{2^{14}\cdot 173},\ \overset{\mathbf{2A}}{\mathbf{0}},\ \overset{\mathbf{2B}}{\mathbf{0}}$, 512, -352, -64, 0, 0, 0, 0, 32, 32, 0, 0, 0, 0, 0, -8, 0, 0, 0, -4, -4, 0, 0, -4, 0, 0, 0, 0, 0, 3, 3, 0, -4, -4, 2, 0, 0, 0, 1, 1, 0].
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This group of order $|O'N| = \mathbf{2^9} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$ was discovered by O'Nan. It has subgroups:

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In addition to the principal block, there is a block of defect 3. Separating the character values corresponding to characters from the principal block, we find:

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For block 2, we find

- $\widetilde{L}_{b2}^{O'N} = [\overbrace{1386240}^{2^8 \cdot 5415}, \overset{\textbf{2A}}{\mathbf{1792}}$, 1140, 0, 0, 60, 28, 324, -12, 0, 0, -28, -24, 0, 28, 60, 60, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -18, -18].

Either test shows us that this part is non-projective, verifying the refined Benson-Smith conjecture.

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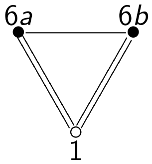

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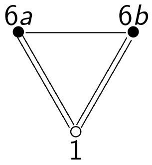

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Restricted to the principal block, we have:

- $\widetilde{L}_{pr}^{He} = [\overbrace{802176}^{2^7 \cdot 6267}, \overset{\underline{\mathbf{2A}}}{\mathbf{2656}}, \overset{\underline{\mathbf{2B}}}{\mathbf{448}}, 348, \text{-}69, 32, 32, 16, \text{-}49, \text{-}8, \text{-}5, \text{-}3, \text{-}3, 46, \text{-}10, \text{-}10, 0, 11, 2, \text{-}1, \text{-}11, \text{-}11, 0, 0, \text{-}7, 14, 14, \text{-}2, \text{-}2, \text{-}6, \text{-}6, \text{-}3, \text{-}3\ ]$.

Thus as with $O'N$, the principal block part of $\widetilde{L}_{He}$ is non-projective.

For block 2 (of defect 3), we have:

- $\widetilde{L}_{b2}^{He} = [\overbrace{60160}^{2^8 \cdot 235}, \overset{\underline{\mathbf{2A}}}{\mathbf{640}}, \overset{\underline{\mathbf{2B}}}{\mathbf{0}}, 58, 40, 16, 0, 0, 10, 10, 0, 2, 2, 37, 16, 16, 0, \text{-}10, \text{-}2, 0, 10, 10, 0, 0, \text{-}2, \text{-}3, \text{-}3, \text{-}5, \text{-}5, 5, 5, 2, 2\ ]$

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Restricted to the principal block, we have:

- $\widetilde{L}^{He}_{pr} = [\overbrace{802176}^{2^7 \cdot 6267}, \overset{\mathbf{2A}}{\mathbf{2656}}, \overset{\mathbf{2B}}{\mathbf{448}}, 348, -69, 32, 32, 16, -49, -8, -5, -3, -3, 46, -10, -10, 0, 11, 2, -1, -11, -11, 0, 0, -7, 14, 14, -2, -2, -6, -6, -3, -3\ ].$

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This group of order $|Co_3| = \mathbf{2^{10}} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ was discovered by Conway. Its subgroups are:

$$H_1 \cong 2^{\cdot} Sp_6(2), \; H_2 \cong 2^2.[2^7.3^2].S_3, \; H_4 \cong 2^{4\cdot} A_8,$$

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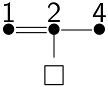

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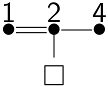

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Co_3 Principal Block

This time there are three blocks, all of nonzero defect. For the principal block, we have:

- $\widetilde{L}_{pr}^{Co_3} = [\ \overbrace{34263040}^{2^{12}\cdot 8365},\ \overset{\textbf{2A}}{\mathbf{0}},\ \overset{\textbf{2B}}{\mathbf{0}}$, -11840, 544, -512, 0, 0, 440, -40, 0, 0, 0, 0, 0, -56, 0, 0, 0, 28, 106, 0, 0, 20, 20, 0, 0, 0, 0, -40, 44, 0, 0, 0, -8, 0, 0, -37, -37, 0, 0, 0].

This passes both tests, and in fact we can use the decomposition matrix to see that it has the character of a projective module:

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Co_3 Blocks 2 and 3

For block 2 (of defect 3), we have:

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This fails either test, so is non-projective, verifying the refined Benson-Smith conjecture. This time we have a third block of defect 1:

- $\widetilde{L}_{b3}^{Co_3} = [\ \overbrace{3108864}^{2^{12}\cdot 759}, \overline{\mathbf{0}}^{\mathbf{2A}}, \overline{\mathbf{0}}^{\mathbf{2B}}$, -1536, 1056, 192, 0, 0, -336, -96, 0, 0, 0, 0, 0, 24, 0, 0, 0, -24, -24, 0, 0, 0, 0, 0, 0, 0, 0, 24, -24, 0, 0, 0, 24, 0, 0, 0, 0, 0, 0, 0].

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$$H_1 \cong 2 \cdot 2^{4+6}{:}S_5,\ H_3 \cong 2^{3+8}{:}L_3(2),\ H_6 \cong 2^6 G_2(2).$$

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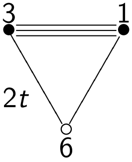

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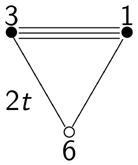

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There are only two blocks. For the principal block, we find:

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Summary

Of the 26 sporadic groups, 19 of them can now be categorized according to projectivity of their Lefschetz modules for their 2-local geometries.

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II: Principal Block Projective	M_{12}, J_2, HS, **Suz**, **Co$_3$**, **Ru**
III: Principal Block Non-projective	**O'N**, **He**

- The sporadic groups Co_1, Fi_{22}, Fi_{23}, Fi'_{24}, HN, B, and M have yet to be classified, because their 2-modular decomposition matrices are not yet known for $p = 2$.
- We do know by the p-test that their Lefschetz modules are non-projective, so they will be in Class II or Class III.

Future Directions

There are several directions to pursue from this point.

- Examine patterns to see what separates the 8 sporadics with non-projective Lefschetz modules from the 11 with projective Lefschetz modules.
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