HOMEWORK #11
SOLUTIONS

(1) Suppose that $f, g : [a, b] \to \mathbb{R}$, and suppose that f is integrable on $[a, b]$. Suppose also that there are finitely many points $c_1, \ldots, c_k \in [a, b]$ so that for all $y \in [a, b] \smallsetminus \{c_1, \ldots, c_k\}$ we have $f(y) = g(y)$. Prove that g is integrable on $[a, b]$ and that

$$\int_a^b f = \int_a^b g.$$

Solution:

Let $\epsilon > 0$ be arbitrary, and choose $n \in \mathbb{N}$ so that $n > \frac{1}{\epsilon}$. There is a dissection $\mathcal{D}_n$ of $[a, b]$ so that

$$\mathcal{U}(f, \mathcal{D}_n) - \mathcal{U}(f, \mathcal{D}_n) < \epsilon.$$

Define a new dissection $\mathcal{D}'_n = \mathcal{D}_n \cup \{c_1, \ldots, c_k\}$. Since $\mathcal{D}'_n$ is a refinement of $\mathcal{D}$, we have

$$\mathcal{U}(f, \mathcal{D}'_n) \leq \mathcal{U}(f, \mathcal{D}_n)$$

and

$$\mathcal{L}(f, \mathcal{D}'_n) \geq \mathcal{L}(f, \mathcal{D}_n).$$

This implies that

$$\mathcal{U}(f, \mathcal{D}'_n) - \mathcal{L}(f, \mathcal{D}'_n) < \frac{1}{n} < \epsilon.$$

This proves that $\lim_{n\to\infty} \mathcal{L}(f, \mathcal{D}'_n) = \lim_{n\to\infty} \mathcal{U}(f, \mathcal{D}'_n) = \int_a^b f$.

Suppose that $\mathcal{D}' = \{x_1, \ldots, x_n\}$ and define

$$m_i = \inf\{g(x) \mid x \in (x_{i-1}, x_i)\}, M_i = \sup\{g(x) \mid x \in (x_{i-1}, x_i)\}.$$

Since there are none of the c_j in the interval (x_{i-1}, x_i), we have

$$m_i = \inf\{f(x) \mid x \in (x_{i-1}, x_i)\}, M_i = \sup\{f(x) \mid x \in (x_{i-1}, x_i)\}.$$

Therefore, $\mathcal{L}(g, \mathcal{D}'_n) = \mathcal{L}(f, \mathcal{D}'_n)$ and $\mathcal{U}(g, \mathcal{D}'_n) = \mathcal{U}(f, \mathcal{D}'_n)$.

This implies that

$$\begin{aligned} \lim_{n\to\infty} \mathcal{L}(g, \mathcal{D}'_n) &= \lim_{n\to\infty} \mathcal{U}(g, \mathcal{D}'_n) \\ &= \lim_{n\to\infty} \mathcal{U}(f, \mathcal{D}'_n) \\ &= \int_a^b f \end{aligned}$$

The first equation proves that g is integrable on $[a,b]$, whilst the final equation proves that

$$\int_a^b g = \int_a^b f,$$

as required.

(2) Use the Riemann integral to evaluate the following limit (where $p = \frac{1}{2}$ and where $p \in \mathbb{N}$ is arbitrary):

$$\lim_{n\to\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}}.$$

Solution:

Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x^p$.

For $n \in \mathbb{N}$, define a dissection $\mathbb{D}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$. Then, since f is an increasing function on $[0,1]$ we have

$$\inf\{f(x) \mid x \in (\frac{i-1}{n}, \frac{i}{n})\} = f(\frac{i-1}{n}) = \frac{(i-1)^p}{n^p},$$

and

$$\sup\{f(x) \mid x \in (\frac{i-1}{n}, \frac{i}{n})\} = f(\frac{i}{n}) = \frac{i^p}{n^p}.$$

Therefore,

$$\mathcal{L}(f, \mathcal{D}_n) = \sum_{i=1}^{n} \frac{1}{n} \frac{(i-1)^p}{n^p},$$

and

$$\begin{aligned} \mathcal{U}(f, \mathcal{D}_n) &= \sum_{i=1}^{n} \frac{1}{n} \frac{i^p}{n^p} \\ &= \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}} \end{aligned}$$

We have

$$\mathcal{U}(f, \mathcal{D}_n) - \mathcal{L}(f, \mathcal{D}_n) = \frac{1}{n}.$$

Thus, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}} &= \lim_{n\to\infty} \mathcal{U}(f, \mathcal{D}_n) \\ &= \lim_{n\to\infty} \mathcal{L}(f, \mathcal{D}_n) \\ &= \int_0^1 f. \end{aligned}$$

But, as we know,

$$\int_0^1 f = \int_0^1 x^p dx = \frac{1}{p+1} 1^{p+1} - \frac{1}{p+1} 0^{p+1} = \frac{1}{p+1}$$

(3) For $n \in \mathbb{N}$, define a function $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \frac{nx^{n-1}}{1+x}.$$

(a) Show that f_n is integrable on $[0,1]$.

(b) Prove that for all $y \in (0,1)$ we have $\lim\limits_{n\to\infty} f_n(y) = 0$.

(c) Prove that $\lim\limits_{n\to\infty} \int_0^1 f_n \neq 0$.

Solution:

(a): We know that $f_n(x)$ is continuous on all of $[0,1]$ since the denominator is never 0. Therefore, it is integrable on $[0,1]$.

(b): Suppose that $y \in (0,1)$. It is sufficient to prove that $\lim_{n\to\infty} ny^{n-1} = 0$. We'll consider consecutive terms of the sequence. Suppose that $n > \frac{y}{1-y}$, then

$$\begin{aligned}
\frac{(n+1)y^n}{ny^{n-1}} &= \frac{n+1}{n}y \\
&= (1+\frac{1}{n})y \\
&< (1+\frac{1-y}{y})y \\
&= 1.
\end{aligned}$$

Therefore, by the Ratio Test, we have $\lim_{n\to\infty} ny^{n-1} = 0$, which we already noted implies that $\lim_{n\to\infty} f_n(y) = 0$.

(c): It's probably possible to work out how to integrate this function directly, but we don't need to.

Note that for all $x \in [0,1]$ we have $\frac{1}{1+x} \geq \frac{1}{2}$. Therefore, for all $x \in [0,1]$ we have

$$\frac{nx^{n-1}}{1+x} \geq \frac{nx^{n-1}}{2}.$$

Therefore, we have (by Theorem 5.14 of Howie)

$$\int_0^1 f \geq \int_0^1 \frac{nx^{n-1}}{2} dx = \frac{1}{2}.$$

(4) For $n \in \mathbb{N}$, let $f_n : [a,b] \to \mathbb{R}$ be functions integrable on $[a,b]$. Suppose that there is a (bounded) function $f : [a,b] \to \mathbb{R}$ so that f is integrable on $[a,b]$ and so that for all $x \in [a,b]$ we have

$$\lim_{n\to\infty} \left[\sup_{x\in[a,b]} \{|f_n(x) - f(x)|\} \right] = 0.$$

Prove that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Solution: (Sorry about the missing |.| in the posted questions, of course the question is false without them. But with them...)

[Of course, this follows from Theorem 7.6 from Howie, but it's easier than this.]

Let $\epsilon > 0$ be arbitrary. We want to show that there is $N \in \mathbb{N}$ so that for all $n \geq N$ we have

$$|\int_a^b f_n - \int_a^b f| < \epsilon.$$

Let N be such that for all $n \geq N$ we have

$$\sup\{|f_n(x) - f(x)| \mid x \in [a,b]\} < \frac{\epsilon}{b-a}.$$

Then, for $n \geq N$ we have

$$\begin{aligned} |\int_a^b f_n - \int_a^b f| &\leq \int_a^b |f_n - f| \\ &\leq \int_a^b \frac{\epsilon}{b-a} = \epsilon. \end{aligned}$$

This proves that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f,$$

as required.