adopt a section 6.4
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Math 310
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6.4. 1, 2, 5

1. For each of the following pairs of vectors $\mathbf{z}$ and $\mathbf{w}$ in C^2, compute (i) $\|\mathbf{z}\|$, (ii) $\|\mathbf{w}\|$, (iii) $\langle \mathbf{z}, \mathbf{w} \rangle$, and (iv) $\langle \mathbf{w}, \mathbf{z} \rangle$.

(a) $\mathbf{z} = \begin{bmatrix} 4+2i \\ 4i \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 2+i \end{bmatrix}$

(b) $\mathbf{z} = \begin{bmatrix} 1+i \\ 2i \\ 3-i \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2-4i \\ 5 \\ 2i \end{bmatrix}$

a) i) $\|\alpha\| = \sqrt{a^2+b^2}$ Let $\alpha = a+bi$

$\|z\| = \sqrt{\Sigma(a^2+b^2)}$

$= \sqrt{4^2+2^2+4^2} = \sqrt{36} = 6$

ii) $\|w\| = \sqrt{(-2)^2+(2^2+1^2)} = \sqrt{9} = 3$

iii) $\langle z, w \rangle = w^H z = \overline{w}^T z = (-2, 2-i)\begin{pmatrix} 4+2i \\ 4i \end{pmatrix} = -8-4i+8i-4(-1) = -4+4i$

iv) $\langle w, z \rangle = \overline{z}^T w = (4-2i, -4i)\begin{pmatrix} -2 \\ 2+i \end{pmatrix} = -8-4i+(-8-4(-1)) = -4-4i$

b) i) $\|z\| = \sqrt{(1^2+1^2)+2^2+(3^2+(-1)^2)} = \sqrt{16} = 4$

ii) $\|w\| = \sqrt{(2^2+(-4)^2)+5^2+2^2} = \sqrt{49} = 7$

iii) $\langle z, w \rangle = \overline{w}^T z = (2+4i, 5, -2i)\begin{pmatrix} 1+i \\ 2i \\ 3-i \end{pmatrix}$

$= (2+6i+4(-1))+10i+(-6i+2(-1)) = -4+10i$

iv) $\langle w, z \rangle = \overline{z}^T w = (1-i, -2i, 3+i)\begin{pmatrix} 2-4i \\ 5 \\ 2i \end{pmatrix}$

$= [2-6i+8(-1)]-10i+6i+2(-1) = -4-10i$

2. Let

$$\mathbf{z}_1 = \begin{bmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{bmatrix} \quad \text{and} \quad \mathbf{z}_2 = \begin{bmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(a) Show that $\{\mathbf{z}_1, \mathbf{z}_2\}$ is an orthonormal set in C^2.

(b) Write the vector $\mathbf{z} = \begin{bmatrix} 2+4i \\ -2i \end{bmatrix}$ as a linear combination of $\mathbf{z}_1$ and $\mathbf{z}_2$.

a) To be an orthnormal set $\langle z_i, z_j \rangle = 0$ when $i \neq j$.

$$\langle z_1, z_2 \rangle = z_2^H z_1 = \left(\frac{-i}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) \begin{pmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{pmatrix} = \frac{-i+1}{2\sqrt{2}} + \frac{i-1}{2\sqrt{2}} = 0$$

$$\langle z_1, z_1 \rangle = z_1^H z_1 = \left(\frac{1-i}{2}, \frac{1+i}{2} \right) \begin{pmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{pmatrix} = \frac{1+1}{4} + \frac{1+1}{4} = 1$$

$$\langle z_2, z_2 \rangle = z_2^H z_2 = \left(\frac{-i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \begin{pmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} + \frac{1}{2} = 1$$

b)

$$z = c_1 z_1 + c_2 z_2$$

$$\begin{pmatrix} 2+4i \\ -2i \end{pmatrix} = c_1 \begin{pmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{pmatrix} + c_2 \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \frac{1+i}{2} & \frac{i}{\sqrt{2}} & 2+4i \\ \frac{1-i}{2} & -\frac{1}{\sqrt{2}} & -2i \end{array} \right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2\cdot\sqrt{2} \end{array} \right)$$

$$z = 4 z_1 + 2\sqrt{2} z_2$$

4) a. $\begin{bmatrix}\overline{1-i} & \overline{2}\\ \overline{2} & \overline{3}\end{bmatrix}^T = \begin{bmatrix}1+i & 2\\ 2 & 3\end{bmatrix} \neq \begin{bmatrix}1-i & 2\\ 2 & 3\end{bmatrix}$

b. $\begin{bmatrix}\overline{1} & \overline{2-i}\\ \overline{2+i} & \overline{-1}\end{bmatrix}^T = \begin{bmatrix}1 & 2-i\\ 2+i & -1\end{bmatrix} = \begin{bmatrix}1 & 2-i\\ 2+i & -1\end{bmatrix}$

it is a Hermitian matrix

c. $\begin{bmatrix}\overline{1/\sqrt{2}} & \overline{-1/\sqrt{2}}\\ \overline{1/\sqrt{2}} & \overline{1/\sqrt{2}}\end{bmatrix}^T = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix} \neq \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$

d. $\begin{bmatrix}\overline{\frac{1}{\sqrt{2}}i} & \overline{1/\sqrt{2}}\\ \overline{1/\sqrt{2}} & \overline{-\frac{1}{\sqrt{2}}i}\end{bmatrix}^T = \begin{bmatrix}-\frac{1}{\sqrt{2}}i & 1/\sqrt{2}\\ 1/\sqrt{2} & \frac{1}{\sqrt{2}}i\end{bmatrix} \neq \begin{bmatrix}\frac{1}{\sqrt{2}}i & 1/\sqrt{2}\\ 1/\sqrt{2} & -\frac{1}{\sqrt{2}}i\end{bmatrix}$

e. $\begin{bmatrix}\overline{0} & \overline{i} & \overline{1}\\ \overline{i} & \overline{0} & \overline{-2+i}\\ \overline{-1} & \overline{2+i} & \overline{0}\end{bmatrix}^T = \begin{bmatrix}0 & -i & -1\\ -i & 0 & -2-i\\ 1 & -2-i & 0\end{bmatrix} \neq \begin{bmatrix}0 & i & 1\\ i & 0 & -2+i\\ -1 & 2+i & 0\end{bmatrix}$

f. $\begin{bmatrix}\overline{3} & \overline{1+i} & \overline{i}\\ \overline{1-i} & \overline{1} & \overline{3}\\ \overline{-i} & \overline{3} & \overline{1}\end{bmatrix}^T = \begin{bmatrix}3 & 1+i & i\\ 1-i & 1 & 3\\ -i & 3 & 1\end{bmatrix} = \begin{bmatrix}3 & 1+i & i\\ 1-i & 1 & 3\\ -i & 3 & 1\end{bmatrix}$

hermitian matrix

a. $\begin{bmatrix}1-i & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}1+i & 2\\ 2 & 3\end{bmatrix} = \begin{bmatrix}6 & 8-2i\\ 8+2i & 13\end{bmatrix}$

$\begin{bmatrix}1+i & 2\\ 2 & 3\end{bmatrix}\begin{bmatrix}1-i & 2\\ 2 & 3\end{bmatrix} = \begin{bmatrix}6 & 8+2i\\ 8-2i & 13\end{bmatrix}$

not equal not normal

b. $\begin{bmatrix} 1 & 2-i \\ 2+i & -1 \end{bmatrix} \begin{bmatrix} 1 & 2-i \\ 2+i & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2-i \\ 2+i & -1 \end{bmatrix} \begin{bmatrix} 1 & 2-i \\ 2+i & -1 \end{bmatrix}$

normal

c. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

normal

d. $\begin{bmatrix} \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \end{bmatrix} = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

$\begin{bmatrix} -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \end{bmatrix} = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$

not equal not normal

e. $\begin{bmatrix} 0 & i & 1 \\ i & 0 & -2+i \\ -1 & 2+i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & -1 \\ -i & 0 & -2-i \\ 1 & -2-i & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2-i & 1+2i \\ -2+i & 6 & -i \\ 1-2i & i & 6 \end{bmatrix}$

$\begin{bmatrix} 0 & -i & -1 \\ -i & 0 & -2-i \\ 1 & -2-i & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 1 \\ i & 0 & -2+i \\ -1 & 2+i & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2-i & 1+2i \\ -2+i & 6 & -i \\ 1-2i & i & 6 \end{bmatrix}$

normal matrix

f. $\begin{bmatrix} 3 & 1+i & i \\ 1-i & 1 & 3 \\ -i & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1+i & i \\ 1-i & 1 & 3 \\ -i & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1+i & i \\ 1-i & 1 & 3 \\ -i & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1+i & i \\ 1-i & 1 & 3 \\ -i & 3 & 1 \end{bmatrix}$

normal

b) They must be real or the values of the diagonal entries will change when $\bar{z}$ is transposed

$$z = \begin{bmatrix} \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \end{bmatrix} \qquad z^H = \begin{bmatrix} -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \end{bmatrix}$$

z^H is not a Hermitian matrix since the values of the diagnol changed

5. Find an orthogonal or unitary diagonalizing matrix for each of the following:

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(g) $\begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}$

a) $A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$ $\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = 1$

$\lambda_1 = 3$; $A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \overset{rref}{\Rightarrow} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = x_2 = \alpha$, $x_2 = \alpha$

$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector $= \frac{1}{\|x\|} x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda_2 = 1$: $A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \overset{rref}{\Rightarrow} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = -x_2 = -\alpha$, $x_2 = \alpha$

$\alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ eigenvector $= \frac{1}{\|x\|} x = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

5b) $\begin{pmatrix} 1 & 3+i \\ 3-i & 4 \end{pmatrix}$ $\quad A - \lambda I = \begin{pmatrix} 1-\lambda & 3+i \\ 3-i & 4-\lambda \end{pmatrix}$ C.E $\lambda^2 - 5\lambda - 6 = 0$

$\lambda_1 = 6$

$\lambda_2 = -1$

$\lambda_1 = 6:\ A - 6I = \begin{pmatrix} -5 & 3+i \\ 3-i & -2 \end{pmatrix}$

$\begin{pmatrix} -5 & 3+i \\ 3-i & -2 \end{pmatrix} X = 0 \xRightarrow{\text{rref}} \begin{pmatrix} 1 & -\frac{1}{5}(3+i) & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 - \frac{1}{5}(3+i)x_2 = 0$

$x_1 = \frac{1}{5}(3+i)x_2 = \frac{1}{5}(3+i)\alpha$

$x_2 = \alpha$

$X = \begin{pmatrix} \frac{1}{5}(3+i) \\ 1 \end{pmatrix}$

$\|X\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + 1} = \sqrt{\frac{7}{5}} = \frac{\sqrt{35}}{5}$

$q_1 = \frac{1}{\|X\|} X = \frac{5}{\sqrt{35}} \begin{pmatrix} \frac{1}{5}(3+i) \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{(3+i)}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \end{pmatrix}$

$\lambda_1 = -1:\ (A+I)X = \begin{pmatrix} 2 & 3+i \\ 3-i & 5 \end{pmatrix} X = 0 \xRightarrow{\text{rref}} \begin{pmatrix} 1 & \frac{1}{2}(3+i) & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 = -\frac{1}{2}(3+i)\alpha \qquad \|X\| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2} = \sqrt{\frac{7}{2}} = \frac{\sqrt{14}}{2}$

$x_2 = \alpha$

$X = \begin{pmatrix} -\frac{1}{2}(3+i) \\ 1 \end{pmatrix} \qquad q_2 = \frac{1}{\|X\|} X = \frac{2}{\sqrt{14}} \begin{pmatrix} -\frac{1}{2}(3+i) \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{3+1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{pmatrix}$

$Q = \begin{pmatrix} \frac{(3+i)}{\sqrt{35}} & -\frac{3+1}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & \frac{2}{\sqrt{14}} \end{pmatrix}$

6.4.5 c) $\begin{pmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $A-\lambda I = \begin{pmatrix} 2-\lambda & i & 0 \\ -i & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$ C.E. $= \det(A-\lambda I)$

$\lambda_1 = 3$

$\lambda_2 = 2$

$\lambda_3 = 1$

$\lambda_1 = 3$: $\begin{pmatrix} -1 & i & 0 \\ -i & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \xRightarrow{rref} \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = ix_2 = i\alpha$, $x_2 = \alpha$, $x_3 = 0$ $\quad x = \alpha \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$

$\|x\| = \sqrt{i^2 + 1^2} = \sqrt{2}$

$q_1 = \frac{1}{\|x\|} x = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

$\lambda_2 = 2$: $\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xRightarrow{rref} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = 0$, $x_2 = 0$, $x_3 = \alpha$ $\quad X = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\|x\| = \sqrt{1^2} = 1$ $\quad q_2 = \frac{1}{\|x\|} X = \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_3 = 1$: $\begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xRightarrow{rref} \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = -ix_2 = -i\alpha$, $x_2 = \alpha$, $x_3 = 0$ $\quad X = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$

$\|x\| = \sqrt{(-i)^2 + 1^2} = \sqrt{2}$ $\quad q_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

$$Q = \begin{pmatrix} \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

6.4.5d) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{pmatrix}$ $A-\lambda I = \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & -2 \\ 1 & -2 & 3-\lambda \end{pmatrix}$; $\det(A-\lambda I) = -\lambda(\lambda-5)(\lambda-3)$

$\lambda_1 = 5$
$\lambda_2 = 3$
$\lambda_3 = 0$

$\lambda = 5$: $\begin{pmatrix} -3 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = 0$, $x_2 = -x_3 = -\alpha$, $x_3 = \alpha$ $\quad X = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\|X\| = \sqrt{(-1)^2 + 1} = \sqrt{2}$ $\quad q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\lambda = 3$: $\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = 2x_3 = 2\alpha$, $x_2 = x_3 = \alpha$, $x_3 = \alpha$ $\quad X = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\|X\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$ $\quad q_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$

$\lambda = 0$: $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = -x_3 = -\alpha$, $x_2 = x_3 = \alpha$, $x_3 = \alpha$ $\quad X = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\|X\| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$ $\quad q_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$$Q = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

6.4.5.e) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $A - \lambda I = \begin{pmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix}$ $\det = -(\lambda - 1)(\lambda^2 - 1)$ $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 1$

$\lambda = -1$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \xRightarrow{rref} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = -x_3 = -\alpha$, $x_2 = 0$, $x_3 = \alpha$ $\quad X = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\|X\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ $\quad q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

$\lambda = 1$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \xRightarrow{rref} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = x_3 = \alpha$, $x_2 = \alpha$, $x_3 = \beta$ $\quad X = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\|X\| = \sqrt{1^2 + 1^2} = \sqrt{2}$ $\quad q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

$\|X\| = \sqrt{1^2} = 1$ $\quad q_3 = \frac{1}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$Q = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

6.4.5f) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $A-\lambda I: \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix}$ $\det = -y^2(y-3)$

$\lambda_1 = 3$

$\lambda_2 = \lambda_3 = 0$

$\lambda = 3: \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xRightarrow{rref} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = x_3 = \alpha$, $x_2 = x_3 = \alpha$, $x_3 = \alpha$ $\quad x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\|x\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ $\quad q_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$\lambda = 0: \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xRightarrow{rref} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = \overset{-\alpha}{-x_2} \overset{-\beta}{-x_3}$, $x_2 = \alpha$, $x_3 = \beta$ $\quad x = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\|x\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ $\quad q_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

$\|x\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ $\quad q_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

6.5.4 g) $\begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}$ $A-\lambda I = \begin{pmatrix} 4-\lambda & 2 & -2 \\ 2 & 1-\lambda & -1 \\ -2 & -1 & 1-\lambda \end{pmatrix}$; $\det = -y^2(y-6)$ $\lambda_1 = 6$ $\lambda_2 = \lambda_3 = 0$

$\lambda = 6$: $\begin{pmatrix} -2 & 2 & -2 \\ 2 & -5 & -1 \\ -2 & -1 & -5 \end{pmatrix} \overset{rref}{\Longrightarrow} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = -2x_3 = -2\alpha$, $x_2 = -x_3 \; -\alpha$, $x_3 = \alpha$ $\quad X = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

$\|X\| = \sqrt{(-2)^2 + (-1)^2 + 1^2} = \sqrt{6}$ $\quad q_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$

$\lambda = 0$: $\begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix} \overset{rref}{\Longrightarrow} \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 = -\frac{1}{2}x_2 + \frac{1}{2}x_3$, $x_2 = \alpha$, $x_3 = \beta$ $\quad X = \alpha \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$

$\|X\| = \sqrt{\left(-\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}$ $\quad q_2 = \frac{2}{\sqrt{5}} \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{pmatrix}$

$\|X\| \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}$ $\quad q_3 = \frac{2}{\sqrt{5}} \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} = \begin{matrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{matrix}$

$$Q = \begin{pmatrix} -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$