1 Vector Spaces- 9-27-06

A set $\mathbf{V}$ is called a vector space if:

I. For each $\mathbf{x}, \mathbf{y} \in \mathbf{V}$, $\mathbf{x} + \mathbf{y}$ is an element of $\mathbf{V}$. (Addition)

II. For each $\mathbf{x} \in \mathbf{V}$ and $a \in \mathbb{R}$, $a\mathbf{x}$ is an element of $\mathbf{V}$. (Multiplication by scalar)

The two operations satisfy the following laws:

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$, commutative law

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$, associative law

A3. $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for each $\mathbf{x}$, neutral element $\mathbf{0}$

A4. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$, unique anti element

A5. $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$ for each $\mathbf{x}, \mathbf{y}$, distributive law

A6. $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$, distributive law

A7. $(ab)\mathbf{x} = a(b\mathbf{x})$, distributive law

A8. $1\mathbf{x} = \mathbf{x}$.

corollary: $0\mathbf{x} = \mathbf{0}$ neutral element:
$0\mathbf{x} = (0 + 0)\mathbf{x} = 0\mathbf{x} + 0\mathbf{x} \Rightarrow$
$\mathbf{0} = 0\mathbf{x} - 0\mathbf{x} = (0\mathbf{x} + 0\mathbf{x}) - 0\mathbf{x} = 0\mathbf{x}.$

Examples:

1. $\mathbb{R}$ - Real Line

2. $\mathbb{R}^2$ = Plane

3. $\mathbb{R}^3$ - Three dimensional space

4. $\mathbb{R}^n$ - n-dimensional space

5. $\mathbb{R}^{m\times n}$ - Space of $m \times n$ matrices

6. Spaces of upper triangular, lower triangular and diagonal matrices

7. $\mathcal{P}_n$ - Space of polynomials of degree less than n:
$\mathcal{P}_n := \{p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1x + a_0\}.$

8. $C[a, b]$ - Space of continuous functions on the interval $[a, b]$.

Note. The examples 1 - 7 are finite dimensional vector spaces. 8 - is infinite dimensional vector space.

Note. In this course all vector spaces are finite dimensional and isomorphic to $\mathbb{R}^n$ (or $\mathbb{C}^n$ as in Chapter 6).

Additional Properties of Vector Spaces

Use the vector space axioms to prove these three properties.

(i) $\mathbf{0x} = \mathbf{0}$.

(ii) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ implies $\mathbf{y} = -\mathbf{x}$.

(iii) $(-1)\mathbf{x} = -\mathbf{x}$.

Proof of (i): Axioms A6 and A8 tell us that

$\mathbf{x} = 1\mathbf{x} = (1 + 0)\mathbf{x} = 1\mathbf{x} + 0\mathbf{x} = \mathbf{x} + 0\mathbf{x}$. Thus

$-\mathbf{x} + \mathbf{x} = -\mathbf{x} + (\mathbf{x} + 0\mathbf{x}) = (-\mathbf{x} + \mathbf{x}) + 0\mathbf{x}$ (A2).

Thus $\mathbf{0} = \mathbf{0} + 0\mathbf{x} = 0\mathbf{x}$, proving (i).

Proof of (ii): Suppose $\mathbf{x} + \mathbf{y} = \mathbf{0}$. Then

$-\mathbf{x} = -\mathbf{x} + \mathbf{0} = -\mathbf{x} + (\mathbf{x} + \mathbf{y})$.

Therefore, $-\mathbf{x} = (-\mathbf{x} + \mathbf{x}) + \mathbf{y} = \mathbf{0} + \mathbf{y} = \mathbf{y}$ by A1, A2, A3, and A4.

Proof of (iii): From part (i) and A6, we know that

$\mathbf{0} = 0\mathbf{x} = (1 + (-1))\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x}$.

Thus by A8, $\mathbf{x} + (-1)\mathbf{x} = \mathbf{0}$.

So from part (ii), this tells us that $(-1)\mathbf{x} = -\mathbf{x}$.

§3.2 Subspaces

Let $\mathbf{V}$ be a vector space. A subset $\mathbf{W}$ of $\mathbf{V}$ is called a subspace of $\mathbf{V}$ if the following three conditions hold:

(i) $\boldsymbol{W}$ is nonempty,

(ii) for any $\mathbf{x}, \mathbf{y} \in \mathbf{W} \Rightarrow \mathbf{x} + \mathbf{y} \in \mathbf{W}$,

(iii) for any $\mathbf{x} \in \mathbf{W},\ \boldsymbol{a} \in \mathbb{R} \Rightarrow \boldsymbol{a}\mathbf{x} \in \mathbf{W}$.

Equivalently: $\mathbf{W} \subset \mathbf{V}$ is a subspace $\iff$ $\mathbf{W}$ is a vector space with respect to the addition and the multiplication by a scalar defined in $\mathbf{V}$.
In other words, $\boldsymbol{W}$ is a vector space by itself!

Note: The zero vector $\mathbf{0} \in \mathbf{W}$ always.
Proof: By (i) we know $\boldsymbol{W}$ is nonempty, so it contains a vector. (We'll call this vector $\mathbf{x}$.) By (iii), we know that $\mathbf{0x} \in \mathbf{W}$. But $\mathbf{0x} = \mathbf{0}$, so $\mathbf{0} \in \boldsymbol{W}$.

Every vector space $\mathbf{V}$ has the following two "non-proper" subspaces:

1. $\mathbf{V}$.

2. The trivial subspace consisting of the zero element: $\mathbf{W} = \{\mathbf{0}\}$.

Examples of subspaces

1. Subspaces of the plane $\mathbb{R}^{\mathbf{2}}$:
●the whole space, ●lines through the origin*,
● the trivial subspace$= \{\mathbf{0}\}$.

*Remember, we think of vectors as segments from the origin to an endpoint. Thus by a“line,” we mean the **set of vectors** whose **endpoints** lie on the line.

2. Subspaces of 3-dimensional space $\mathbb{R}^{\mathbf{3}}$:
●the whole space, ● planes through the origin,
●lines through the origin, ●the trivial subspace.

(Read many more examples in the text.)

Note: If we already know that $\boldsymbol{V}$ is a subspace, then to show that $\boldsymbol{W} \subset \boldsymbol{V}$ is a subspace only requires showing $\boldsymbol{W}$ is: (i) nonempty, (ii) closed under scalar multiplication, and (iii) closed under addition.
We DON'T have to check all 8 axioms, because we know the axioms already held for everything in the bigger space $\boldsymbol{V}$!

So what's NOT a subspace?

One example: A line that does not go through the origin.

Important example of a subspace:

The **nullspace**

Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$. The **nullspace** of $\boldsymbol{A}$, denoted $\boldsymbol{N}(\boldsymbol{A})$, is a subspace of $\mathbb{R}^n$ consisting of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $\boldsymbol{A}\mathbf{x} = \mathbf{0}$.

Why is $\boldsymbol{N}(\boldsymbol{A})$ a subspace?

- $\mathbf{0} \in \boldsymbol{N}(\boldsymbol{A})$, so it is nonempty.
- For $\mathbf{x} \in \boldsymbol{N}(\boldsymbol{A})$ and $\boldsymbol{\alpha} \in \mathbb{R}$, $\boldsymbol{A}(\boldsymbol{\alpha}\mathbf{x}) = \boldsymbol{\alpha}\boldsymbol{A}\mathbf{x} = \boldsymbol{\alpha}\mathbf{0} = \mathbf{0}$. Thus $\boldsymbol{\alpha}\mathbf{x} \in \boldsymbol{N}(\boldsymbol{A})$.
- For $\mathbf{x}, \mathbf{y} \in \boldsymbol{N}(\boldsymbol{A})$, then $\boldsymbol{A}(\mathbf{x}+\mathbf{y}) = \boldsymbol{A}\mathbf{x} + \boldsymbol{A}\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Thus $\mathbf{x}+\mathbf{y} \in \boldsymbol{N}(\boldsymbol{A})$.

Find $\boldsymbol{N}(\boldsymbol{A})$ for $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix}$.

$$N(A) = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

In general, to find $N(A)$, row reduce A first.

SUMMARY for finding Nullspace

●Row Reduce.

●Find Free variables.

●Solve for Dependent variables in terms of free variables.

●Put answer in set notation.

2 Linear combination & span 9-29-06

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbf{V}$ and $a_1, \ldots, a_k \in \mathbb{R}$.

The vector $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

The set of **all** linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called the **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and denoted by $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

Note: $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is a linear subspace of $\mathbf{V}$.

Fact: All subspaces in a finite dimensional vector space can be written as $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ for some corresponding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Examples:

1. Any line through the origin in $1, 2,$ or 3 dimensional space is spanned by any nonzero vector on the line.

2. Any plane through the origin in 2 or 3 dimensional space is spanned by any two nonzero vectors not lying on a line, i.e. non collinear vectors.

3. $\mathbb{R}^3$ spanned by any 3 non planar vectors.

The set $\{\mathbf{v_1}, \ldots, \mathbf{v}_n\}$ is a **spanning set** for V if and only if every vector in V can be written as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v}_n$.

Example: $\mathbb{R}^3$. We define the standard vectors:

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The set $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is a spanning set for $\mathbb{R}^3$.

The set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is not a spanning set for $\mathbb{R}^3$.