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Key for 3.6
10/13/06

3.6

1 a. $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\left\{ \begin{matrix} (1,3,2) \\ (0,1,0) \end{matrix} \right\}$ = basis for the row space of A

$\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} \right\}$ = basis for the column space of A

$x_1 = -2\lambda$
$x_2 = 0$
$x_3 = \lambda$

$N(A) = \lambda \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \lambda \in \mathbb{R}$

b. $A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 14 & 1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$\left\{ \begin{matrix} (1, 2, -1, -2) \\ (0, 1, 0, -2/7) \\ (0, 0, 1, 0) \end{matrix} \right\}$ = basis for the row space of A

$\left\{ \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \right\}$ = basis for the column space of A

$x_1 = 10/7\lambda$
$x_2 = 2/7\lambda$
$x_3 = 0$
$x_4 = \lambda$

$\left\{ \begin{pmatrix} 10/7 \\ 2/7 \\ 0 \\ 1 \end{pmatrix} \lambda \right\}$ = basis for $N(A)$

c. $A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & -5 & 11 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix}$

$\underset{\text{ref}}{\begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & -7/5 & 0 \\ 0 & 0 & 1 & 3/4 \end{bmatrix}} \to \underset{\text{rref}}{\begin{bmatrix} 1 & 0 & 0 & -13/20 \\ 0 & 1 & 0 & 21/20 \\ 0 & 0 & 1 & 3/4 \end{bmatrix}}$

$\left\{ \begin{matrix} (1, 3, -2, 1) \\ (0, 1, -7/5, 0) \\ (0, 0, 1, 3/4) \end{matrix} \right\}$ = form a basis for the row space of A.

$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} \right\}$ = form a basis for the column space of A.

$x_1 = \frac{13}{20}\lambda$

$x_2 = \frac{-21}{20}\lambda$

$x_3 = -3/4\,\lambda$

$x_4 = \lambda$

basis for $N(A) = \lambda \begin{pmatrix} \frac{13}{20} \\ \frac{-21}{20} \\ \frac{-3}{4} \\ 1 \end{pmatrix}$

2 a. $X = \begin{bmatrix} 1 & 2 & -3 \\ -2 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & -3 \\ 0 & 0 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & \frac{-3}{2} \\ 0 & 0 & 1 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

$x_2 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$

$x_3 = \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}$

3 columns of X will form a basis for the column space of X

Thus, the dim Span $(x_1, x_2, x_3) = 3$

c. $A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & -5 & 11 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix}$

$\underset{\text{ref}}{\begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & -7/5 & 0 \\ 0 & 0 & 1 & 3/4 \end{bmatrix}} \rightarrow \underset{\text{rref}}{\begin{bmatrix} 1 & 0 & 0 & -13/20 \\ 0 & 1 & 0 & 21/20 \\ 0 & 0 & 1 & 3/4 \end{bmatrix}}$

$\left\{ \begin{matrix} (1, 3, -2, 1) \\ (0, 1, -7/5, 0) \\ (0, 0, 1, 3/4) \end{matrix} \right\}$ = form a basis for the row space of A.

$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} \right\}$ = form a basis for the column space of A.

$x_1 = \frac{13}{20}\lambda$

$x_2 = \frac{-21}{20}\lambda$

$x_3 = -3/4\,\lambda$

$x_4 = \lambda$

basis for $N(A) = \lambda \begin{pmatrix} \frac{13}{20} \\ \frac{-21}{20} \\ \frac{-3}{4} \\ 1 \end{pmatrix}$

2 a.

$X = \begin{bmatrix} 1 & 2 & -3 \\ -2 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & -3 \\ 0 & 0 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & \frac{-3}{2} \\ 0 & 0 & 1 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

3 columns of X will form a basis for the column space of X

Thus, the dim Span $(x_1, x_2, x_3) = 3$

$x_2 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$

$x_3 = \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}$

b. $X = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, x_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

3 columns of X will form a basis for the column space of X. Thus, dim span (x_1, x_2, x_3) equals 3.

c. $A = \begin{bmatrix} 1 & -2 & 3 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & -4 & 5 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2 columns of A will form a basis for the column space of X.

Thus, dimension equals 2

$x_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad x_3 = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$

3 a. $A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 2 & 4 & 5 & 5 & 4 & 9 \\ 3 & 6 & 7 & 8 & 5 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 & -3 \end{bmatrix}$

$\rightarrow U = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 & 4 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Free variables

$u_2 = (2, 0, 0)^T \qquad u_2 = 2u_1$

$u_4 = (3, -1, 0)^T \qquad u_4 = 5u_1 - u_3$

$u_5 = (1, 2, 0)^T \qquad u_5 = -3u_1 + 2u_2$

b. Column vectors of A that correspond to the lead variables of U: $x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $x_3 = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$ $x_5 = \begin{pmatrix} 4 \\ 9 \\ 9 \end{pmatrix}$

$$x_2 = 2x_1$$
$$x_4 = 5x_1 - x_3$$
$$x_5 = -3x_1 + 2x_3$$

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Section 3.6
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Math 310

4 a. A linear system $Ax=b$ is consistent if and only if b is in the column space of b.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

a basis for the column space of A $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$4\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

b is in the column space of A, thus $Ax=b$ is consistent.

b. $A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a basis for the column space of A $= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

There is no way to express b as a linear combination of our basis for the column space A, therefore $Ax=b$ is inconsistent.

c. $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 \\ 0 & 5/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

a basis for the column space of A $= \left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}\right]$

$$2\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

thus, b is in the column space of A. Furthermore, $Ax=b$ is consistent.

d. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

a basis for the column space of A $= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

There is no way to express b as a linear combination of the basis for the column space of A.

b is not in the column space of A, therefore, $Ax=b$ is inconsistent.

e. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$

$\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$ = a basis for the column space of A

$5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$ b is in the column space of A therefore $Ax = b$ is consistent.

f. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ = a basis for the column space of A

$5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$ b is in the column space of A, therefore $Ax = b$ is consistent

7 a. If A is an $m \times n$ matrix, then the rank of A plus the nullity of A equals n.

A is an 6×5 matrix

$5 = \text{rank} + \dim N(A) = 3 + 2$

rank of $A = 3$

B is an 6×5 matrix

$5 = \text{rank} + \dim N(B) = 4 + 1$

$\dim N(B) = \underline{1}$

9 a. Because A and B are row equivalent, they have the same row space. Furthermore, the dimension of the row space of A equals the dimension of the row space of B. By theorem 3.6.3 the dimension of the row space of a matrix equals the dimension of the column space of a matrix. Thus, the dimension of the column space of A equals the dimension of the column space of B.

b. The column spaces of two row equivalent matrices are <u>not</u> necessarily the same; however, their column vectors satisfy the same dependency relations.

For example, U and A are row equivalent matrices.

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

Column vectors of A — a_1 and a_3 are linearly independent.

$a_2 = 2a_1$
$a_4 = 3a_1 + 2a_3$

$$U = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For U the column vectors — u_1 and u_3 are linearly independent.

$u_2 = 2u_1$
$u_4 = 3u_1 + 2u_3$