PRESCRIBED VIRTUAL HOMOLOGICAL TORSION OF 3-MANIFOLDS

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ABSTRACT. We prove that given any finite abelian group A and any irreducible 3–manifold M with empty or toroidal boundary which is not a graph manifold there exists a finite cover $M' \to M$ so that A is a direct factor in $H_1(M', \mathbb{Z})$. This generalizes results of Sun [Sun15] and of Friedl–Herrmann [FH17].

1. INTRODUCTION

In [Sun15], Sun showed that any closed hyperbolic 3–manifold virtually contains any prescribed finite subgroup in homological torsion. Sun used the immersed almost-Fuchsian surfaces of Kahn and Markovic [KM12] to construct immersed π_1–injective 2–complexes. By using Agol's result that the fundamental groups of closed hyperbolic 3–manifolds are virtually compact special [Ago13] and the implications on virtual retractions to quasi-convex subgroups, for any closed hyperbolic 3–manifold Sun [Sun15, Theorem 1.5] finds a finite cover containing the prescribed finite abelian group as a direct factor in homology.

Since the Kahn-Markovic construction requires that the manifolds be closed, Sun's results do not apply to hyperbolic 3–manifolds with cusps. Indeed, Sun asked whether his result applied also to finite-volume hyperbolic 3-manifolds with cusps. In this paper, we extend the results of Sun to a larger class of 3–manifolds which includes all finite-volume hyperbolic 3–manifolds, giving a positive answer to [Sun15, Question 1.8].

Theorem 1.1. *Suppose that M is an irreducible 3–manifold with empty or toroidal boundary which is not a graph manifold and that A is a finite abelian group. There is a finite cover $M' \to M$ so that $H_1(M'; \mathbb{Z})$ has a direct factor isomorphic to A.*

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Prior to Theorem 1.1, Friedl and Herrmann used [Sun15] and a result of Hadari [Had15] to show that for any such M and any $k > 0$ there is finite cover $N \to M$ with $|H_1(N; \mathbb{Z})| > k$ [FH17, Theorem 1.3]. Independently, Liu showed that any such M admits a finite *regular* cover $N' \to M$ with $|H_1(N'; \mathbb{Z})| \neq 0$ [Liu17, Corollary 1.4].

A *hybrid* hyperbolic manifold is constructed either by inbreeding (c.f. [Ago06, BT11]) or interbreeding (c.f. [GPS88]) arithmetic hyperbolic manifolds. For $n > 3$ every arithmetic hyperbolic n–manifold N of simplest type contains a totally geodesic arithmetic hyperbolic 3–manifold M (coming from restrictions of the associated quadratic form). By [BHW11, §9], we get the following corollary (some of these cases follow from [Sun15]).

Corollary 1.2. *Suppose that $n > 3$ and N is a finite-volume hyperbolic n–manifold which is either arithmetic of simplest type or a hybrid. If A is a finite abelian group then there is a finite cover $N' \to N$ so that $H^1(N'; \mathbb{Z})$ has a direct factor isomorphic to A.*

The bulk of this paper is devoted to the case of Theorem 1.1 where M is a finite-volume hyperbolic 3–manifold. We follow the strategy of [Sun15] but give an independent proof which simplifies and generalizes Sun's arguments, recovering Sun's results in the closed hyperbolic setting. We replace Sun's use of the results of Kahn and Markovic [KM12] with those of Kahn and Wright [KW18] and replace some arguments of Sun with an elementary argument using coverings of surfaces. We begin in Section 2 by recording some facts about quasi-isometries and hyperbolic spaces. In Section 3 we apply the construction of Kahn-Wright to build an almost-Fuchsian surface in M. In Section 4 we use the Kahn-Wright surface to construct a 2-complex $X_n \looparrowright M$. We then apply virtual retraction properties to complete the proof of Theorem 1.1 in the finite-volume hyperbolic case in Section 5. Finally, in Section 6 we deduce the general case of Theorem 1.1 from that of finite-volume hyperbolic 3–manifolds.

We remark that independent from Kahn and Wright, Cooper and Futer [CF19] obtained similar results on constructing many closed immersed π_1–injective quasi-Fuchsian surfaces in finite-volume hyperbolic 3–manifolds with cusps. However, our arguments rely on the additional control on the quasi-conformal constants and on the holonomies in the Kahn-Wright constructions.

2. QUASI-ISOMETRIC EMBEDDINGS

In this section we record some elementary facts about quasi-isometries and hyperbolic spaces.

Definition 2.1. *Let k, λ, c be constants, and let X, Y be metric spaces. A map $f\colon X \to Y$ is a k–*local (λ, c)–quasi-isometric embedding *if for all $x \in X$ the restriction to the ball of radius k*

$$f|_{B_k(x)}\colon B_k(x) \to Y$$

is a (λ, c)–quasi-isometric embedding.

The following is essentially [KW18, Theorem A.20].

Proposition 2.2. *For all δ, for all $c \geq 0$ and all $\lambda \geq 1$ there exist k, λ', c' so that if Y is a δ–hyperbolic metric space and X is a geodesic metric space then any k–local (λ, c)–quasi-isometric embedding is a (λ', c')–quasi-isometric embedding.*

Proof. Since X and Y are geodesic metric spaces, distances in X and Y are calculated by geodesics. Therefore, we can apply the standard local-to-global result for quasi-geodesics (see, for example, [CDP90, Theorem 3.1.4, p.25]). □

2.1. **Half-planes.** Let $\theta \in (0, \pi]$. Let P_θ be the subspace of $\mathbb{H}^3$ obtained from gluing two totally geodesic half-planes together along their boundary geodesic, meeting at angle θ. There is a natural embedding $p_\theta\colon \mathbb{H}^2 \to \mathbb{H}^3$ taking $\mathbb{H}^2$ to P_θ given by mapping the imaginary axis to the boundary geodesic of the two half-planes (we consider $\mathbb{H}^2$ in the upper half-space model as a subset of $\mathbb{C}^2$). The image of these boundary geodesics is the *pleating locus* for p_θ.

Lemma 2.3. *Given $\theta \in (0, \pi]$ there exists $c_\theta \geq 0$ so that for all $\eta \in [\theta, \pi]$ the map p_η is a $(1, c_\theta)$–quasi-isometric embedding.*

Proof. We show that it suffices to take

$$c_\theta = 2 \cdot \operatorname{arccosh}\left(\frac{1}{\sin\left(\frac{\theta}{2}\right)}\right).$$

Indeed, suppose that $x, y \in \mathbb{H}^2$ and let $\overline{x} = p_\eta(x)$ and $\overline{y} = p_\eta(y)$, and consider the image of $[x, y]$ in $p_\eta(\mathbb{H}^2)$. If the sign of the real parts of x and y are the same, then $[x, y]$ maps to a geodesic in $\mathbb{H}^3$ and $d_{\mathbb{H}^3}(\overline{x}, \overline{y}) = d_{\mathbb{H}^2}(x, y)$ in this case.

Suppose then that the signs of the real parts of x and y are different, and let $z \in \mathbb{H}^2$ be the point where $[x, y]$ meets the y–axis. Let $\overline{z} = p_\eta(z)$. Then $p_\eta([x, y])$ consists of two geodesic segments $[\overline{x}, \overline{z}]$ and $[\overline{z}, \overline{y}]$ meeting at some angle $\alpha \geq \eta \geq \theta$.

Consider the geodesic triangle Δ in $\mathbb{H}^3$ with vertices $\overline{x}, \overline{y}, \overline{z}$, and let e be the distance from $\overline{z}$ to the geodesic $\gamma = [\overline{x}, \overline{y}]$. The geodesic from $\overline{z}$ to γ cuts Δ into two right-angled hyperbolic triangles, one of

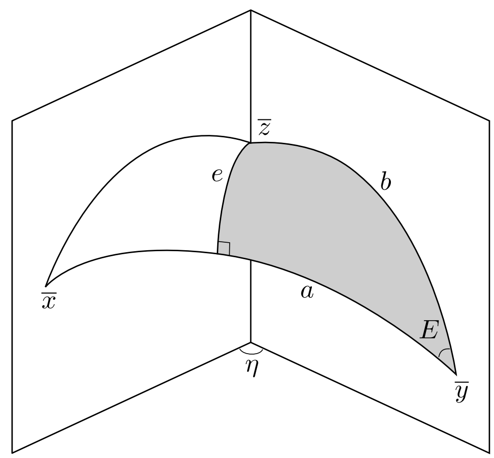

FIGURE 1. The proof of Lemma 2.3

which has angle at $\overline{z}$ at least $\frac{\theta}{2}$. We thus have a hyperbolic triangle with side lengths e, a, b, say, where the angle opposite b is $\frac{\pi}{2}$, and the angle opposite a is at $\overline{z}$ and is $A \geq \frac{\theta}{2}$. Let E be the angle opposite the side of length e.

The second hyperbolic law of cosines says

$$\cos(E) = -\cos(A)\cos\left(\frac{\pi}{2}\right) + \sin(A)\sin\left(\frac{\pi}{2}\right)\cosh(e),$$

so

$$\cosh(e) = \frac{\cos(E)}{\sin(A)} \leq \frac{1}{\sin\left(\frac{\theta}{2}\right)}.$$

Let $d_1 = d_{\mathbb{H}^3}(\overline{x}, \overline{z})$ and $d_2 = d_{\mathbb{H}^3}(\overline{z}, \overline{y})$. Observe that $d_{\mathbb{H}^2}(x, y) = d_1 + d_2$. It is clear that

$$d_1 + d_2 - 2\operatorname{arccosh}\left(\frac{1}{\sin\left(\frac{\theta}{2}\right)}\right) \leq d_{\mathbb{H}^3}(\overline{x}, \overline{y}) \leq d_1 + d_2,$$

and the result follows. □

3. KAHN-WRIGHT SURFACES

From this section until the end of Section 5, let N be a finite-volume hyperbolic 3–manifold. We remark that the arguments work in the closed setting as well as in the cusped setting, hence recovering Sun's results.

The set of closed geodesics in $N = \mathbb{H}^3/\Gamma$ is in 1-to-1 correspondence with the set of conjugacy classes of loxodromic elements in Γ. For a closed geodesic α in N (with corresponding conjugacy class

$[\gamma] \subset \Gamma$) let $\ell(\alpha)$ denote the length of α (the translation length γ) and $\theta(\alpha)$ the holonomy class of α (the rotation angle of γ around its axis).

3.1. **Pre-good curves.** Later in the section, we give a brief discussion of the construction of surfaces due to Kahn and Wright in [KW18]. However, we first give a lemma which proves the existence of certain well-behaved geodesics whose n^{th} powers will become part of the Kahn–Wright surface. See [KW18, §3] for the definition of height in the following statement. The following is an analogue in the finite-volume case of Sun's [Sun15, Lemma 2.9]. In order to use this geodesic in Kahn and Wright's construction, it is important to control the height.

Lemma 3.1. *For $n \in \mathbb{N}, \epsilon > 0$, $h > 0$, there exists R_0 so that for all $R > R_0$ there exists a geodesic α_0 in N of height at most h such that $\left|\ell(\alpha_0) - \frac{2R}{n}\right| < \frac{\epsilon}{n}$ and $\left|\theta(\alpha_0) - \frac{2\pi}{n}\right| < \frac{\epsilon}{n}$.*

Proof. For a closed subset Ω of $\mathrm{SO}(2)$ and $T > 0$, let

$$\mathcal{G}(T, \Omega) = \{\alpha : \alpha \text{ is a closed geodesic in } N, \ell(\alpha) \leq T, \theta(\alpha) \in \Omega\}.$$

As noted in [KW18, §3.1], an application of the Margulis argument shows that

$$\#\mathcal{G}(T, \Omega) \sim \frac{e^{2T}}{2T}\|\Omega\| \text{ as } T \to \infty \tag{1}$$

which in this case follows, for example, from [MMO14, Theorem 1.1] by setting $\varphi := 1_\Omega$ the indicator function on $\mathrm{SO}(2)$ (see also [GW80]).

Considering geodesics $\alpha \in \mathcal{G}(2R/n + \epsilon/n, \Omega) \setminus \mathcal{G}(2R/n - \epsilon/n, \Omega)$ where Ω is the interval $(\frac{2\pi}{n} - \frac{\epsilon}{n}, \frac{2\pi}{n} + \frac{\epsilon}{n})$, we have

$$\#\left\{\alpha : \left|\ell(\alpha_0) - \frac{2R}{n}\right| < \frac{\epsilon}{n} \text{ and } \left|\theta(\alpha_0) - \frac{2\pi}{n}\right| < \frac{\epsilon}{n}\right\} \sim c_\epsilon \frac{e^{4R}}{4R}. \tag{2}$$

The arguments in the proof of [KW18, Lemma 3.1] apply to show that as R grows, the proportion of those α with height larger than h shrinks. In particular, for sufficiently large R one can find α_0 as needed. □

Note that α_0 may be chosen to be primitive. In the language of Kahn and Wright, α_0^n is an (R, ϵ)*–good curve*.

Definition 3.2. *Fix $n \in \mathbb{N}$, and also R, ϵ. An (R, ϵ, n)–pre-good curve in N is a geodesic α_0 satisfying the conclusion of Lemma 3.1 for some $h \in (0, 1)$.*

We remark that Kahn and Wright allow curves to have height at most $50\log(R)$ before needing to be "cut-off". We assume that $R > e$ so certainly curves of height less than 1 are fine. Lemma 3.1 asserts that for fixed n and ϵ, for large enough R there exists an (R, ϵ, n)–pre-good curve (in fact there are many).

3.2. **The construction of Kahn and Wright.** In [KW18], Kahn and Wright build certain quasi-Fuchsian immersed surfaces in N out of pieces called *good pants* and *umbrellas*. In turn, the umbrellas are assembled out of *good hamster wheels*. Each good pant and good hamster wheel is immersed in N, and has geodesic boundary components, which are referred to as *cuffs*.

The construction in [KW18] depends on choices of parameters R (sufficiently large) and $\epsilon > 0$ (sufficiently small). We postpone for now the choice of these parameters to discuss the construction. Kahn and Wright also specify another pair of parameters called *cutoff heights*, and the purpose of Lemma 3.1 above is to ensure that we can find an α_0 whose height stays below the cutoff heights, and whose n^{th} power is a good curve.

Suppose that we find a curve α_0 as in Lemma 3.1 so that $\alpha = \alpha_0^n$ is an (R, ϵ)–good curve, and so that the height of α_0 (and hence α) is at most 1. Then Kahn and Wright build a surface S out of good pants and good hamster wheels, and α appears as a cuff on at least two (in fact many) of these pieces.

Consider $\alpha_0\colon\ \mathbb{S}^1 \to N$ as a map from the circle to N parametrized proportional to arc length, and let $u = \alpha_0(1)$. Let $\phi_n\colon\ \mathbb{S}^1 \to \mathbb{S}^1$ be the connected n–fold covering map, and let $\alpha\colon\ \mathbb{S}^1 \to N$ be the composition $\alpha_0 \circ \phi_n$. Suppose $f\colon\ S_0 \looparrowright N$ is a Kahn–Wright surface and that there is a map $c\colon\ \mathbb{S}^1 \to S_0$ so $\alpha = f \circ c$. Let $\phi_n^{-1}(u) = \{u_1, \ldots, u_n\}$ and note that there are n different points $\{x_1, \ldots, x_n\}$ on S_0 so $f(x_i) = \alpha(u_i)$ for each i.

Choose a basepoint $y \in \mathbb{H}^3$ and let $\pi\colon\ (\mathbb{H}^3, y) \to (N, u)$ be the based universal covering map. Fix a basepoint $z \in \mathbb{H}^2$, and for each i, let $\tau_i\colon\ (\mathbb{H}^2, z) \to (S_0, x_i)$ be a based universal cover.

The map f elevates to n distinct (based) maps:

$$\widetilde{f}_i\colon\ (\mathbb{H}^2, z) \to (\mathbb{H}^3, y)$$

so that for each i we have $\widetilde{f}_i \circ \pi = \tau_i \circ f$.

Let $H_R = \{z \in \mathbb{H}^2 \mid \mathrm{Re(z)} \geq 0\}$ and $H_L = \{z \in \mathbb{H}^2 \mid \mathrm{Re(z)} \leq 0\}$, and let $m = \{z \in \mathbb{H}^2 \mid \mathrm{Re(z)} = 0\} = \mathrm{H_R} \cap \mathrm{H_L}$.

Now, for a pair $i \neq j$ from $\{1, \ldots, n\}$ we have $\widetilde{f}_i(m) = \widetilde{f}_j(m)$. Thus, we can take the two maps $\widetilde{f}_i|_{H_R}$ and $\widetilde{f}_j|_{H_R}$ and glue them together via an orientation-preserving isometry along the boundary to get a continuous map $\widetilde{f}_{i,f}^{H_R} \colon \mathbb{H}^2 \to \mathbb{H}^3$, and similarly for the two maps restricted to H_L to get a continuous map $\widetilde{f}_{i,j}^{H_L}$.

Kahn and Wright prove that for appropriate choices of parameters, their surface, built as a *good assembly* of pants and hamster wheels, is close to a *perfect assembly*, and that the map which takes the good assembly to the perfect assembly is *compliant* (see [KW18, §A.5]), which in particular means that it takes cuffs to cuffs. For a perfect assembly with cuff α, the construction analogous to the $\widetilde{f}_{i,j}$ leads to pairs of totally geodesic half-planes glued along their boundary geodesic, namely to a map p_θ for some θ. Thus, the map that takes the good assembly to the perfect assembly induces a map between $\widetilde{f}_{i,j}^{H_R} \colon \mathbb{H}^2 \to \mathbb{H}^3$ and some map $p_\theta \colon \mathbb{H}^2 \to \mathbb{H}^3$, and this map takes m to the pleating locus for p_θ.

Our first task is to bound θ away from 0, and our second is to show that the two maps are close. The sense in which they are close will be that of [KW18, p.51] – being of ϵ_0*–bounded distortion to distance* D for appropriate choice of ϵ_0 and D.

Denote the angles of the maps p_θ induced by i, j and H_R by $\theta(i, j, H_R)$, and for i, j and H_L by $\theta(i, j, H_L)$.

The following is a summary of the above discussion, and also of [KW18, Theorem A.18]. Note that it follows from the proof of [KW18, Theorem A.18] that the chosen maps which we denote by $g_{i,j}^{H_R}$ and $g_{i,j}^{H_K}$ in the following statement are compliant. In the following statement, R_0 is the constant from the statement of Lemma 3.1.

Theorem 3.3. *Fix $n \in \mathbb{N}$. For all D there exist C, ϵ_0 and $R_1 > R_0$ so that for all $\epsilon \in (0, \epsilon_0)$ and all $R > R_1$ and any (n, R, ϵ)–pre-good curve α_0 there exists a Kahn-Wright surface $f \colon S_0 \looparrowright N$ containing $\alpha = \alpha_0^n$ as a cuff, constructed as an (R, ϵ)–good assembly.*

For each $i, j \in \{1, \ldots, n\}$ with $i \neq j$, there are maps $g_{i,j}^{H_R}, g_{i,j}^{H_L} \colon \mathbb{H}^3 \to \mathbb{H}^3$ so that $\widetilde{f}_{i,j}^{H_R} \circ g_{i,j}^{H_R} = p_{\theta(i,j,H_R)}$ and $\widetilde{f}_{i,j}^{H_L} \circ g_{i,j}^{H_L} = p_{\theta(i,j,H_L)}$. Moreover, we have:

$$\theta(i, j, H_R), \theta(i, j, H_L) \in \left(\frac{\pi}{n}, \pi\right),$$

and $g_{i,j}^{H_R}$ is a D–local $(1 + C\epsilon, C\epsilon)$–quasi-isometry.

The following is an easy consequence of Theorem 3.3 and Lemma 2.3. In the following statement R_0 is the constant from Lemma 3.1 and R_1 is the constant from Theorem 3.3.

Corollary 3.4. *Fix $n \in \mathbb{N}$. There exist λ, κ so that for any D there exist $R_D > R_1, R_0$ and $\epsilon_D > 0$, so that for any α_0 and $f\colon S_0 \looparrowright N$ as in Theorem 3.3 with $R > R_D$ and any $\epsilon \in (0, \epsilon_D)$ the maps $\widetilde{f}_{i,j}^{H_R}$ and $\widetilde{f}_{i,j}^{H_L}$ are D–local (λ, κ)–quasi-isometric embedding.*

Now, choose D, λ_1, κ_1 so that any D–local (λ, κ)–quasi-isometric embedding from $\mathbb{H}^2$ to $\mathbb{H}^3$ is a global (λ_1, κ_1)–quasi-isometric embedding (see Proposition 2.2). This D then gives R_D and ϵ_D as above.

Lemma 3.1 proves that there is an (n, R_D, ϵ_D)–pre-good curve α_0, and the construction from [KW18] proves that there is an $f\colon S_0 \to N$ with $\alpha = \alpha_0^n$ as a cuff satisfying the conclusions of Theorem 3.3 and Corollary 3.4, with $R = R_D$.

We fix this map $f\colon S_0 \looparrowright N$, along with n, D, R_D, ϵ_D, α_0, $\alpha = \alpha_0^n$, κ, and ϵ as chosen above for the next two sections.

4. THE SPACE X_n

By standard separability properties of surface groups, we may find a cover $S \to S_0$ to which α lifts as a non-separating simple closed curve, and so that:

1. The injectivity radius of S is at least $\max\{2D, \lambda_1\kappa_1\}$; and
2. The lift of α to S is contained in an embedded collar of width at least $\max\{2D, \lambda_1\kappa_1\}$.

Given the surface S, we build a space X_n which immerses into N, exactly as in [Sun15]. Passing from S_0 to S before constructing X_n makes the proof that X_n is π_1–injective with quasi-convex image much simpler than Sun's proof from [Sun15, §4]. Let C denote the image of α in S, and let $\phi_C^n\colon C \to \mathbb{S}^1$ be an n–to–1 covering map, and let $\tau_C\colon C \to C$ be a deck transformation. We may choose ϕ_C^n so that τ is an isometry.

Definition 4.1. *The space $X_n(S, C)$ is defined by cutting S along C to get a surface S_1 with two boundary components, denoted C_1 and C_2, and taking the quotient of S_1 by the relation generated by $c \sim \tau_{C_i}(c)$ for $c \in C_i$ and $i = 1, 2$.*

Suppose that S is equipped with a hyperbolic metric, and consider the induced metric on S_1. Since the maps τ_{C_i} are isometries, there is a natural induced quotient metric on $X_n(S, C)$, which is locally isometric to $\mathbb{H}^2$ away from the images of the C_i.

The following result is clear from the construction of S from S_0.

Lemma 4.2. *The injectivity radius of X_n is at least $\max\{2D, \lambda_1\kappa_1\}$.*

Let S_1 be the surface obtained from S by cutting along C, and let C_1, C_2 be the boundary components of S_1. Let $q\colon S_1 \to X_n$ be the defining quotient map and let $\overline{C_i} = q(C_i)$ for $i = 1, 2$.

Because in S the curve C has an embedded collar of width at least $2D$, for any $i, j \in \{1, 2\}$, any two distinct elevations of $\overline{C_i}$ and $\overline{C_j}$ to $\widetilde{X_n}$ are at distance at least $4D$ from each other.

Definition 4.3. *Suppose that $\mathcal{A} = \{Z_1, \ldots, Z_m\}$ is a finite collection of metric spaces and that $k > 0$. A metric space Z is k–*modeled on $\mathcal{A}$ *if for every $z \in Z$ there is an i so that the ball of radius k about z is isometric to a ball in Z_i.*

Recall $H_R = \{z \mid \mathrm{Re}(z) \geq 0\}$ is the (closed) half-hyperbolic plane (in the upper half-space model). Let W_n be the space obtained from n copies of H_R glued along the boundary geodesics (by an isometry).

Lemma 4.4. *The space $\widetilde{X_n}$ is D–modeled on W_n.*

Proof. Let $x \in \widetilde{X_n}$ and consider the covering map $\pi\colon \widetilde{X_n} \to X_n$.

Case 1: $d(\pi(x), \{\overline{C_1}, \overline{C_2}\}) \leq D$.

In this case, in $\widetilde{X_n}$ there is a unique elevation of some $\overline{C_i}$ which lies within D of x. Let y be a point in this elevation so $d(x, y) \leq D$. Then $B_D(x) \subseteq B_{2D}(y)$, and $B_{2D}(y)$ is isometric to a ball of radius $2D$ in W_n.

Case 2: $d(\pi(x), \{\overline{C_1}, \overline{C_2}\}) > D$.

In this case there is no elevation of either $\overline{C_i}$ which lies within D of x, and $B_D(x)$ is isometric to a ball of radius D in $\mathbb{H}^2$ (and so in W_n). □

By construction the immersion $f_1\colon S \to N$ obtained from composing the covering map $S \to S_0$ with $f\colon S_0 \looparrowright N$ yields an immersion $g\colon X_n \looparrowright N$. Let $\widetilde{g}\colon \widetilde{X_n} \to \mathbb{H}^3$ be the induced map on universal covers.

Two points x, y in $\widetilde{X_n}$ at distance at most D either lie in an isometrically embedded copy of a half-space from $\mathbb{H}^2$, or else in two different "sheets" of a copy of W_n. In either case, it follows immediately from Corollary 3.4 that $\widetilde{g}$ is a D–local (λ, κ)–quasi-isometric embedding. Thus,

Theorem 4.5. *The map $\widetilde{g}\colon \widetilde{X_n} \to \mathbb{H}^3$ is a D–local (λ, κ)–quasi-isometric embedding, and hence is a (global) (λ_1, κ_1)–quasi-isometric embedding.*

In particular, since the injectivity radius of X_n is at least $\lambda_1\kappa_1$ the map g is π_1–injective, and $g_*(\pi_1(X_n))$ is relatively quasi-convex in $\pi_1(N)$. Moreover, $g_*(\pi_1(X_n))$ does not intersect any (conjugate of) the cusp subgroups of $\pi_1(N)$.

5. Virtual retractions and the proof of Theorem 1.1 in the hyperbolic case

In this section we prove Theorem 1.1 in case of a finite-volume hyperbolic 3–manifold N. Let $\Gamma = \pi_1(N)$. By [Ago13, Theorem 1.1] and [Wis, Theorem 14.29] $\pi_1(N_0)$ is the fundamental group of a virtually special cube complex X. Let $\Gamma_1 \leq \Gamma$ be a finite-index subgroup so that the cover of X corresponding to Γ_1 is special, and let N_1 be the cover of N corresponding to Γ_1. As in Section 4, construct an immersion $g\colon X_n \to N_1$. Note that $(\Gamma_1, \mathcal{P})$ is relatively hyperbolic where $\mathcal{P}$ consists of the (abelian) cusp subgroups.

Let $H = g_*(\pi_1(X_n)) \leq \pi_1(N_1) = \Gamma_1$. The subgroup H is relatively quasi-convex in Γ_1, and so by [HW08, Corollary 6.7] (with the formulation as in [PW18, Theorem 6.3]) we have that H is a virtual retract of Γ_1. Let Γ_2 be a finite-index subgroup of Γ which retracts onto H. Let N_2 be the finite cover of N_1 corresponding to Γ_2. As in [Sun15, Proposition 3.7], we have the induced maps on homology

$$H_1(X_n;\mathbb{Z}) \xrightarrow{g_*} H_1(N_2;\mathbb{Z}) \xrightarrow{r_*} H_1(X_n;\mathbb{Z}).$$

Therefore, since $r \circ g_* = id_H$, $H_1(X_n;\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}^k$ is a direct factor of $H_1(N_2;\mathbb{Z})$. In particular, $\mathbb{Z}/n\mathbb{Z}$ is a direct factor of $H_1(N_2;\mathbb{Z})$ and this proves the hyperbolic case of Theorem 1.1 in the case that A is finite cyclic.

Given a finite abelian group A, induction on the rank k of A also works as in [Sun15, Proposition 3.9] as follows. Let $A = \oplus_{i=1}^{k-1}\mathbb{Z}/n_i\mathbb{Z}$ and $A' = A \oplus \mathbb{Z}/n_k\mathbb{Z}$. Suppose by induction that $H \leq \Gamma_1$ is a relatively quasi-convex free product of images of $\pi_1(X_{n_i})$ (for $i = 1, \ldots, k-1$) and that $H' = (g_k)_*(\pi_1(X_{n_k})) \leq \Gamma_1$. Choose any $\gamma \in \Gamma_1$ whose fixed points in $\partial\mathbb{H}^3$ are disjoint from both limit sets $\Lambda(H)$ and $\Lambda(H')$. Then after conjugating H' by some sufficiently high power γ^m, the first Klein–Maskit combination theorem [Mas93] applies (note that by [Hru10, Corollary 1.3] a subgroup is relatively quasi-convex if and only if it is geometrically finite). Since H' is relatively quasi-convex, the free product $H * gH'g^{-1}$ is also a relatively quasi-convex subgroup of Γ_1 isomorphic to the abstract group $H * H'$. The proof now follows exactly as above, completing the proof of Theorem 1.1 in the finite-volume hyperbolic case.

6. NON-HYPERBOLIC MANIFOLDS

We now prove Theorem 1.1 in general. To that end, suppose that M is an irreducible 3–manifold which is not a graph manifold, and that A is a finite abelian group. By [PW18, Theorem 1.1] there exists a $\mathrm{CAT}(0)$ cube complex X equipped with a free $\pi_1(M)$–action so that there are finitely many orbits of hyperplanes and so that $\pi_1(M)\backslash X$ has a finite special cover. Let $\Gamma_1 \leq \pi(M)$ be a finite-index subgroup corresponding to the finite special cover of $\pi_1(M)\backslash X$, and let M_1 be the finite cover of M corresponding to Γ_1. Let M_h be a hyperbolic piece in the geometric decomposition of M_1 and let $\Gamma_h := \pi_1(M_h) \leq \pi_1(M_1)$ (basepoints/conjugacy classes are not important here). According to the construction in the previous sections, there is a relatively quasi-convex subgroup H of Γ_h so that A is a direct factor of $H^1(H;\mathbb{Z})$. Theorem 1.1 follows immediately from the following result. This result is presumably known to the experts, but we were unable to find it in the literature.

Proposition 6.1. *There is a finite-index subgroup $\Gamma_2 \leq \Gamma_1$ so that $H \leq \Gamma_h$ and H is a retract of Γ_2.*

Proof. Let $\widetilde{M_h} \leq \widetilde{M}$ be the (Γ_h–invariant) universal cover of M_h inside the universal cover of M. The space X is built via a *wallspace* construction on $\widetilde{M}$ as in [HW14]. As in [HW14, §2.9], we can associate to $\widetilde{M_h}$ a *hemiwallspace* consisting of those half-spaces in $\widetilde{M}$ which intersect $\widetilde{M_h}$. This builds a Γ_h–invariant convex sub-complex X_h of X by [HW14, Lemma 2.29].

By [PW18, Theorem 2.1], the surfaces of the cubulation intersecting M_h all intersect M_h in a geometrically finite surface. Therefore, by [HW14, Theorem 6.12] the Γ_h–action on X_h is (free and) *co-sparse* (see [SW15, Definition 7.1]).

According to [SW15, Theorem 7.2] inside of X_h there is an H–invariant convex sub-complex Z upon which H acts co-sparsely. In fact, since H does not intersect any of the parabolic subgroups of Γ_h, the sub-complex Z found in [SW15, Theorem 7.2] is H–cocompact (this follow immediately from the proof).

Since $\Gamma_1\backslash X$ is special, it follows from [HW08, Corollary 6.7] (we use the formulation as in [PW18, Theorem 6.3]) that H is a virtual retract of Γ_1. □

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