HOMEWORK #10
SOLUTIONS

(1) Prove that the function $f(x) = x^3$ is (Riemann) integrable on $[0,1]$ and show that

$$\int_0^1 x^3 dx = \frac{1}{4}.$$

(Without using formulae for integration that you learnt in previous calculus classes...)

You may use the identity $\sum_{i=1}^{n} i^3 = \frac{1}{4}(n^4 + 2n^3 + n^2)$.

Solution:

Let $n \in \mathbb{N}$ and define the dissection

$$\mathcal{D}_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}.$$

The function $f(x) = x^3$ is increasing between 0 and 1. Therefore the supremum of the values on an interval (x_{i-1}, x_i) is $f(x_i) = x_i^3$, and the infimum is $f(x_{i-1}) = x_{i-1}^3$. Thus we can calculate the lower and upper sums of f with respect to $\mathcal{D}_n$:

$$\begin{aligned}
\mathcal{L}(f, \mathcal{D}_n) &= \sum_{i=1}^{n} (\frac{i-1}{n})^3 . \frac{1}{n} \\
&= \frac{1}{n^4} \sum_{i=1}^{n} (i-1)^3 \\
&= \frac{1}{4n^4}(n^4 + 2n^3 + n^2) - \frac{n^3}{n^4} \\
&= \frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2}
\end{aligned}$$

A similar calculation gives

$$\mathcal{U}(f, \mathcal{D}_n) = \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}.$$

As $n \to \infty$ we have $\mathcal{L}(f, \mathcal{D}_n) \to \frac{1}{4}$ and also $\mathcal{U}(f, \mathcal{D}_n) \to \frac{1}{4}$. From this it follows that f is integrable on $[0,1]$ and that

$$\int_0^1 x^3 dx = \frac{1}{4},$$

as required.

(2) Suppose that $g(x)$ is a continuous function on an interval $[a, b]$ such that $g(x) > 0$ for all x. Show that

$$\int_a^b g(x)dx > 0.$$

Solution

Since $g(x) \neq 0$ on $[a, b]$ the function $\frac{1}{g}$ is defined and continuous on $[a, b]$. Hence there is $M > 0$ so that $\frac{1}{g(x)} < M$ for all x. This means that $g(x) > \frac{1}{M} > 0$ for all x in $[a, b]$. Let $\mathcal{D} = \{a, b\}$. Then $\mathcal{L}(f, \mathcal{D}) \leq \int_a^b f$. However,

$$\mathcal{L}(f, \mathcal{D}) > \frac{1}{M}|b - a| > 0,$$

so we're done.

(3) Let $f : [1, 3] \to \mathbb{R}$ be defined by:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 2 \\ 0 & \text{if } x \in (2, 3] \cap \mathbb{Q} \\ 1 & \text{if } x \in (2, 3] \setminus \mathbb{Q} \end{cases}$$

Prove that f is not Riemann integrable.

Solution:

f is integrable on $[1, 3]$ if and only if it is integrable on $[1, 2]$ and also on $[2, 3]$. Thus, it is enough to show that f is not integrable on $[2, 3]$.

Well, for any dissection $\mathcal{D} = \{2 = x_0, \ldots, x_n = 3\}$ of $[2, 3]$ we have

$$\begin{aligned} \sup\{f(x) \mid x \in (x_{i-1}, x_i)\} &= 1, \text{ and} \\ \inf\{f(x) \mid x \in (x_{i-1}, x_i)\} &= 0. \end{aligned}$$

Therefore

$$\mathcal{L}(f, \mathcal{D}) = 0,$$

and

$$\underline{\int_2^3} f = 0.$$

Similarly,

$$\mathcal{U}(f, \mathcal{D}) = 1,$$

and

$$\overline{\int_2^3} f = 1.$$

This shows that f is not integrable on $[2, 3]$ and hence it is not integrable on $[1, 3]$.

(4) Define $p : [0,2] \to \mathbb{R}$ as follows:

$$p(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

Prove that $p(x)$ is Riemann integrable on $[0,2]$ and determine

$$\int_0^2 p(x)dx.$$

Solution:

f is continuous so integrable on $[0,2]$. We have

$$\int_0^2 f = \int_0^1 f + \int_1^2 f.$$

Howie works out $\int_0^1 f = \frac{1}{2}$.

On $[1,2]$, f is identically 1, so it is easy to see that all lower and upper sums (with respect to any dissection) are equal to 1, which means that

$$\int_1^2 f = 1.$$

Therefore, $\int_0^2 f = 1\frac{1}{2}$.

(5) Suppose that $f : [a,b] \to \mathbb{R}$ is (Riemann) integrable on $[a,b]$. Prove that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right).$$

Solution: Fix $n \in \mathbb{N}$. Consider the following dissection:

$$\mathcal{D}_n = \{a, a+\frac{b-a}{2n}+\frac{b-a}{n}, a+\frac{b-a}{2n}+\frac{2(b-a)}{n}, \ldots, a+\frac{b-a}{2n}+\frac{(n-1)(b-a)}{n}, b\}$$

This is a partition of $[a,b]$ into n intervals, one of length $\frac{b-a}{2n}$, one of length $\frac{b-a}{2n} + \frac{b-a}{n}$ and $n-2$ of length $\frac{b-a}{n}$.

It is more natural to define the partition without the $+\frac{b-a}{2n}$ points, but we want the point $a + \frac{i(b-a)}{n}$ to be in the interior of the i^{th} interval, which it is.

Therefore, it the intervals are $I_{1,n}, \ldots, I_{n,n}$, and

$$\begin{aligned} m_{i,n} &= \inf\{f(x) \mid x \in I_{i,n}\} \\ M_{i,n} &= \sup\{f(x) \mid x \in I_{i,n}\}, \end{aligned}$$

from which it follows that (for $1 \le i < n$)

$$m_{i,n} \le f(a + i\frac{b-a}{n}) \le M_{i,n}.$$

Therefore, the terms of the lower and the upper sum of $\mathcal{D}_n$ match up to give an inequality with the required sum $\frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$, except for the first and lat terms where the lengths of the intervals are wrong.

The best thing to do is note that these terms are small, and going to 0 as $n \to \infty$. Therefore, if we prove that there is a number L so that

$$L = \lim_{n\to\infty} \mathcal{L}(f, \mathcal{D}_n) = \lim_{n\to\infty} \mathcal{U}(f, \mathcal{D}_n) \tag{1}$$

then we'll see that $L = \int_a^b f$ and that $L = \lim_{n\to\infty} \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$, which is what we are required to prove. Thus, we are left to prove Equation (1).

To do this, note that we know that f is integrable.

Let $\epsilon > 0$ be arbitrary. There is a dissection $\mathcal{D}$ of $[a, b]$ so that

$$\mathcal{U}(f, \mathcal{D}) - \mathcal{L}(f, \mathcal{D}) < \frac{\epsilon}{2}.$$

Suppose that there are k intervals in $\mathcal{D}$ and let these intervals be $J_1, \dots, J_k$. Suppose also that

$$M = \sup\{|f(x) \mid x \in [a, b]\}.$$

(We'll assume that $M > 0$, since otherwise there's almost nothing to prove.)

For any n, any given term in the sum $\frac{b-a}{n}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$ has size at most $M\frac{b-a}{n}$.

Choose $N \in \mathbb{N}$ so that $N > \frac{2kM(b-a)}{\epsilon}$. Then, if $n > N$ we consider

$$\mathcal{U}(f, \mathcal{D}_n) - \mathcal{L}(f, \mathcal{D}_n).$$

This has n terms in it, involving the difference between the supremum and the infimum of f on a given interval. All but at most k of the intervals $I_{i,n}$ are entirely contained within some interval J_s. Since the intervals J_s are bigger, the difference between the supremum and infimum of f on the interval J_s can only be bigger than on the corresponding $I_{i,n}$.

The choice of n guarantees that each of the other terms in the difference (those corresponding to intervals not entirely contained in a J_s) have size at most $\frac{\epsilon}{2k}$, and there are at most k of these terms. Thus, we split the sum

$$\mathcal{U}(f, \mathcal{D}_n) - \mathcal{L}(f, \mathcal{D}_n)$$

into two sets of terms. The first set of terms has total size no bigger than

$$\mathcal{U}(f, \mathcal{D}) - \mathcal{L}(f, \mathcal{D}) < \frac{\epsilon}{2},$$

while the second set of terms has size at most

$$k.\frac{\epsilon}{2k} = \frac{\epsilon}{2}.$$

Thus, putting these together we see that

$$\mathcal{U}(f, \mathcal{D}_n) - \mathcal{L}(f, \mathcal{D}_n) < \epsilon$$

which proves Equation (1), and finishes the proof.