Math 313 - Analysis I **Spring 2009**

HOMEWORK #2
Model Solutions

(1) Give the definition of the real numbers $\mathbb{R}$ in terms of *Dedekind cuts*, and define the inclusion map from the rational numbers to the real numbers, $f : \mathbb{Q} \to \mathbb{R}$. (That is, if a is a rational number, show how to identify a with a particular Dedekind cut.)

Solution: Let $\mathbb{Q}$ be the set of rational numbers. A *Dedekind cut* is a pair (A, B) of subsets of $\mathbb{Q}$ satisfying:

(i) $A \cup B = \mathbb{Q}$, $A \cap B = \emptyset$;

(ii) If $x, y \in Q$, $x < y$ and $y \in A$ then $x \in A$. Conversely, if $x, y \in \mathbb{Q}$, $x < y$ and $x \in B$ then $y \in B$.

(iii) A has no greatest member (that is to say there is no $a \in A$ so that for all $x \in A$ we have $x \leq a$).

We define the real numbers, as a set, to be the set of all Dedekind cuts.

The inclusion map $f : \mathbb{Q} \to \mathbb{R}$ is defined by $f(a) = (\{x \in \mathbb{Q} \mid x < a\}, \{y \in \mathbb{Q} \mid y \geq a\})$.

NOTE: If (A, B) is Dedekind cut then condition (i) above ensures that $B = \mathbb{Q} \setminus A$. Therefore, a Dedekind cut is determined by the first element of a pair.

For the remainder of the solutions, we'll just refer to a Dedekind cut as a set A which satisfies the following two conditions:

(a) A is nonempty, and has no greatest element; and

(b) If $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$ satisfies $y < x$ then $y \in \mathbb{Q}$.

We say that A *determines a Dedekind cut*, and hence also that A determines a real number.

Aside (not required for solution to the problem): A real number α, determined by a set A, is *irrational* if $\mathbb{Q} \setminus A$ does not have a least element, and rational exactly when $\mathbb{Q} \setminus A$ *does* have a least element.

(2) Define the order '$<$' on Dedekind cuts. You do not need to prove anything about this order.

Solution: Let the sets A_1 and A_2 determine Dedekind cuts, and hence real numbers α_1 and α_2 respectively. We define $\alpha_1 < \alpha_2$ to mean $A_1 \subset A_2$.

(3) Define the negative of a Dedekind cut, and also addition and multiplication of Dedekind cuts. (Remember to define multiplication $a \times b$ when one or both of a and b is negative.)

Solution: If A determines a Dedekind cut, and hence a real number α, then the real number $-\alpha$ is the Dedekind cut determined by the set

$$\{x \in \mathbb{Q} \mid -x \notin A, \text{ and } -x \text{ is not the least element of } \mathbb{Q} \setminus A\}.$$

(NOTE: It is not necessarily the case that $\mathbb{Q} \setminus A$ *has* a least element. If it doesn't, then the condition that $-x$ is not the least element of $\mathbb{Q} \setminus A$ is trivially true.)

If A determines a real number α and B determines a real number β then we define $\alpha + \beta$ to be the Dedekind cut determined by the set:

$$\{x + y \mid x \in A, y \in B\}.$$

If A determines a real number α, and B determines a real number β then we define $\alpha.\beta$ as follows:

First suppose that $\alpha \geq 0$ and $\beta \geq 0$ (this means that $0 \in A$ or $A = \{x \in \mathbb{Q} \mid x < 0\}$, and similarly for B).[1] Then we define $\alpha \times \beta$ to be the Dedekind cut determined by the set

$$\{x \times y \mid x \in A, x \geq 0, y \in B, y \geq 0\} \cup \{a \in \mathbb{Q} \mid a < 0\}.$$

Now suppose that $\alpha \geq 0$ and $\beta < 0$. Then the above definition applies to α and $-\beta$ and we define

$$\alpha \times \beta = -(\alpha \times (-\beta)).$$

Similarly, if $\alpha < 0$ and $\beta \geq 0$, we define

$$\alpha \times \beta = -((-\alpha) \times \beta),$$

whereas if $\alpha < 0$ and $\beta < 0$ we define

$$\alpha \times \beta = (-\alpha) \times (-\beta).$$

(4) Prove that if a and b are rational numbers, then addition and multiplication of a and b as Dedekind cuts agrees with the usual addition and multiplication of rational numbers. That is, if the map f is as defined in (1), prove that for all $a, b \in \mathbb{Q}$ we have:

$$f(a + b) = f(a) + f(b), \text{ and } f(a \times b) = f(a) \times f(b).$$

(In each equation above, the operation on the left hand side is the usual one in $\mathbb{Q}$, whereas the one on the right hand side is the one you defined above in (3).)

[1] When we say $\alpha \geq 0$, we are considering the 0 in $\mathbb{R}$ to be equal to $f(0)$.

Solution: Let a and b be rational numbers.

Let $A = \{x \in \mathbb{Q} \mid x < a\}$, and $B = \{y \in \mathbb{Q} \mid y < b\}$. Then A determines the Dedekind cut $f(a)$ and B determines the Dedekind cut $f(b)$.

Now, $f(a+b)$ is the Dedekind cut determined by $C = \{z \in \mathbb{Q} \mid z < a+b\}$.

By the definition of addition of Dedekind cuts above we have $f(a) + f(b)$ is the Dedekind cut determined by the set $\{x + y \mid x \in A, y \in B\}$.

Well, the set of rational numbers of the form $x + y$ where $x < a$ and $y < b$ and $x, y \in \mathbb{Q}$ is exactly the set of rational numbers less than $a + b$. Therefore,

$$\{x + y \mid x \in A, y \in B\} = C,$$

so $f(a) + f(b) = f(a+b)$. This proves that addition of 'rational' Dedekind cuts agrees with the usual addition of rational numbers.

We'll now move on to multiplication.

First suppose that $a \geq 0$ and $b \geq 0$. Then $f(a) \times f(b)$ is the Dedekind cut determined by the set

$$D = \{xy \mid x \in A, x \geq 0, y \in B, y \geq 0\} \cup \{a \in \mathbb{Q} \mid a < 0\},$$

as in the definition in Part (iii). It is now easiest to consider the complement of D:

$$\mathbb{Q} \setminus D = \{x \times y \mid x \in \mathbb{Q} \setminus A, y \in \mathbb{Q} \setminus B\}.$$

(This is the correct definition of $\mathbb{Q} \setminus D$ only because $a \geq 0$ and $b \geq 0$.)

Note that $\mathbb{Q} \setminus D$ has a smallest element, and it is $a \times b$. Also, $\mathbb{Q} \setminus D$ is upwards closed (since D is downwards closed). Therefore, we must have

$$D = \{z \in \mathbb{Q} \mid z < a \times b\},$$

which is the set determining the Dedekind cut $f(a \times b)$, by the definition of f.

Thus, in case $a, b \geq 0$ we have $f(a) \times f(b) = f(a \times b)$, as required.

In order to deal with the other cases, it is convenient to consider the negative, since we defined the multiplication of negative numbers using the negative.

Now, $-f(a)$ is determined by the set

$$D = \{x \mid -x \notin A, -x \text{ not the least member of } \mathbb{Q} \setminus A\}.$$

Since $A = \{x \in Q \mid x < a\}$, we have

$$\begin{aligned} D &= \{x \mid -x \notin A, -x \text{ not the least member of } \mathbb{Q} \setminus A\} \\ &= \{x \in \mathbb{Q} \mid -x > a\} \\ &= \{y \in \mathbb{Q} \mid y < -a\}. \end{aligned}$$

This is exactly the set which determines the Dedekind cut $f(-a)$, which is to say that $-f(a) = f(-a)$.

Now, suppose that $a \geq 0$ and $b < 0$. Then we have $f(a) \geq 0$, $f(b) < 0$. Therefore, by the definition of multiplication, of f and the above calculations:

$$\begin{aligned} f(a) \times f(b) &= -(f(a) \times (-f(b))) \\ &= -(f(a) \times f(-b)) \\ &= -f(a \times (-b)) \\ &= f(-(a \times (-b))) \\ &= f(a \times b), \end{aligned}$$

as required.

Similarly, if $a < 0$ and $b \geq 0$ we have $f(a) < 0$, $f(b) \geq 0$, and so

$$\begin{aligned} f(a) \times f(b) &= -((-f(a)) \times f(b)) \\ &= -(f(-a) \times f(b)) \\ &= -f((-a) \times b) \\ &= f(-(-a \times b)) \\ &= f(a \times b), \end{aligned}$$

as required.

Finally, if $a < 0$ and $b < 0$ we have $f(a) < 0$ and $f(b) < 0$. Then we have

$$\begin{aligned} f(a) \times f(b) &= (-f(a)) \times (-f(b)) \\ &= f(-a) \times f(-b) \\ &= f((-a) \times (-b)) \\ &= f(a \times b), \end{aligned}$$

as required.

Therefore, in all cases we have $f(a) \times f(b) = f(a \times b)$, which is what we were required to prove.