(1) Prove that the function $f(x) = x^3$ is (Riemann) integrable on $[0, 1]$ and show that

$$\int_0^1 x^3 \, dx = \frac{1}{4}.$$  

(Without using formulae for integration that you learnt in previous calculus classes...)

You may use the identity $\sum_{i=1}^n i^3 = \frac{1}{4}(n^4 + 2n^3 + n^2)$.

Solution:

Let $n \in \mathbb{N}$ and define the dissection

$$D_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}.$$

The function $f(x) = x^3$ is increasing between 0 and 1. Therefore the supremum of the values on an interval $(x_{i-1}, x_i)$ is $f(x_i) = x_i^3$, and the infimum is $f(x_{i-1}) = x_{i-1}^3$. Thus we can calculate the lower and upper sums of $f$ with respect to $D_n$:

$$L(f, D_n) = \sum_{i=1}^n (\frac{i-1}{n})^3 \cdot \frac{1}{n} = \frac{1}{n^4} \sum_{i=1}^n (i-1)^3 = \frac{1}{4n^4}(n^4 + 2n^3 + n^2) - \frac{n^3}{n^2} = \frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2}.$$

A similar calculation gives

$$U(f, D_n) = \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}.$$

As $n \to \infty$ we have $L(f, D_n) \to \frac{1}{4}$ and also $U(f, D_n) \to \frac{1}{4}$. From this it follows that $f$ is integrable on $[0, 1]$ and that

$$\int_0^1 x^3 \, dx = \frac{1}{4},$$

as required.
(2) Suppose that $g(x)$ is a continuous function on an interval $[a, b]$ such that $g(x) > 0$ for all $x$. Show that

$$\int_a^b g(x)dx > 0.$$ 

**Solution**

Since $g(x) \neq 0$ on $[a, b]$ the function $\frac{1}{g}$ is defined and continuous on $[a, b]$. Hence there is $M > 0$ so that $\frac{1}{g(x)} < M$ for all $x$. This means that $g(x) > \frac{1}{M} > 0$ for all $x$ in $[a, b]$. Let $D = \{a, b\}$. Then $\mathcal{L}(f, D) \leq \int_a^b f$. However,

$$\mathcal{L}(f, D) > \frac{1}{M}|b - a| > 0,$$

so we’re done.

(3) Let $f : [1, 3] \to \mathbb{R}$ be defined by:

$$f(x) = \begin{cases} 
0 & \text{if } x \leq 2 \\
0 & \text{if } x \in (2, 3] \cap \mathbb{Q} \\
1 & \text{if } x \in (2, 3] \setminus \mathbb{Q}
\end{cases}$$

Prove that $f$ is not Riemann integrable.

**Solution:**

$f$ is integrable on $[1, 3]$ if and only if it is integrable on $[1, 2]$ and also on $[2, 3]$. Thus, it is enough to show that $f$ is not integrable on $[2, 3]$.

Well, for any dissection $D = \{2 = x_0, \ldots, x_n = 3\}$ of $[2, 3]$ we have

$$\sup \{f(x) \mid x \in (x_{i-1}, x_i)\} = 1, \text{ and}$$

$$\inf \{f(x) \mid x \in (x_{i-1}, x_i)\} = 0.$$

Therefore

$$\mathcal{L}(f, D) = 0,$$

and

$$\int_2^3 f = 0.$$

Similarly,

$$\mathcal{U}(f, D) = 1,$$

and

$$\int_2^3 f = 1.$$

This shows that $f$ is not integrable on $[2, 3]$ and hence it is not integrable on $[1, 3]$. 

(4) Define \( p : [0, 2] \to \mathbb{R} \) as follows:
\[
p(x) = \begin{cases} 
x, & \text{if } x \leq 1 \\
1 & \text{if } x > 1
\end{cases}
\]
Prove that \( p(x) \) is Riemann integrable on \([0, 2]\) and determine
\[
\int_0^2 p(x) \, dx.
\]
**Solution:**
\( f \) is continuous so integrable on \([0, 2]\). We have
\[
\int_0^2 f = \int_0^1 f + \int_1^2 f.
\]
Howie works out \( \int_0^1 f = \frac{1}{2} \).
On \([1, 2]\), \( f \) is identically 1, so it is easy to see that all lower and upper sums (with respect to any dissection) are equal to 1, which means that
\[
\int_1^2 f = 1.
\]
Therefore, \( \int_0^2 f = 1 + \frac{1}{2} \).

(5) Suppose that \( f : [a, b] \to \mathbb{R} \) is (Riemann) integrable on \([a, b]\). Prove that
\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^n f \left( a + \frac{i(b-a)}{n} \right).
\]
**Solution:** Fix \( n \in \mathbb{N} \). Consider the following dissection:
\[
D_n = \{ a + \frac{b-a}{2n}, a + \frac{b-a}{n}, a + \frac{b-a}{2n} + \frac{2(b-a)}{n}, \ldots, a + \frac{b-a}{2n} + \frac{(n-1)(b-a)}{n}, b \}
\]
This is a partition of \([a, b]\) into \( n \) intervals, one of length \( \frac{b-a}{2n} \), one of length \( \frac{b-a}{n} \) and \( n - 2 \) of length \( \frac{b-a}{2n} \).
It is more natural to define the partition without the \( + \frac{b-a}{2n} \) points, but we want the point \( a + \frac{i(b-a)}{n} \) to be in the interior of the \( i^{th} \) interval, which it is.
Therefore, the intervals are \( I_{1,n}, \ldots, I_{n,n} \), and
\[
m_{i,n} = \inf \{ f(x) \mid x \in I_{i,n} \}
\]
\[
M_{i,n} = \sup \{ f(x) \mid x \in I_{i,n} \},
\]
from which it follows that (for \(1 \leq i < n\))
\[
m_{i,n} \leq f(a + \frac{b - a}{n}) \leq M_{i,n}.
\]

Therefore, the terms of the lower and the upper sum of \(\mathcal{D}_n\) match up to give an inequality with the required sum \(\frac{b-a}{n} \sum_{i=1}^{n} f(a + \frac{b-a}{n})\), except for the first and last terms where the lengths of the intervals are wrong.

The best thing to do is note that these terms are small, and going to 0 as \(n \to \infty\). Therefore, if we prove that there is a number \(L\) so that
\[
L = \lim_{n \to \infty} \mathcal{L}(f, \mathcal{D}_n) = \lim_{n \to \infty} \mathcal{U}(f, \mathcal{D}_n) \tag{1}
\]
then we’ll see that \(L = \int_{a}^{b} f\) and that \(L = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + \frac{b-a}{n})\), which is what we are required to prove. Thus, we are left to prove Equation (1).

To do this, note that we know that \(f\) is integrable.

Let \(\epsilon > 0\) be arbitrary. There is a dissection \(\mathcal{D}\) of \([a, b]\) so that
\[
\mathcal{U}(f, \mathcal{D}) - \mathcal{L}(f, \mathcal{D}) < \frac{\epsilon}{2}.
\]

Suppose that there are \(k\) intervals in \(\mathcal{D}\) and let these intervals be \(J_1, \ldots, J_k\).

Suppose also that
\[
M = \sup \{|f(x)| \mid x \in [a, b]\}.
\]

(We’ll assume that \(M > 0\), since otherwise there’s almost nothing to prove.)

For any \(n\), any given term in the sum \(\frac{b-a}{n} \sum_{i=1}^{n} f(a + \frac{b-a}{n})\) has size at most \(M \frac{b-a}{n}\).

Choose \(N \in \mathbb{N}\) so that \(N > \frac{2kM(b-a)}{\epsilon}\). Then, if \(n > N\) we consider
\[
\mathcal{U}(f, \mathcal{D}_n) - \mathcal{L}(f, \mathcal{D}_n).
\]

This has \(n\) terms in it, involving the difference between the supremum and the infimum of \(f\) on a given interval. All but at most \(k\) of the intervals \(I_{i,n}\) are entirely contained within some interval \(J_s\). Since the intervals \(J_s\) are bigger, the difference between the supremum and infimum of \(f\) on the interval \(J_s\) can only be bigger than on the corresponding \(I_{i,n}\).

The choice of \(n\) guarantees that each of the other terms in the difference (those corresponding to intervals not entirely contained in a \(J_s\)) have size at most \(\frac{\epsilon}{2k}\), and there are at most \(k\) of these terms. Thus, we split the sum
\[
\mathcal{U}(f, \mathcal{D}_n) - \mathcal{L}(f, \mathcal{D}_n)
\]
into two sets of terms. The first set of terms has total size no bigger than
\[
\mathcal{U}(f, \mathcal{D}) - \mathcal{L}(f, \mathcal{D}) < \frac{\epsilon}{2},
\]
while the second set of terms has size at most

\[ \frac{\epsilon}{2k} = \frac{\epsilon}{2}. \]

Thus, putting these together we see that

\[ \mathcal{U}(f, \mathcal{D}_n) - \mathcal{L}(f, \mathcal{D}_n) < \epsilon \]

which proves Equation (1), and finishes the proof.