§6.4 Hermitian Matrices 12/4/06

Let $\alpha \in \mathbb{C}$. Then $\alpha = a + bi$, where $a, b \in \mathbb{R}$. The **length** of α is given by: $|\alpha| = \sqrt{\overline{\alpha}\alpha} = \sqrt{a^2 + b^2}$.

Let $\mathbb{C}^n$ denote the vector space of all n-tuples of complex numbers. Then for a vector $\mathbf{z} = (z_1, \dots, z_n)^T$ in $\mathbb{C}^n$, the length of $\mathbf{z}$ is

$$||\mathbf{z}|| = \sqrt{(|z_1|^2 + \cdots + |z_n|^2)}$$
$$= \sqrt{(\overline{z_1}z_1 + \cdots + \overline{z_n}z_n)} = \sqrt{(\overline{\mathbf{z}}^T \mathbf{z})}$$

Notation: Conjugate Transpose. Write $\mathbf{z}^H$ for $\overline{\mathbf{z}}^T$, the transpose of $\overline{\mathbf{z}}$. Then $||\mathbf{z}|| = \sqrt{(\mathbf{z}^H \mathbf{z})}$.

Definition

Let $\mathbf{V}$ be a vector space over $\mathbb{C}$. For $\mathbf{z}, \mathbf{w} \in V$, the **inner product** $\langle \mathbf{z}, \mathbf{w} \rangle$ is a complex number satisfying the following conditions:

- $\langle \mathbf{z}, \mathbf{z} \rangle \geq 0$ with equality if and only if $\mathbf{z} = 0$. (Positive)
- $\langle \mathbf{z}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{z} \rangle}$ for all $\mathbf{z}, \mathbf{w} \in V$. (***Conjugate*** Symmetric)
- $\langle \alpha\mathbf{z} + \mathbf{w}, \mathbf{u} \rangle = \alpha\langle \mathbf{z}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle$ for any scalar $\alpha \in \mathbb{C}$. (Linear)

Just as we defined $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ as a the standard inner product for the vector space $\mathbb{R}^n$, we can define an inner product on $\mathbb{C}^n$ by $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^H \mathbf{z}$. This satisfies the inner product axioms.

Example: Find $\mathbf{w}^H \mathbf{z}$, $\mathbf{z}^H \mathbf{z}$, and $\mathbf{w}^H \mathbf{w}$ for

$$\mathbf{z} = \begin{pmatrix} 5+i \\ 1-3i \end{pmatrix}, \ \mathbf{w} = \begin{pmatrix} 2+i \\ -2+3i \end{pmatrix}.$$

Complex Matrix Rules

Let $A, B, M \in \mathbb{C}^{m \times n}$, $C \in \mathbb{C}^{n \times r}$.

- Definition: if $M = A + iB$, then $\overline{M} = A - iB$.
- $(A^H)^H = A$.
- $(\alpha A + \beta B)^H = \overline{\alpha} A^H + \overline{\beta} B^H$.
- $(AC)^H = C^H A^H$.

Definition

A matrix M is **Hermitian** if $M = M^H$.

Example: $M = \begin{pmatrix} 3 & 2-i \\ 2+i & 4 \end{pmatrix}$. Check...

Note: If $M \in \mathbb{R}^{m \times n}$, then $\overline{M} = M$, and so in this case $M^H = M^T$. So if M is both real and also *symmetric* ($M^T = M$), then M is always Hermitian. We can think of Hermitian matrices as the complex analog of real symmetric matrices, and they have lots of convenient properties...

Theorem

The eigenvalues of a Hermitian matrix are all real. Furthermore, eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof: Later if we have time. (See p. 347)

Definition

An $n \times n$ matrix U is **unitary** if its column vectors form an orthonormal set in $\mathbb{C}^n$.

Lemma: U is unitary $\iff U^H U = I$.

Proof

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be the column vectors of U. Since U is unitary, then $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \delta_{ij}$. But remember, this inner product is defined in $\mathbb{C}^n$, so $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \mathbf{u}_i^H \mathbf{u}_j$.
Thus $U^H U = I$. (Note: this is equivalent to $U^{-1} = U^H$.)

Corollary If the eigenvalues of a Hermitian matrix A are distinct, then there exists a unitary matrix U that diagonalizes A.

Example...

The above corollary is actually true even if the eigenvalues of A are not distinct. This is called the **Spectral Theorem**. We will not prove this in full detail here, but it can be proved by factoring A into UTU^H, where U is unitary and T is an upper triangular matrix. The factorization $A = UTU^H$ is called the *Schur decomposition* of A. It comes from the following theorem.

Schur's Theorem

For any $n \times n$ matrix A, there exists a unitary matrix U such that the matrix $T = U^H AU$ is upper triangular.

In the case that A is Hermitian, the matrix T will actually be **diagonal**, which is proved by the Spectral Theorem.

Spectral Theorem

If A is Hermitian, then there exists a unitary matrix U that diagonalizes A.

In the case that A is real and symmetric, then its eigenvalues and eigenvectors must be real. Thus the diagonalizing matrix U must be orthogonal.

Corollary

If A is a real symmetric matrix, then there is an orthogonal matrix U that diagonalizes A, that is, $U^T AU = D$, where D is diagonal.

Example...

There are non-Hermitian matrices that have complete orthonormal sets of eigenvectors. For example...

Definition. A Matrix A is **skew Hermitian** if $A^H = -A$.

Skew Hermitian matrices have complete orthonormal sets of eigenvectors. But other sets of matrices do also.

Question: What is the most general way to describe matrices with complete orthonormal sets of eigenvectors?

Definition. A Matrix A is **normal** if $AA^H = A^H A$.

Theorem

A matrix A is normal if and only if A has a complete orthonormal set of eigenvectors.