IMMUTABILITY IS NOT UNIFORMLY DECIDABLE IN HYPERBOLIC GROUPS

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Abstract. A finitely generated subgroup H of a torsion-free hyperbolic group G is called *immutable* if there are only finitely many conjugacy classes of injections of H into G. We show that there is no uniform algorithm to recognize immutability, answering a uniform version of a question asked by the authors.

In [4] we introduced the following notion which is important for the study of conjugacy classes of solutions to equations and inequations over torsion-free hyperbolic groups, and also for the study of limit groups over (torsion-free) hyperbolic groups.

Definition 1. [4, Definition 7.1] Let G be a group. A finitely generated subgroup H of G is called *immutable* if there are finitely many injective homomorphisms $\phi_1, \ldots, \phi_N \colon H \to G$ so that any injective homomorphism $\phi \colon H \to G$ is conjugate to one of the ϕ_i.

We gave the following characterization of immutable subgroups.

Lemma 2. [4, Lemma 7.2] *Let Γ be a torsion-free hyperbolic group. A finitely generated subgroup of Γ is immutable if and only if it does not admit a nontrivial free splitting or an essential splitting over $\mathbb{Z}$.*

The following corollary is immediate.

Corollary 3. *Let Γ be a torsion-free hyperbolic group and suppose that H is a finitely generated subgroup. If for every action of H on a simplicial tree with trivial or cyclic edge stabilizers H has a global fixed point then H is immutable.*

If Γ is a torsion-free hyperbolic group then the immutable subgroups of Γ form some of the essential building blocks of the structure of Γ–limit groups. See [4] and [5] for more information.

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In [4, Theorem 1.4] we proved that given a torsion-free hyperbolic group Γ it is possible to recursively enumerate the finite tuples of Γ which generate immutable subgroups. This naturally lead us to ask the following

Question 4. [4, Question 7.12] *Let Γ be a torsion-free hyperbolic group. Is there an algorithm that takes as input a finite subset S of Γ and decides whether or not the subgroup $\langle S\rangle$ is immutable?*

We are not able to answer this question, but we can answer the *uniform* version of this question in the negative, as witnessed by the following result. It is worth remarking that the algorithm from [4, Theorem 1.4] *is* uniform, in the sense that one can enumerate pairs (Γ, S) where Γ is a torsion-free hyperbolic group (given by a finite presentation) and S is a finite subset of words in the generators of Γ so that $\langle S\rangle$ is immutable in Γ.

Theorem 5. *There is no algorithm which takes as input a presentation of a (torsion-free) hyperbolic group and a finite tuple of elements, and determines whether or not the tuple generates an immutable subgroup.*

Proof. Let Γ_0 be a non-elementary, torsion-free, hyperbolic group with Property (T) and let $\{a, b\} \in \Gamma_0$ be such that $\langle a, b\rangle$ is a nonabelian free, malnormal and quasi-convex subgroup of Γ_0. There are many hyperbolic groups with Property (T) (see, for example, [9]). The existence of such a pair $\{a, b\}$ follows immediately from [6, Theorem C]. Throughout our proof, Γ_0 and $\{a, b\}$ are fixed.

Consider a finitely presented group Q with unsolvable word problem (see [7]), and let G be a hyperbolic group that fits into a short exact sequence

$$1 \to N \to G \to Q * \mathbb{Z} \to 1,$$

where N is finitely generated and has Kazhdan's Property (T). Such a G can be constructed using [2, Corollary 1.2], by taking H from that result to be a non-elementary hyperbolic group with Property (T), and recalling that having Property (T) is closed under taking quotients.

Let t be the generator for the second free factor in $Q * \mathbb{Z}$. Given a word u in the generators of Q, define words

$$c_u = tut^{-2}ut,$$

and

$$d_u = utut^{-1}u.$$

Claim 1. *If $u =_Q 1$ then $\langle c_u, d_u\rangle = \{1\}$ in $Q * \mathbb{Z}$. If $u \neq_Q 1$ then $\langle c_u, d_u\rangle$ is free of rank 2 in $Q * \mathbb{Z}$.*

Proof of Claim 1. The first assertion of the claim is obvious, and the second follows from the fact that if u is nontrivial in Q then any reduced word in $\{c_u, d_u\}^{\pm}$ yields a word in $\{t, u\}^{\pm}$ which is in normal form in the free product $Q * \mathbb{Z}$, and hence is nontrivial in $Q * \mathbb{Z}$. □

We lift the elements $c_u, d_u \in Q * \mathbb{Z}$ to elements $\bar{c}_u, \bar{d}_u \in G$.

Claim 2. *Given words c_u and d_u, it is possible to algorithmically find words $w_u, x_u, y_u, z_u \in N$ so that $\langle w_u \bar{c}_u x_u, y_u \bar{d} d_u z_u \rangle$ is quasi-convex and free of rank 2.*

Proof of Claim 2. It is well known (see, for example, [1, Lemma 4.9]) that in a δ-hyperbolic space a path which is made from concatenating geodesics whose length is much greater than the Gromov product at the concatenation points is a uniform-quality quasi-geodesic, and in particular not a loop.

By considering geodesic words representing $\bar{c}_u$ and $\bar{d}_u$, it is possible to find long words in the generators of N as in the statement of the claim so that any concatenation of $(w_u \bar{c}_u x_u)^{\pm}$ and $(y_u \bar{d}_u z_u)^{\pm}$ is such a quasigeodesic. From this, it follows immediately that the free group $\langle w_u \bar{c}_u x_u, y_u \bar{d}_u z_u \rangle$ is quasi-isometrically embedded and has free image in G. This can be done algorithmically because the word problem in G is (uniformly) solvable, so we can compute geodesic representatives for words and calculate Gromov products. □

Let $g_u = w_u \bar{c}_u x_u$ and $h_u = y_u \bar{d}_u z_u$, and let $J_u = \langle g_u, h_u \rangle$. Note that the image of J_u in Q is either trivial (if $u =_Q 1$) or free of rank 2 (otherwise). Therefore, if $u =_Q 1$ then $J_u \cap N = J_u$ and otherwise $J_u \cap N = \{1\}$.

Now consider the group

$$\Gamma_u = G *_{\{g_u = a, h_u = b\}} \Gamma_0.$$

Since $\langle a, b \rangle$ is malnormal and quasiconvex in Γ_0 and $\langle g_u, h_u \rangle$ is quasi-convex in G, the group Γ_u is hyperbolic by the Bestvina–Feighn Combination Theorem [3].

Let $K_u = \langle N, \Gamma_0 \rangle \leq \Gamma_u$. We remark that a presentation for Γ_u and generators for K_u as a subgroup of Γ_u can be algorithmically computed from the presentations of G and Γ_0 and the word u.

Claim 3. *If $u =_Q 1$ then K_u is immutable. If $u \neq_Q 1$ then K_u splits nontrivially over $\{1\}$ and so is not immutable.*

Proof of Claim 3. Let $N_u = N \cap J_u$. We observed above that if $u =_Q 1$ then $N_u = J_u$, and that if $u \neq_Q 1$ then $N_u = \{1\}$. By considering the induced action of K_u on the Bass-Serre tree of the splitting of Γ_u given by the defining amalgam, we see that in case $u =_Q 1$ we have

$$K_u \cong N *_{\{g_u=a,h_u=b\}} \Gamma_0,$$

whereas in case $u \neq_Q 1$ we have

$$K_u \cong N * \Gamma_0.$$

Thus, if $u \neq_Q 1$ then K_u splits nontrivially as a free product, as required.

On the other hand, suppose that $u =_Q 1$, and suppose that K_u acts on a tree T with trivial or cyclic edge stabilizers. Since Property (T) groups have Property (FA) [8], N and Γ_0 must act elliptically on T. However, if they do not have a common fixed vertex, then their intersection (which is free of rank 2) must fix the edge-path between the fixed point sets for N and for Γ_0, contradicting the assumption that edge stabilizers are trivial or cyclic. Thus, there is a common fixed point for N and Γ_0, and so K_u acts on T with global fixed point. It follows from Corollary 3 that K_u is immutable, as required. □

An algorithm as described in the statement of the theorem would (when given the explicit presentation of Γ_u and the explicit generators for K_u) be able to determine whether or not K_u is immutable. In turn, this would decide the word problem for Q, by Claim 3. Since this is impossible, there is no such algorithm, and the proof of Theorem 5 is complete. □

Remark 6. By taking only a cyclic subgroup to amalgamate in the definition of Γ_u, instead of a free group of rank 2, it is straightforward to see that one cannot decide whether non-immutable subgroups split over $\{1\}$ or over $\{\mathbb{Z}\}$.

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