

# Group project: Intro to Proofs

Work in groups. Use your own paper and hand in on Wednesday. (Homework 4 is due Friday.) This project is worth 20 points, with several possibilities for extra credit.

• Here's how to prove a result by *mathematical induction* (or just *induction* for short): First, we show the result is true for a **base step** (such as  $k = 1$ ). Secondly, we do the **induction step**: *assume* that the result is already true for  $k = m$ , and show that therefore it must be true for  $k = m + 1$ . Then we're done! We've shown the result to be true for any integer  $n \in \mathbb{Z}$ .

EXAMPLE: We use mathematical induction to prove the Gauss formula of the sum of all integers from 1 to  $n$ :

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}. \quad \left( \text{Recall that } \sum_{i=1}^n i = 1 + 2 + \cdots + n. \right)$$

**Base step:** We show the result is true for  $n = 1$ :

$$\sum_{i=1}^1 i = \frac{1(1+1)}{2}.$$

Both sides of the equation equal 1, so our base step is proved.

**Induction step:** Now we *assume* that the result is already true for  $k = m$ . This assumption is called our *induction hypothesis*. The meat of our proof is to show that because of this hypothesis for  $k = m$ , the formula must also be true for  $k = m + 1$ . Here we go...

Assuming the formula is true for  $k = m$  means that we already know that this formula is true:

$$\sum_{i=1}^m i = \frac{m(m+1)}{2}.$$

What we *need to show* is that that formula is still true when we substitute  $m + 1$  in for  $m$ :

$$\sum_{i=1}^{m+1} i = \frac{(m+1)((m+1)+1)}{2}.$$

We will show the  $(m + 1)$  formula is true by starting on the left side and using our induction hypothesis to get to the expression on the right side. Here's the meat of the proof...

The sum on the left side is  $\sum_{i=1}^m i$  plus another  $(m + 1)$  term, i.e.

$$\sum_{i=1}^{m+1} i = \sum_{i=1}^m i + (m + 1).$$

Since we already know that

$$\sum_{i=1}^m i = \frac{m(m+1)}{2},$$

we have

$$\sum_{i=1}^{m+1} i = \frac{m(m+1)}{2} + (m + 1).$$

We can use basic arithmetic to get

$$\begin{aligned}\frac{m(m+1)}{2} + (m+1) &= \frac{m(m+1)}{2} + \frac{2(m+1)}{2} = \frac{m(m+1) + 2(m+1)}{2} = \frac{m^2 + m + 2m + 2}{2} \\ &= \frac{m^2 + 3m + 2}{2} = \frac{(m+1)(m+2)}{2} = \frac{(m+1)((m+1)+1)}{2}.\end{aligned}$$

In summary, we have that

$$\sum_{i=1}^{m+1} i = \frac{(m+1)((m+1)+1)}{2}.$$

This is precisely what we wanted to show, so our proof is done. ■ Now it's your turn...

Those who have worked with mathematical induction before can particularly help those who are working with it for the first time. Many of our proofs do not need mathematical induction; proofs that require it will be indicated explicitly.

For these exercises, work together in groups for #1, 2a, and 3. You may use your result for #1 to solve later problems. Try to get as much done as possible on these 3 problems right now in class. Then for #4 and any extra credit, work outside of class, either with groups or on your own.

1. (5 points) See the Summary at the bottom of p. 100 of the text (review the previous page or so to understand why the Summary is true). Now use it to prove that  $\det(AB) = \det(A) \det(B)$ : The determinant of the product of matrices is equal to the product of the determinants. Write up your proof.
2. Let  $\mathbf{A}$  be a nonsingular  $n \times n$  matrix.
  - (a) (5 points) Use induction to prove that  $\mathbf{A}^m$  is nonsingular for any  $m \in \mathbb{Z}^+$ .
  - (b) (Extra Credit) Use induction to prove that  $(\mathbf{A}^m)^{-1} = (\mathbf{A}^{-1})^m$  for any  $m \in \mathbb{Z}^+$

*Hint: break it up into two cases. Case 1:  $B$  is singular. Case 2:  $B$  is nonsingular—this is the case that will use the p. 100 Summary.*

*Solution: See p. 103 of the text. Make sure that you follow the steps there. Discuss the reasons for each step with your group.*

3. (5 points) In your own words, prove Cramer's rule.

*The book outlines a proof on p. 107. Read that proof and discuss it with your group. Give a more detailed proof in your own words.*
4. (5 points) Now that you've proved  $\det(AB) = \det(A) \det(B)$  in #1, write down the proof that we did in class: If  $A$  is invertible, then  $\det(A^{-1}) = \frac{1}{\det A}$ . See if you can do it without referring to your notes.

5. (Extra Credit) Prove that if  $A$  has two identical rows or columns, then  $\det A = 0$ . *This is easy extra credit!*
6. (Extra Credit) Briefly review the beginning of section 2.3, and write up a clear proof in your own words that  $A (\text{adj } A) = (\det A)I$
7. (Extra Credit) Suppose  $\mathbf{A}$  and  $\mathbf{B}$  are  $k \times k$  matrices. We want to show that  $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$  using mathematical induction. We'll do it in 2 steps.
- (a) Show the result is true for  $k = 1$ . That's all we need for our base step, but just to see what's going on, also show that it is true for  $k = 2$ .
- (b) Assume that the result is true for  $k = n$ . (This means: assume that for all  $n \times n$  matrices  $\mathbf{A}$  and  $\mathbf{B}$ ,  $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$ .) Using this assumption, show that the result is true for  $k = n + 1$ ; i.e., for any  $[n + 1] \times [n + 1]$  matrices  $\mathbf{A}$  and  $\mathbf{B}$ ,  $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$ . (This is the meat of the proof right here—see what you can come up with!)