HOMEWORK #2
Model Solutions

(1) Give the definition of the real numbers \( \mathbb{R} \) in terms of Dedekind cuts, and define the inclusion map from the rational numbers to the real numbers, \( f : \mathbb{Q} \to \mathbb{R} \). (That is, if \( a \) is a rational number, show how to identify \( a \) with a particular Dedekind cut.)

**Solution:** Let \( \mathbb{Q} \) be the set of rational numbers. A Dedekind cut is a pair \((A,B)\) of subsets of \( \mathbb{Q} \) satisfying:

(i) \( A \cup B = \mathbb{Q}, \ A \cap B = \emptyset; \)

(ii) If \( x,y \in \mathbb{Q}, \ x < y \) and \( y \in A \) then \( x \in A \). Conversely, if \( x,y \in \mathbb{Q}, \ x < y \) and \( x \in B \) then \( y \in B \).

(iii) \( A \) has no greatest member (that is to say there is no \( a \in A \) so that for all \( x \in A \) we have \( x \leq a \)).

We define the real numbers, as a set, to be the set of all Dedekind cuts.

The inclusion map \( f : \mathbb{Q} \to \mathbb{R} \) is defined by

\[
f(a) = (\{x \in \mathbb{Q} \mid x < a\}, \{y \in \mathbb{Q} \mid y \geq a\}).
\]

NOTE: If \((A,B)\) is Dedekind cut then condition (i) above ensures that \( B = \mathbb{Q} \setminus A \). Therefore, a Dedekind cut is determined by the first element of a pair.

For the remainder of the solutions, we’ll just refer to a Dedekind cut as a set \( A \) which satisfies the following two conditions:

(a) \( A \) is nonempty, and has no greatest element; and

(b) If \( x \in \mathbb{Q} \) and \( y \in \mathbb{Q} \) satisfies \( y < x \) then \( y \in \mathbb{Q} \).

We say that \( A \) determines a Dedekind cut, and hence also that \( A \) determines a real number.

Aside (not required for solution to the problem): A real number \( \alpha \), determined by a set \( A \), is irrational if \( \mathbb{Q} \setminus A \) does not have a least element, and rational exactly when \( \mathbb{Q} \setminus A \) does have a least element.

(2) Define the order ‘<’ on Dedekind cuts. You do not need to prove anything about this order.

**Solution:** Let the sets \( A_1 \) and \( A_2 \) determine Dedekind cuts, and hence real numbers \( \alpha_1 \) and \( \alpha_2 \) respectively. We define \( \alpha_1 < \alpha_2 \) to mean \( A_1 \subset A_2 \).
(3) Define the negative of a Dedekind cut, and also addition and multiplication of Dedekind cuts. (Remember to define multiplication $a \times b$ when one or both of $a$ and $b$ is negative.)

**Solution:** If $A$ determines a Dedekind cut, and hence a real number $\alpha$, then the real number $-\alpha$ is the Dedekind cut determined by the set

$$\{ x \in \mathbb{Q} \mid -x \notin A, \text{ and } -x \text{ is not the least element of } \mathbb{Q} \setminus A \}.$$  

(Note: It is not necessarily the case that $\mathbb{Q} \setminus A$ has a least element. If it doesn’t, then the condition that $-x$ is not the least element of $\mathbb{Q} \setminus A$ is trivially true.)

If $A$ determines a real number $\alpha$ and $B$ determines a real number $\beta$ then we define $\alpha + \beta$ to be the Dedekind cut determined by the set:

$$\{ x + y \mid x \in A, y \in B \}.$$  

If $A$ determines a real number $\alpha$, and $B$ determines a real number $\beta$ then we define $\alpha \times \beta$ as follows:

First suppose that $\alpha \geq 0$ and $\beta \geq 0$ (this means that $0 \in A$ or $A = \{ x \in \mathbb{Q} \mid x < 0 \}$, and similarly for $B$). Then we define $\alpha \times \beta$ to be the Dedekind cut determined by the set

$$\{ x \times y \mid x \in A, x \geq 0, y \in B, y \geq 0 \} \cup \{ a \in \mathbb{Q} \mid a < 0 \}.$$  

Now suppose that $\alpha \geq 0$ and $\beta < 0$. Then the above definition applies to $\alpha$ and $-\beta$ and we define

$$\alpha \times \beta = - (\alpha \times (-\beta)).$$  

Similarly, if $\alpha < 0$ and $\beta \geq 0$, we define

$$\alpha \times \beta = -((-\alpha) \times \beta),$$

whereas if $\alpha < 0$ and $\beta < 0$ we define

$$\alpha \times \beta = (-\alpha) \times (-\beta).$$

(4) Prove that if $a$ and $b$ are rational numbers, then addition and multiplication of $a$ and $b$ as Dedekind cuts agrees with the usual addition and multiplication of rational numbers. That is, if the map $f$ is as defined in (1), prove that for all $a, b \in \mathbb{Q}$ we have:

$$f(a + b) = f(a) + f(b), \text{ and } f(a \times b) = f(a) \times f(b).$$

(In each equation above, the operation on the left hand side is the usual one in $\mathbb{Q}$, whereas the one on the right hand side is the one you defined above in (3).)

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1. When we say $\alpha \geq 0$, we are considering the 0 in $\mathbb{R}$ to be equal to $f(0)$. 
Solution: Let $a$ and $b$ be rational numbers.

Let $A = \{ x \in \mathbb{Q} \mid x < a \}$, and $B = \{ y \in \mathbb{Q} \mid y < b \}$. Then $A$ determines the Dedekind cut $f(a)$ and $B$ determines the Dedekind cut $f(b)$.

Now, $f(a+b)$ is the Dedekind cut determined by $C = \{ z \in \mathbb{Q} \mid z < a+b \}$.

By the definition of addition of Dedekind cuts above we have $f(a) + f(b)$ is the Dedekind cut determined by the set $\{ x + y \mid x \in A, y \in B \}$.

Well, the set of rational numbers of the form $x + y$ where $x < a$ and $y < b$ and $x,y \in \mathbb{Q}$ is exactly the set of rational numbers less than $a + b$.

Therefore, 

$$\{ x + y \mid x \in A, y \in B \} = C,$$

so $f(a) + f(b) = f(a+b)$. This proves that addition of ‘rational’ Dedekind cuts agrees with the usual addition of rational numbers.

We’ll now move on to multiplication.

First suppose that $a \geq 0$ and $b \geq 0$. Then $f(a) \times f(b)$ is the Dedekind cut determined by the set

$$D = \{ xy \mid x \in A, x \geq 0, y \in B, y \geq 0 \} \cup \{ a \in \mathbb{Q} \mid a < 0 \},$$

as in the definition in Part (iii). It is now easiest to consider the complement of $D$:

$$\mathbb{Q} \smallsetminus D = \{ x \times y \mid x \in \mathbb{Q} \smallsetminus A, y \in \mathbb{Q} \smallsetminus B \}.$$

(This is the correct definition of $\mathbb{Q} \smallsetminus D$ only because $a \geq 0$ and $b \geq 0$.)

Note that $\mathbb{Q} \smallsetminus D$ has a smallest element, and it is $a \times b$. Also, $\mathbb{Q} \smallsetminus D$ is upwards closed (since $D$ is downwards closed). Therefore, we must have

$$D = \{ z \in \mathbb{Q} \mid z < a \times b \},$$

which is the set determining the Dedekind cut $f(a \times b)$, by the definition of $f$.

Thus, in case $a,b \geq 0$ we have $f(a) \times f(b) = f(a \times b)$, as required.

In order to deal with the other cases, it is convenient to consider the negative, since we defined the multiplication of negative numbers using the negative.

Now, $-f(a)$ is determined by the set

$$D = \{ x \mid -x \not\in A, -x \text{ not the least member of } \mathbb{Q} \smallsetminus A \}.$$

Since $A = \{ x \in \mathbb{Q} \mid x < a \}$, we have

$$D = \{ x \mid -x \not\in A, -x \text{ not the least member of } \mathbb{Q} \smallsetminus A \}$$

$$= \{ x \in \mathbb{Q} \mid -x > a \}$$

$$= \{ y \in \mathbb{Q} \mid y < -a \}. $$
This is exactly the set which determines the Dedekind cut $f(-a)$, which is to say that $-f(a) = f(-a)$.

Now, suppose that $a \geq 0$ and $b < 0$. Then we have $f(a) \geq 0$, $f(b) < 0$. Therefore, by the definition of multiplication, of $f$ and the above calculations:

$$f(a) \times f(b) = -(f(a) \times (-f(b)))$$
$$= -(f(a) \times f(-b))$$
$$= -f(a \times (-b))$$
$$= f(-(a \times (-b)))$$
$$= f(a \times b),$$

as required.

Similarly, if $a < 0$ and $b \geq 0$ we have $f(a) < 0$, $f(b) \geq 0$, and so

$$f(a) \times f(b) = -((-f(a)) \times f(b))$$
$$= -(f(-a) \times f(b))$$
$$= -f((-a) \times b)$$
$$= f(-(a \times b))$$
$$= f(a \times b),$$

as required.

Finally, if $a < 0$ and $b < 0$ we have $f(a) < 0$ and $f(b) < 0$. Then we have

$$f(a) \times f(b) = (-f(a)) \times (-f(b))$$
$$= f(-a) \times f(-b)$$
$$= f((-a) \times (-b))$$
$$= f(a \times b),$$

as required.

Therefore, in all cases we have $f(a) \times f(b) = f(a \times b)$, which is what we were required to prove.