RELATIVELY GEOMETRIC ACTIONS OF KÄHLER GROUPS ON CAT(0) CUBE COMPLEXES

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ABSTRACT. We prove that for $n \geq 2$, a non-uniform lattice in $\mathrm{PU}(n,1)$ does not admit a relatively geometric action on a CAT(0) cube complex, in the sense of [10]. As a consequence, if Γ is a non-uniform lattice in a non-compact semisimple Lie group G without compact factors that admits a relatively geometric action on a CAT(0) cube complex, then G is commensurable with $\mathrm{SO}(n,1)$. We also prove that if a Kähler group is hyperbolic relative to residually finite parabolic subgroups, and acts relatively geometrically on a CAT(0) cube complex, then it is virtually a surface group.

1. Introduction

A finitely generated group is called *cubulated* if it acts properly cocompactly on a CAT(0) cube complex. Agol [1], building on the work of Wise [29] and many others, proved that cubulated hyperbolic groups enjoy many important properties, and used this to solve several open conjectures in 3-manifold topology, in particular the Virtual Haken and Virtual Fibering Conjectures. Wise [29, §17] proved the Virtual Fibering Conjecture in the non-compact, finite-volume setting, using the relatively hyperbolic structure of the fundamental group.

Einstein–Groves define the notion of a *relatively geometric* action of a group pair $(\Gamma, \mathcal{P})$ on a CAT(0) cube complex [10]. For such an action, elements of $\mathcal{P}$ act elliptically. This allows the possibility that even though the elements of $\mathcal{P}$ might not act properly on any CAT(0) cube complex, there still may be a relatively geometric action. Relatively geometric actions are a natural generalization of proper actions and share many of the same features as in the proper case, especially when Γ is hyperbolic relative to $\mathcal{P}$.

Uniform lattices in $\mathrm{SO}(3,1)$ always act geometrically thus relatively geometrically on CAT(0) cube complexes [5]. Bergeron–Haglund–Wise [4] prove that in higher dimensions, lattices in $\mathrm{SO}(n,1)$ which are arithmetic of simplest type are cubulated. It also follows from this and Wise's quasiconvex hierarchy theorem [29] that many "hybrid" hyperbolic n-manifolds have cubulated fundamental groups. In the relatively geometric setting, using the work of Cooper–Futer [7], Einstein–Groves proved that non-uniform lattices in $\mathrm{SO}(3,1)$ also admit relatively geometric actions, relative to their cusp subgroups [10]. In fact, they prove that if $(G, \mathcal{P})$ is hyperbolic relative to free abelian subgroups and the Bowditch boundary $\partial(G, \mathcal{P})$ is homeomorphic to S^2, then G is isomorphic to a non-uniform lattice in $\mathrm{SO}(3,1)$ if and only if $(G, \mathcal{P})$ admits a relatively geometric action on a CAT(0) cube complex. This result is a relative version of the work of Markovic [25] and Haïssinsky [22] in the convex-cocompact setting, giving an equivalent formulation of the Cannon conjecture in terms of actions on hyperbolic CAT(0) cube complexes. It is not known in general whether the above results extend to all lattices in $\mathrm{SO}(n,1)$ for $n \geq 3$.

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In contrast, work of Delzant–Gromov implies that uniform lattices in $\mathrm{PU}(n,1)$ are not cubulated [8]. Recall that a group Γ is *Kähler* if $\Gamma \cong \pi_1(X)$ for some compact Kähler manifold X. If $\Gamma \leq \mathrm{PU}(n,1)$ is a torsion-free, uniform lattice, then Γ acts freely, properly discontinuously cocompactly on complex hyperbolic n–space $\mathbf{H}^n_{\mathbb{C}}$. The quotient $M = \Gamma \backslash \mathbf{H}^n_{\mathbb{C}}$ is a closed, negatively curved Kähler manifold, and in particular Γ is a hyperbolic, Kähler group. In this context, Delzant–Gromov showed that any infinite Kähler group that is hyperbolic and cubulated is commensurable to a surface group of genus $g \geq 2$ [8]. Thus Γ is not cubulated for $n \geq 2$. Since every uniform lattice in $\mathrm{PU}(n,1)$ is virtually torsion-free, it follows that uniform lattices in $\mathrm{PU}(n,1)$ are not cubulated if $n \geq 2$.

On the other hand, uniform lattices in $\mathrm{PU}(1,1) = \mathrm{SO}(2,1)$, are finite extensions of hyperbolic surface groups, hence are hyperbolic and cubulated. Similarly, non-uniform lattices in $\mathrm{PU}(1,1)$ are the orbifold fundamental groups of surfaces with finitely many cusps, hence virtually free. Such lattices admit both proper cocompact and relatively geometric actions on $\mathrm{CAT}(0)$ cube complexes. Since the cusp subgroups of a non-uniform lattice in $\mathrm{PU}(n,1)$ $(n \geq 2)$ are virtually nilpotent but not virtually abelian, it follows from a result of Haglund [21] that such a lattice does not admit a proper action on a $\mathrm{CAT}(0)$ cube complex (see Proposition 4.3 below).

However, the parabolic subgroups do not yield such an obstruction to the existence of a relativeley geometric action. Thus, this leaves open the question of whether non-uniform lattices in $\mathrm{PU}(n,1)$ admit relatively geometric actions on $\mathrm{CAT}(0)$ cube complexes for $n \geq 2$. Our first result answers this question in the negative.

Theorem 1.1. *Let $\Gamma \leq \mathrm{PU}(n,1)$ be a non-uniform lattice with $n \geq 2$, and let $\mathcal{P}$ be the collection of cusp subgroups of Γ. Then $(\Gamma, \mathcal{P})$ does not admit a relatively geometric action on a $\mathrm{CAT}(0)$ cube complex.*

Corollary 1.2. *Let Γ be a lattice in a non-compact semisimple Lie group G without compact factors. If either*

(1) Γ is uniform and cubulated hyperbolic, or
(2) Γ is non-uniform, hyperbolic relative to its cusp subgroups $\mathcal{P}$, and $(\Gamma, \mathcal{P})$ admits a relatively geometric action on a $\mathrm{CAT}(0)$ cube complex,

then G is commensurable to $\mathrm{SO}(n,1)$ for some $n \geq 1$.

Proof. A uniform lattice (resp. non-uniform lattice) Γ in a semisimple Lie group G is hyperbolic (resp. hyperbolic relative to its cusp subgroups $\mathcal{P}$) if and only if G has rank 1, by a result of Behrstock–Druţu–Mosher [3]. Any rank 1 noncompact semisimple Lie group is commensurable with one of $\mathrm{SO}(n,1)$, $\mathrm{PU}(n,1)$, $\mathrm{Sp}(n,1)$ for $n \geq 2$, or the isometry group of the octonionic hyperbolic plane $\mathbf{H}^2_{\mathbb{O}}$. The latter and $\mathrm{Sp}(n,1)$ have Property (T), while $\mathrm{SO}(n,1)$ and $\mathrm{PU}(n,1)$ do not. Hence if Γ is commensurable with a lattice in $\mathrm{Sp}(n,1)$ or $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{O}})$, then Γ has (T).

By a result of Niblo–Reeves [26], any action of a group with Property (T) on a $\mathrm{CAT}(0)$ cube complex has a global fixed point, so lattices in $\mathrm{Sp}(n,1)$ and $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{O}})$ admit neither geometric nor relatively geometric actions on $\mathrm{CAT}(0)$ cube complexes. Hence if Γ is as in the statement of the result, it must be commensurable to a lattice in either $\mathrm{PU}(n,1)$ or $\mathrm{SO}(n,1)$. For $n \geq 2$, the uniform case of $\Gamma \leq \mathrm{PU}(n,1)$ is eliminated by work Delzant–Gromov [8]. The corollary now follows from Theorem 1.1. □

We say that a relatively hyperbolic group pair $(\Gamma, \mathcal{P})$ is *properly* relatively hyperbolic if $\mathcal{P} \neq \{\Gamma\}$. The following result considers general relatively geometric actions of Kähler relatively hyperbolic groups on $\mathrm{CAT}(0)$ cube complexes (when the peripheral subgroups are residually finite).

Theorem 1.3. *Let $(\Gamma, \mathcal{P})$ be a properly relatively hyperbolic pair such that each element of $\mathcal{P}$ is residually finite. If Γ is Kähler and acts relatively geometrically on a* CAT(0) *cube complex, then Γ is virtually a hyperbolic surface group.*

We will deduce Theorem 1.1 from Theorem 1.3 in Section 4. In fact, non-uniform lattices in $\mathrm{PU}(n,1)$ are Kähler for $n \geq 3$ [28], hence Theorem 1.1 follows immediately from Theorem 1.3 in this range. However, our proof of Theorem 1.1 will work for all $n \geq 2$, and will not use this fact. In [9], Delzant–Py considered actions of Kähler groups on locally finite, finite-dimensional CAT(0) cube complexes that are more general than geometric ones (see Theorem A for precise hypotheses), and showed that every such action virtually factors through a surface group. We remark that the cube complexes appearing in relatively geometric actions will in general not be locally finite.

We conclude the introduction with a sample application of Theorem 1.3.

Corollary 1.4. *Suppose that A and B are infinite residually finite groups which are not virtually free. Any $C'\left(\frac{1}{6}\right)$–small cancellation quotient of $A * B$ is not Kähler.*

Proof. Let Γ be such a small cancellation quotient of $A * B$. According to [13], Γ is residually finite and admits a relatively geometric action on a CAT(0) cube complex. If Γ were Kähler, it would be a virtual surface group, by Theorem 1.3. However, A embeds in Γ as an infinite-index subgroup, and the only infinite index subgroups of virtual surface groups are virtually free. □

Outline: In Section 2, we review the definition of a relatively geometric action of a group pair on a CAT(0) cube complex and the notion of group-theoretic Dehn fillings, then collect some known results about these. In Section 3 we prove Theorem 1.3. In Section 4, after reviewing the Borel–Serre and toroidal compactifications of non-uniform quotients of complex hyperbolic space, we prove Theorem 1.1.

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2. Actions on CAT(0) Cube Complexes

In this section, we review the notion of a relatively geometric action of a group pair $(\Gamma, \mathcal{P})$ on a CAT(0) cube complex, defined by Einstein and Groves in [10]. We then introduce Dehn fillings of group pairs and recall some useful results from [11].

Definition 2.1. Let Γ be a group and $\mathcal{P}$ a collection of subgroups of Γ. An action of Γ on a CAT(0) cube complex X is *relatively geometric with respect to* $\mathcal{P}$ if

(1) $\Gamma\backslash X$ is compact;
(2) Each element of $\mathcal{P}$ acts elliptically on X;
(3) Each cell stabilizer in X is either finite or else conjugate to a finite-index subgroup of an element of $\mathcal{P}$.

Recall that if $(\Gamma, \mathcal{P})$ is a relatively hyperbolic group pair and $\Gamma_0 \leq \Gamma$ has finite-index then $(\Gamma_0, \mathcal{P}_0)$ is also a relatively hyperbolic group pair, where $\mathcal{P}_0$ is the set of representatives of the Γ_0–conjugacy classes of

$$\{P^g \cap \Gamma_0 \mid g \in \Gamma, P \in \mathcal{P}\} \tag{1}$$

Since $[\Gamma\colon \Gamma_0]$ is finite, $\mathcal{P}_0$ is still a finite collection of subgroups. It follows that if Γ admits a relatively geometric action on a CAT(0) cube complex X, then $(\Gamma_0, \mathcal{P}_0)$ also admits a relatively geometric action on X by restriction. Indeed, (2) and (3) in Definition 2.1 follow immediately and (1) follows from that fact that under the natural map $\Gamma_0\backslash X \to \Gamma\backslash X$, each cell of $\Gamma\backslash X$ has at most $[\Gamma\colon \Gamma_0] < \infty$ pre-images. Hence if $\Gamma\backslash X$ is compact, so is $\Gamma_0\backslash X$. We have just proven

Lemma 2.2. *Let $\Gamma_0 \leq \Gamma$ be a finite-index subgroup. If $(\Gamma, \mathcal{P})$ has a relatively geometric action on a* CAT(0) *cube complex X, then the restriction of this action to $(\Gamma_0, \mathcal{P}_0)$ is also relatively geometric, where $\mathcal{P}_0$ is defined as in Equation 1.*

2.1. **Dehn fillings.**

Dehn fillings first appeared in the context of 3-manifold topology and were subsequently generalized to the group-theoretic setting by Osin [27] and Groves–Manning [18]. We now recall the notion of a Dehn filling of a group pair $(G, \mathcal{P})$:

Definition 2.3 (Dehn Filling)**.** Given a group pair $(G, \mathcal{P})$, where $\mathcal{P} = \{P_1, ..., P_m\}$ and a choice of normal subgroups of peripheral groups $\mathcal{N} = \{N_i \trianglelefteq P_i\}$, the *Dehn filling* of $(G, \mathcal{P})$ with respect to $\mathcal{N}$ is the pair $(\overline{G}, \overline{\mathcal{P}})$ where $\overline{G} = G/K$ and $K = \langle\langle \cup N_i \rangle\rangle$ is the normal closure in G of the group generated by $\{\cup_i N_i\}$ and $\bar{\mathcal{P}}$ is the set of images of $\mathcal{P}$ under this quotient. The N_i are called the *filling kernels*. When we want to specify the filling kernels we write $G(N_1, \dots, N_m)$ for the quotient $\overline{G}$.

Definition 2.4 (Peripherally finite)**.** If each normal subgroup N_i has finite-index in P_i, the filling is said to be *peripherally finite*.

Definition 2.5 (Sufficiently long)**.** We say that a property $\mathcal{X}$ holds for all sufficiently long Dehn fillings of $(G, \mathcal{P})$ if there is a finite subset $B \subset G \smallsetminus \{1\}$ so that whenever $N_i \cap B = \emptyset$ for all i, the corresponding Dehn filling $G(N_1, ..., N_n)$ has property $\mathcal{X}$.

The proof of the next theorem relies on the notion of a $\mathcal{Q}$–filling of a collection of subgroups $\mathcal{Q}$ of G. Recall from [17] that given a subgroup $Q < G$, the quotient $G(N_1, \dots, N_m)$ is a *Q–filling* if for all $g \in G$, and $P_i \in \mathcal{P}$, $|Q \cap P_i^g| = \infty$ implies $N_i^g \subseteq Q$. If $\mathcal{Q} = \{Q_1, \dots, Q_l\}$ is a family of subgroups, then $G(N_1, \dots, N_m)$ is a *$\mathcal{Q}$–filling* if it is a Q–filling for every $Q \in \mathcal{Q}$.

Let $\mathcal{Q}$ be a collection of finite-index subgroups of elements of $\mathcal{P}$ so that any infinite cell stabilizer contains a conjugate of an element of $\mathcal{Q}$. The following is proved in [11].

Theorem 2.6 (Proposition 4.1 and Corollary 4.2 of [11])**.** *Let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic pair such that the elements of $\mathcal{P}$ are residually finite. If $(\Gamma, \mathcal{P})$ admits a relatively geometric action on a* CAT(0) *cube complex X then*

1. *For sufficiently long $\mathcal{Q}$–fillings $\Gamma \to \overline{\Gamma} = \Gamma/K$, the quotient $\overline{X} = K\backslash X$ is a* CAT(0) *cube complex; and*
2. *Any sufficiently long, peripherally finite $\mathcal{Q}$–filling of Γ is hyperbolic and virtually special.*

The following result is implicit in [11]. For completeness, we provide a proof.

Lemma 2.7. *In the context of Theorem 2.6.(1), the action of $\overline{\Gamma}$ on $\overline{X}$ is relatively geometric.*

Proof. Since $\overline{\Gamma}\backslash\overline{X} = \Gamma\backslash X$ the action is cocompact. Let $\overline{\mathcal{P}}$ be the induced peripheral structure on $\overline{\Gamma}$ (the image of $\mathcal{P}$). The fact that elements of $\overline{\mathcal{P}}$ act elliptically on $\overline{X}$ follows from the fact that elements of $\mathcal{P}$ act elliptically on X. Because each cell-stabilizer of $\Gamma \curvearrowright X$ is either finite or conjugate to a finite-index of subgroup of some $P_i \in \mathcal{P}$, this implies that the cell-stabilizers of $\overline{\Gamma} \curvearrowright \overline{X}$ are conjugate to finite-index subgroups of $P_i/(K \cap P_i)$ (the elements of $\overline{\mathcal{P}}$). Thus the action of Δ on Y is relatively geometric. □

3. Relatively geometric actions: the Kähler case

In this section, we apply Theorem 2.6 to prove Theorem 1.3. The main idea is to use Dehn filling to produce a minimal action of a finite-index subgroup of Γ on a tree with finite kernel. A deep result of Gromov–Schoen implies that any Kähler group admitting a minimal acting on tree with finite kernel must be virtually a hyperbolic surface group [16].

Proof of Theorem 1.3. Suppose that $(\Gamma, \mathcal{P})$ acts relatively geometrically on a CAT(0) cube complex. Since the elements of $\mathcal{P}$ are residually finite, there exists a sufficiently long, peripherally finite $\mathcal{Q}$–filling $\Gamma \to \overline{\Gamma} = \Gamma/K$ which satisfies the hypotheses of Theorem 2.6.((2)), so $\overline{\Gamma}$ is hyperbolic and $\overline{X} = K\backslash X$ is a CAT(0) cube complex. Let $\Gamma_0 \le \Gamma$ be a finite-index subgroup such that Γ_0 is torsion-free and $\Gamma_0\backslash X$ is special, which exists by [11, Theorem 1.4].

Cutting along an embedded essential two-sided hyperplane H in $\Gamma_0\backslash X$ yields a splitting of Γ_0 according to the complex of groups version of van Kampen's Theorem [6, III.$\mathcal{C}$.3.11.(5), III.$\mathcal{C}$.3.12, p.552].[1] The edge group of such a splitting is a hyperplane stabilizer for the Γ_0–action on X, which is relatively quasi-convex by [12, Corollary 4.11], and infinite-index since the hyperplane is essential. The action of Γ_0 on the Bass–Serre tree T associated to this splitting has finite kernel, since any normal subgroup contained in an infinite-index relatively quasi-convex subgroup is finite. Let F denote the kernel of the action of Γ_0 on T.

By [16], the induced action of Γ_0 on T factors through a surjective homomorphism $\varphi\colon \Gamma_0 \to \Delta_0$, where $\Delta_0 \le \mathrm{PSL}_2(\mathbb{R})$ is a cocompact lattice. The kernel of φ is contained in F, hence finite, so Γ_0 is commensurable up to finite kernels with Δ_0, which is itself virtually a hyperbolic surface group. Since any group commensurable up to finite kernels with a hyperbolic surface group is virtually a hyperbolic surface group, this means that Γ_0, and hence Γ, is virtually a hyperbolic surface group, as desired. □

4. Relatively geometric actions: Lattices in PU$(n,1)$

Let Γ be a non-uniform lattice in $\mathrm{PU}(n,1)$. Then Γ acts properly discontinuously on complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ and the quotient, which we henceforth denote by $M = \Gamma\backslash\mathbf{H}^n_{\mathbb{C}}$, is a non-compact orbifold of finite volume with finitely many cusps. Each cusp corresponds to a conjugacy class of subgroups stabilizing a parabolic fixed point in $\partial_\infty \mathbf{H}^n_{\mathbb{C}}$. Farb [14] proved that Γ is hyperbolic relative to the collection of these cusp subgroups, which we denote by $\mathcal{P}$. In this section, we prove Theorem 1.1, namely that $(\Gamma, \mathcal{P})$ does not admit a relatively geometric action on a CAT(0) cube complex.

Throughout the course of the proof, we pass freely to finite-index subgroups by invoking Lemma 2.2. In order to streamline the exposition, we do not refer to Lemma 2.2 each time. First we reduce the Theorem 1.1 to the case where Γ is torsion-free.

Lemma 4.1. *Γ has a torsion-free subgroup of finite index.*

[1] One can also see this tree directly by considering the dual tree to the collection of hyperplanes of X which project to H. See [17, Remark 1.1] for more details.

Proof. We have a short exact sequence

$$1 \to \mathbb{Z}/(n+1)\mathbb{Z} \to \mathrm{SU}(n,1) \to \mathrm{PU}(n,1) \to 1.$$

Restricting to Γ, we get a short exact sequence

$$1 \to \mathbb{Z}/(n+1)\mathbb{Z} \to \Lambda \to \Gamma \to 1,$$

where Λ is the pre-image of Γ in $\mathrm{SU}(n,1)$. Since Γ is finitely generated and $\mathbb{Z}/(n+1)\mathbb{Z}$ is finite, Λ is finitely generated. As $\mathrm{SU}(n,1)$ is linear, Selberg's lemma implies that Λ has a finite-index torsion-free subgroup, say, Λ_0. Thus $\Lambda_0 \cap \mathbb{Z}/(n+1)\mathbb{Z} = 1$ and hence it is mapped isomorphically to finite-index subgroup $\Gamma_0 \le \Gamma$. □

Following Lemma 4.1, for the remainder of this section we assume that $\Gamma \le \mathrm{PU}(n,1)$ is torsion-free.

4.1. **The structure of cusps.** We now briefly review the geometric structure of cusps in M. For more details see [15]. Recall that up to scaling each horosphere in $\mathbf{H}^n_{\mathbb{C}}$ is isometric to $\mathcal{H}_{2n-1}(\mathbb{R})$, the $(2n-1)$–dimensional real Heisenberg group, equipped with a left-invariant metric. The Heisenberg group is a central extension

$$1 \to \mathbb{R} \to \mathcal{H}_{2n-1}(\mathbb{R}) \to \mathbb{R}^{2n-2} \to 1 \tag{2}$$

with extension 2–cocycle equal to the standard symplectic form

$$\omega = 2\sum_{i=1}^{n-1} dx_i \wedge dy_i,$$

where $(x_1, y_1, \ldots, x_{n-1}, y_{n-1})$ are coordinates on $\mathbb{R}^{2n-2}$. The Lie algebra $\mathfrak{h}_{2n-1}$ is 2–step nilpotent with basis $\{X_1,\ Y_1, \ldots, X_n,\ Y_n,\ Z\}$ where

$$[X_i, Y_i] = Z$$

and all other brackets vanish. Thus Z generates the center of $\mathfrak{h}_{2n-1}$ representing the kernel $\mathbb{R}$ in Equation (2), while the remaining coordinates project to the generators of $\mathbb{R}^{2n-2}$. Choosing the identity matrix I_{2n-1} as the inner product on $\mathfrak{h}_{2n-1}$, we see that the isometry group of $\mathcal{H}_{2n-1}(\mathbb{R})$ is isomorphic to $\mathcal{H}_{2n-1}(\mathbb{R}) \rtimes U(n-1)$, where the $\mathcal{H}_{2n-1}(\mathbb{R})$ factor is the action of $\mathcal{H}_{2n-1}(\mathbb{R})$ on itself by left translation, and the unitary group $U(n-1)$ is the stabilizer of the identity. Indeed, any isometry which fixes $1 \in \mathcal{H}_{2n-1}(\mathbb{R})$ must also be a Lie algebra isomorphism; it therefore preserves the center $\langle Z \rangle$ and induces an isometry of $\mathbb{R}^{2n-2} \cong \langle X_1, Y_1, \cdots, X_{n-1}, Y_{n-1} \rangle$ preserving ω. We conclude that such an isometry lies in $U(n-1) = O_{2n-2}(\mathbb{R}) \cap \mathrm{Sp}_{2n-2}(\mathbb{R})$.

Definition 4.2. Let $\pi\colon \mathcal{H}_{2n-1}(\mathbb{R}) \rtimes U(n-1) \to U(n-1)$ be the projection. For any $g \in \mathcal{H}_{2n-1}(\mathbb{R}) \rtimes U(n-1)$, we call $\pi(g)$ the *rotational part* of g.

Since the center of $\mathcal{H}_{2n-1}(\mathbb{R})$ is invariant under any isometry we have a short exact sequence

$$1 \to \mathbb{R} = Z(\mathcal{H}_{2n-1}(\mathbb{R})) \to \mathcal{H}_{2n-1}(\mathbb{R}) \rtimes U(n-1) \to \mathbb{R}^{2n-2} \rtimes U(n-1) \to 1 \tag{3}$$

Since Γ is torsion-free, each cusp subgroup $P \le \Gamma$ is isomorphic to a discrete, torsion-free, cocompact subgroup of $\mathrm{Isom}(\mathcal{H}_{2n-1}(\mathbb{R}))$. In particular, $P_0 = P \cap \mathcal{H}_{2n-1}(\mathbb{R})$ is a discrete cocompact subgroup and $P \cap Z(\mathcal{H}_{2n-1}(\mathbb{R})) \cong \mathbb{Z}$. By Equation 3, P fits into a short exact sequence

$$1 \to \mathbb{Z} = P \cap Z(\mathcal{H}_{2n-1}(\mathbb{R})) \to P \to \Lambda \to 1 \tag{4}$$

where Λ is a discrete cocompact subgroup of $\mathbb{R}^{2n-2} \rtimes U(n-1)$. It follows that Λ has a finite-index subgroup Λ_0 isomorphic to $\mathbb{Z}^{2n-2}$, which is the image of P_0.

On the level of quotient spaces, the sequence in Equation (4) has the following translation. The quotient space $\mathcal{O} = \Lambda\backslash\mathbb{R}^{2n-2}$ is a Euclidean orbifold finitely covered by the $(2n-2)$–dimensional torus $T = \Lambda_0\backslash\mathbb{C}^{n-1}$, and $\Sigma = P\backslash\mathcal{H}_{2n-1}(\mathbb{R})$ is the total space of an S^1–bundle over $\mathcal{O}$, *i.e.*, there is a fiber sequence

$$S^1 \hookrightarrow \Sigma \to \mathcal{O} \tag{5}$$

Since $\mathcal{O}$ need not be smooth, this is not generally a locally trivial fibration. However, as P is torsion-free, Σ is smooth. Passing to the torus cover, we obtain an actual fiber bundle

$$S^1 \hookrightarrow \widehat{\Sigma} \to T$$

The finite group $F = P/P_0$ acts on $\widehat{\Sigma}$ preserving the fibration, hence defines a finite group of isometries of T. Thus the stabilizer of a point in T acts freely on the S^1 fiber. Since the action of F on $\widehat{\Sigma}$ is free, it follows that point stabilizers in T must be cyclic of finite order, and act by rotations on the fiber. Since $F \le U(n-1)$, any abelian subgroup is diagonalizable. Thus, locally each point in N has a neighborhood of the form $(S^1 \times \mathbb{D}^{n-1})/(\mathbb{Z}/m\mathbb{Z})$ where $\mathbb{D} \subset \mathbb{C}$ is the open unit disk, and $\mathbb{Z}/m\mathbb{Z}$ acts on S^1 by rotation by $2\pi/m$ and on the polydisk $\mathbb{D}^{n-1}$ by a diagonal unitary matrix $\Delta = \operatorname{diag}\left(e^{\frac{2\pi k_1}{m}}, \dots, e^{\frac{2\pi k_{n-1}}{m}}\right)$, where at least one k_i is coprime to m. See Figure 1 for a schematic. Since F acts by rotation on each fiber, Σ is the boundary of a disk bundle over $\mathcal{O}$, which we denote by $E_{\mathcal{O}}$.

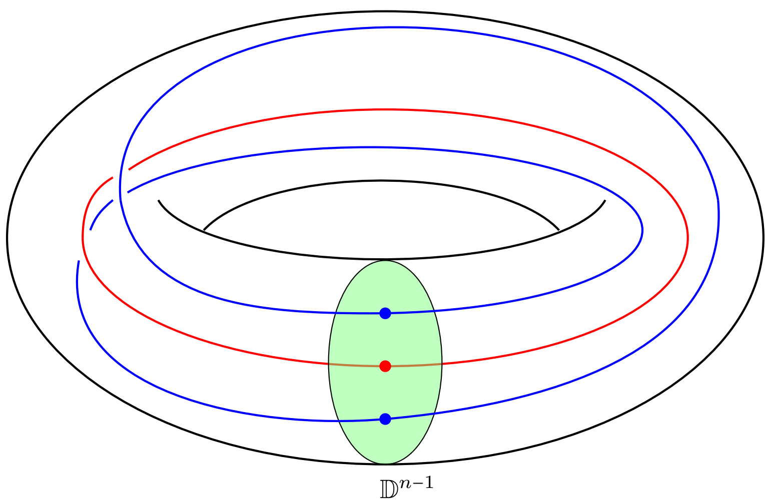

FIGURE 1. Local picture of the fibration in Equation 5 near a singular point of $\mathcal{O}$. A nonsingular fiber, shown in blue, winds $m = 2$ times around the singular fiber, shown in red.

Recall that the center of $\mathcal{H}_{2n-1}(\mathbb{R})$ is quadratically distorted. It follows that the center of P is quadratically distorted as well. By [21, Theorem 1.5], there is no proper action of P on a CAT(0) cube complex. Therefore, we have:

Proposition 4.3. *Let $\Gamma \le \mathrm{PU}(n,1)$ be a non-uniform lattice, and suppose Γ acts on a $\mathrm{CAT}(0)$ cube complex X. The action of each cusp subgroup of Γ is not proper. In particular, Γ is not cubulated.*

4.2. **The toroidal compactification of M.** Another natural compactification of M fills in the cusps with the Euclidean orbifolds described in Section 4.1. Let $\mathcal{O}_i$ be the Euclidean orbifold quotient of Σ_i, with corresponding disk bundle E_i. Thus, we can identify $E_i \smallsetminus \mathcal{O}_i$ with the cusp $\mathcal{C}_i$, then compactify M by adding $\sqcup_i \mathcal{O}_i$ at infinity. The result is a Kähler orbifold $\mathcal{T}(M)$ with boundary divisor $D = \sqcup_i \mathcal{O}_i$. The pair $(\mathcal{T}(M), D)$ is called the *toroidal compactification of M*. See [24, 2] for more details.

When the parabolic elements in Γ have trivial rotational part, then each $\mathcal{O}_i$ is a $(2n-2)$–dimensional torus, $\mathcal{T}(M)$ is a smooth Kähler manifold and D is a smooth divisor in $\mathcal{T}(M)$. Moreover, Hummel–Schroeder show that $\mathcal{T}(M)$ admits a nonpositively curved Riemannian metric [24]. In particular, $\mathcal{T}(M)$ is aspherical; if $\Delta = \pi_1(\mathcal{T}(M))$ then $\mathcal{T}(M)$ is a $K(\Delta,1)$. The following lemma ensures that we can always find a finite cover of M whose toroidal compactification is smooth.

Lemma 4.4. *Let $\Gamma \le \mathrm{PU}(n,1)$ be torsion-free and let $M = {}_{\Gamma}\backslash \mathbf{H}^n_{\mathbb{C}}$ be the quotient. There exists a finite cover $M' \to M$ such that the toroidal compactification of M' is smooth.*

Proof. By the main theorem of [23] (p. 2453), there exists a finite subset $F \subset \Gamma$ of parabolic isometries such that if a $N \trianglelefteq \Gamma$ is a normal subgroup satisfying $F \cap N = \varnothing$, then any parabolic isometry in N has no rotational part. Since Γ is residually finite and F is finite, we can find a finite-index normal subgroup $\Gamma' \trianglelefteq \Gamma$ such that $\Gamma' \cap F = \varnothing$. Therefore the finite cover $M' := {}_{\Gamma'}\backslash \mathbf{H}^n_{\mathbb{C}}$ of M admits a toroidal compactification which is smooth. □

For the rest of this section, we assume that $\mathcal{T}(M)$ is smooth. Since $M_0 \smallsetminus \partial M_0 \cong M \cong \mathcal{T}(M) \smallsetminus D$, there is a natural map of pairs $f\colon (M_0, \partial M_0) \to (\mathcal{T}(M), D)$ that is a diffeomorphism on the interior of M_0 and sends $\partial M = \sqcup \Sigma_i \to D = \sqcup_i \mathcal{O}_i$ via the fibering in Equation 5.

4.3. **Proof of Theorem 1.1.** We now have all the ingredients necessary to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 4.1 and Lemma 4.4, we may assume that $\Gamma \le \mathrm{SU}(n,1)$ is torsion-free, and that the toroidal compactification $\mathcal{T}(M)$ is smooth. In particular, Γ and all of its peripheral subgroups are residually finite.

Suppose $(\Gamma, \mathcal{P})$ admits a relatively geometric action on a $\mathrm{CAT}(0)$ cube complex X. Given a finite-index subgroup $\Gamma_0 \le \Gamma$, let $\mathcal{P}_0$ be the induced peripheral structure on Γ_0, and let Δ_0 be $\pi_1(\mathcal{T}(M_0))$, where $M_0 = {}_{\Gamma_0}\backslash \mathbf{H}^n_{\mathbb{C}}$. Since the kernel of the quotient map $\Gamma_0 \to \Delta_0$ is normally generated by subgroups in $\mathcal{P}_0$, we get an induced peripheral structure $(\Delta_0, \mathcal{A}_0)$, where $\mathcal{A}_0$ is the collection of images of elements of $\mathcal{P}_0$. Our strategy is to show that there exists a finite-index subgroup $\Gamma_0 \le \Gamma$ so that the pair $(\Delta_0, \mathcal{A}_0)$ is relatively hyperbolic and admits a relatively geometric action on a $\mathrm{CAT}(0)$ cube complex. Since $\mathcal{T}(M_0)$ is smooth (since $\mathcal{T}(M)$ is), Δ is Kähler. Thus, as $n \ge 2$, we will get a contradiction by Theorem 1.3.

Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be the induced peripheral structure on Γ. Now let $Z(P_i)$ be the center of P_i. We apply Theorem 2.6(1) to a sufficiently long $\mathcal{Q}$–filling $\mathcal{Z} = \{Z_1, \ldots, Z_k\}$ where $Z_i \le Z(P_i)$ is a finite-index subgroup. We then obtain a Dehn filling $\psi\colon \Gamma \to \Delta = \Gamma/K$ determined by the Z_i so that $Y = {}_{K}\backslash X$ is a $\mathrm{CAT}(0)$ cube complex.

Let $(\Delta, \mathcal{A})$ be the induced peripheral structure on Δ. By Theorem 2.6, we know that $(\Delta, \mathcal{A})$ is relatively hyperbolic. Lemma 2.7 implies that the action of Δ on Y is relatively geometric.

Finally, we claim that there exists a finite-index subgroup $\Delta_0 \le \Delta$ that is torsion-free. Since the elements of $\mathcal{A}$ are virtually abelian, hence residually finite, we know that Δ is also residually finite by Corollary 1.7 of [11]. Since Γ is torsion-free, by [19, Theorem 4.1] so long as the filling $\Gamma \to \Delta$ is long enough (which we may assume without loss of generality), any element of finite order in Δ is conjugate into some element of $\mathcal{A}$. As there are finitely many elements of $\mathcal{A}$, each of which has only finitely many conjugacy classes of finite order elements, we can find a finite-index subgroup $\Delta_0 \le \Delta$ which avoids each of these conjugacy classes, hence is torsion-free. The induced peripheral structure $(\Delta_0, \mathcal{A}_0)$ is relatively hyperbolic and $\Delta_0 \curvearrowright Y$ is relatively geometric by Lemma 2.2. Let $\Gamma_0 = \psi^{-1}(\Delta_0)$ and let $\mathcal{P}_0 = \{P_{0,1}, \dots, P_{0,r}\}$ be the induced peripheral structure on Γ_0. Then $K \le \Gamma_0$, and since Δ_0 is torsion-free, this implies $K \cap P_{0,i} = Z(P_{0,i})$ for each i. As the $\mathcal{P}$ is the collection of cusp subgroups of $M_0 = {}_{\Gamma_0}\backslash \mathbf{H}^n_{\mathbb{C}}$, we conclude that $\Delta_0 = \pi_1(\mathcal{T}(M))$. Thus, Δ_0 is Kähler and acts relatively geometrically on Y. By Theorem 1.3, we conclude that Δ_0 is virtually a hyperbolic surface group, which is impossible because Δ_0 contains a subgroup isomorphic to $\mathbb{Z}^{2n-2}$ and $n \ge 2$. This contradiction completes the proof. □

Remark 4.5. In [20, Definition 1.9], Groves–Manning introduce the notion of a *weakly relatively geometric action* on a CAT(0) cube complex. We can replace "relatively geometric" with "weakly relatively geometric" in Theorem 1.1 using similar arguments. Indeed, after performing the toroidal filling of $(\Gamma, \mathcal{P})$ to land in the Kähler setting, we can perform a further peripherally finite filling to obtain a hyperbolic quotient, which is virtually special [20, Theorem 4.5].

In this case the cube complex for the quotient is not ${}_K\backslash X$ where K is the kernel of the filling homomorphism $\Gamma \to \overline{\Gamma}$. Indeed, the action of $\overline{\Gamma}$ on ${}_K\backslash X$ in general has cell stabilizers that are virtually free. Nevertheless, Theorem D of [17] implies that in this case $\overline{\Gamma}$ is still cubulated hyperbolic. The arguments from the remainder of the proof of Theorem 1.1 still apply.

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